Econ 101A – Solution to Final Exam
F 13 December.

Problem 1. Cost functions (18 points) Consider the cost functions in Figure 1a and 1b.

1. Take the total cost function in Figure 1a and draw the marginal cost function c'_y and the average cost function $c(y)/y$. What is the supply function, that is, the quantity $y^*(p)$ that a perfectly-competitive firm will produce as a function of the price p? What does the firm produce when the price p is higher than the marginal cost $c'(y)$? (6 points)
2. Take the total cost function in Figure 1b and draw the marginal cost function c'_y and the average cost function $c(y)/y$. Draw the supply function, that is, the quantity $y^*(p)$ that a perfectly-competitive firm will produce as a function of the price p. Here you do not need to solve analytically. (6 points)
3. Draw the industry supply function for firms that have cost function as in Figure 1b if there are 3 firms in the market. (5 points)

Solution to Problem 1.

1. The marginal cost $c'_y(y)$ in this case equals the average cost $c(y)/y$ and is equal to a constant c. The supply function $y(p)$ equals 0 if the price p is lower than the marginal cost c and is equal to any quantity if the price p equals c, since in this case the profits are identically 0 no matter what the quantity produced y. Finally, for $p > c$ the firm makes positive profits $p - c$ for each unit produced, and therefore will want to increase production as much as possible, theoretically up to infinity.
2. See Appendix.
3. See Appendix.

Problem 2. Short answer problems. (points) In the following problems, you are required to give a short answer.

1. Annibal has homework to do. Instead he goes out with friends and gets a D on the homework. Does this imply that Annibal is time-inconsistent? (4 points)
2. Annabel does not buy icecream boxes from a shop because she knows that if she buys them, she will eat them. Today Annabel came home from school and discovered that her sister has bought icecream. Annable eats half of the 1gallon icrecream box. Does this imply that Annabel is time-inconsistent? (4 points)
3. Find the *pure-strategy* Nash Equilibria of the following simultaneous game [do NOT look for mixed-strategy equilibria]: (4 points)

1\2	Left	Middle	Right
Up	$1, 1$	$3, 0$	$2, 1$
Middle	$1, 0$	$0, -1$	$0, -2$
Down	$0, 5$	$1, 1$	$1, 1$

Solution to Problem 2.

1. The fact that Annibal did not do the homework does *not* imply time inconsistency. He may just be very impatient (high discounting δ) or may like games more than the net benefit from going to school.
2. Unlike Annibal, Annabel is for sure time-inconsistent. If she has icecream in from of her, she cannot resist temptation and she eats it. On the other hand, if she has to make a decision for the future (i.e., whether to purchase an icecream box), she is patient and prefers no icecream around.
3. In the matrix above I underlined the best responses:

1\2	Left	Middle	Right
Up	$\underline{1}, \underline{1}$	$\underline{3}, 0$	$\underline{2}, \underline{1}$
Middle	$\underline{1}, \underline{0}$	$0, -1$	$0, -2$
Down	$0, \underline{5}$	$1, 1$	$1, 1$

The Nash equilibria are (Up, Left), (Up, Right), and (Middle, Left).

Problem 3. Public good contribution (33 points) In this exercise, we consider the problem of contribution to public goods. Assume that in a community of n individuals each individual i decides a contribution g_i toward a public good. The total quantity of public good provided will equal $G = \sum_{j=1}^{n} g_j$. Each individual pays a cost of effort $\gamma (g_i)^2/2$ for the contribution. Therefore the utility function of individual i is

$$U_i (g_i, g_{-i}) = G - \gamma \frac{(g_i)^2}{2} = \sum_{j=1}^{n} g_j - \gamma \frac{(g_i)^2}{2}.$$

You can think as being public radio ('funded by our listeners'), the garbage collection, or contribution to a charity like Doctors without Borders.

1. Consider first what the socially optimal solution. Assume that the social welfare measure V is the sum of the individual utility functions $V = \sum_{j=1}^{n} U_j (g_j, g_{-j})$. Maximize V with respect to g_i for $i = 1, ..., n$. Using the first order conditions, determine the socially optimal $g^*_{SO,i}$ for $i = 1, ..., n$ and G^*_{SO} (5 points)

2. How do you know that the solutions for $g^*_{SO,i}$ are a maximum? Argue that, even though you may not know how to compute the determinant of an n-by-n Hessian, what you found is indeed an optimum. (6 points)

3. Consider now the problem of **simultaneous contribution** to public goods in the community. Find the Nash Equilibrium in the contribution level. As in Cournot duopoly, each individual maximizes holding the contribution of others g^*_{-i} constant. What is the quantity contributed g^*_i? (4 points)

4. Compare the contributions in the Nash equilibium to the social optimum quantity $g^*_{SO,i}$? In particular, compute $g^*_i / g^*_{SO,i}$. How does this vary with the number of individuals n? In which communities is the problem of underprovision of public goods more serious? (3 points)

5. Assume now altruistic individuals that maximize $U_i (g_i, g_{-i}) + \alpha \sum_{j \neq i} U_j (g_j, g_{-j})$. That is, individuals put weight $\alpha > 0$ on the utility of other people in the community. Recompute the Nash equilibrium g^*_i for the case of altruism. What is the comparative static with respect to α? (6 points)

6. Suppose now that the government wishes to attain the socially optimal level of contribution $g^*_{SO,i}$. Some marketing groups just discovered that with appropriate advertisement campaigns it is possible to change the level of altruism α of people. Suppose that the campaign is costless for the government. Compute the level of α the the government with induce. (5 points)

7. What does economics suggest in this case? Is it better if people are nice to each other or selfish? Would you choose to live in a society with high or low α? 4 points)

Solution to Problem 3.

1. The social welfare function equals the sum of the individual utility functions. Therefore,

$$V = \sum_{i=1}^{n} \left(G - \gamma \frac{(g_i)^2}{2} \right) = nG - \frac{\gamma}{2} \sum_{i=1}^{n} (g_i)^2 = n \sum_{i=1}^{n} g_i - \frac{\gamma}{2} \sum_{i=1}^{n} (g_i)^2 .$$

   The first order condition of V with respect to g_i is

$$n - \gamma g^*_i = 0$$

   or $g^*_{SO,i} = n/\gamma$. The total quantity of the public good provided is $G^*_{SO} = n^2/\gamma$.

2. Notice that in principle we should write down the n-by-n Hessian matrix of second derivatives. It is easy to show that this matrix would have all zero elements off the diagonal, and $-\gamma$ on the diagonal. Assuming $\gamma > 0$, this implies that the determinants of the minors alternate sign starting from a negative

minor, which is the condition needed for a maximum. But you were not supposed to know this. The alternative way to get this is just to notice that the maximization problem is separable in each of the variables. That is, the value of g_j does not affect the optimal value of g_i for $i \neq j$. To see this, rewrite V as

$$V = n\sum_{i=1}^{n} g_i - \frac{\gamma}{2}\sum_{i=1}^{n}(g_i)^2 = \sum_{i=1}^{n}\left(ng_i - \gamma\frac{(g_i)^2}{2}\right)$$

where the last expression makes clear that one can solve the maximization for each of the terms in parenthesis separately. To conclude, notice that each of the terms in parenthesis have second order condition $-\gamma < 0$, necessary condition for a minimum.

3. The individual maximizes

$$U_i\left(g_i, g_{-i}^*\right) = G - \gamma\frac{(g_i)^2}{2} = g_i + g_{-i}^* - \gamma\frac{(g_i)^2}{2}.$$

The first order condition with respect to i g_i is

$$1 - \gamma g_i^* = 0$$

or $g_i^* = 1/\gamma$.

4. To compare the Nash equilibrium with the social optimum, we compute

$$g_i^*/g_{SO,i}^* = \frac{1/\gamma}{n/\gamma} = \frac{1}{n}.$$

The ratio is clearly decreasing in n. Therefore the problem of underprovision of public good becomes dramatically more severe as the number of persons in the community n increases.

5. The altruistic indiviual now maximizes

$$\begin{aligned} U_i(g_i, g_{-i}) + \alpha\sum_{j\neq i} U_i(g_i, g_{-i}) &= G - \gamma\frac{(g_i)^2}{2} + \alpha\left[(n-1)G - \gamma\sum_{j\neq i}\frac{(g_j)^2}{2}\right] = \\ &= [1 + \alpha(n-1)]\left(g_i + g_{-i}^*\right) - \gamma\left[\frac{(g_i)^2}{2} + \alpha\sum_{j\neq i}\frac{\left(g_j^*\right)^2}{2}\right]. \end{aligned}$$

The first order condition with respect to g_i is

$$[1 + \alpha(n-1)] - \gamma g_i^* = 0$$

or $g_i^* = [1 + \alpha(n-1)]/\gamma$. The optimal contribution g_i^* is increasing in the altruism parameter α.

6. The government knows that altruistic individuals contribute $[1 + \alpha(n-1)]/\gamma$ to the public good. The government would like to attain the level of contribution $g_{SO,i}^* = n/\gamma$. It is not hard to notice that the two levels of contribution are equal if

$$1 + \alpha(n-1) = n$$

or $\alpha = 1$. Not surprisingly, if the individuals put equal weight on the utility of others as on own utility, we obtain the socially optimal allocation.

7. In this case it is certainly nice to be in a society where other people are altruistic, since this increases the level of contribution of everyone and allowsconsumers to alleviate (or eliminate of $\alpha = 1$) the public good problem. Notice that even a selfish individual would prefer to be surrounded to nice people.

Problem 4. Hotelling model of spatial competition (36 points) In this exercise, we consider the problem of political parties that seeks to determine the optimal placement on the Left-right spectrum. Denote the placement of party i as $t_i \in [0,1]$, where $t = 0$ indicates left wing and $t = 1$ indicates right wing. The parties seek to maximize the number of votes received. The voter political views are uniformly distributed between 0 and 1. Voters vote for the political party that is closer to them and, if two parties are equally close, they randomize.

1. Consider first the case of two parties. Assume that party 1 chooses to place at $t_1 < t_2$, the placement of party 2. Show that the share of votes that party 1 receives is $t_1 + (t_2 - t_1)/2$. [Consequently, the share of votes received by party 2 is $1 - (t_1 + (t_2 - t_1)/2)$] Similarly, show that if $t_1 = t_2$, the share of votes of party 1 is 1/2. Finally, for $t_1 > t_2$, show that the share of votes for party 1 is $1 - t_1 + (t_1 - t_2)/2$ [and the share of votes of party 2 is $t_1 - (t_1 - t_2)/2$] (5 points)

2. Consider first the case of **sequential decision**. Assume that in period 1 party 1 has chosen a placement at t_1. What placement t_2^* will party 2 choose (as a function of t_1) in period 2 in order to maximize its share of votes? (5 points)

3. Now that you have determined the decision of party 2 in period 2, determine the optimal decision t_1^* of party 1 in period 1.(5 points)

4. What features does this subgame-perfect equilibrium have? Do you think that it reflects some feature of the American two-party system? (4 points)

5. Consider now the case of **simultaneous decision**. Now parties choose simultaneously where to locate on the policy space. Show that $t_1^* = t_2^* = 1/2$ is a Nash Equilibrium. You do not need to find all the Nash Equilibria. However, I can tell you that it is unique. (5 points)

6. We now investigate whether the solution $t_1^* = t_2^* = 1/2$ is also socially optimal. Suppose that a voter with policy preference t has a disutility $(t - t^*)^2$ from voting a politician with policy stand t^*. That is, if I am conservative, I do not mind too much voting for a middle-of-the-road politician, but I really hate voting for Ralph Nader. Call t_1^* and t_2^* the two positions of the politicians and assume $t_1^* \leq t_2^*$. What is the average disutility U associated with the politician choices? I answer this question for you: it is

$$\begin{aligned} U &= \int_0^{(t_1^*+t_2^*)/2} (t - t_1^*)^2 \, dt + \int_{(t_1^*+t_2^*)/2}^{1} (t - t_2^*)^2 \, dt = \left[\frac{1}{3}(t - t_1^*)^3\right]_0^{(t_1^*+t_2^*)/2} + \left[\frac{1}{3}(t - t_2^*)^3\right]_{(t_1^*+t_2^*)/2}^{1} = \\ &= \frac{1}{3}\left\{\frac{1}{8}(t_2^* - t_1^*)^3 + (t_1^*)^3 + (1 - t_2^*)^3 - \frac{1}{8}(t_1^* - t_2^*)^3\right\} = \frac{1}{3}\left\{(t_1^*)^3 + (1 - t_2^*)^3 + \frac{1}{4}(t_2^* - t_1^*)^3\right\}. \end{aligned}$$

   Now take the final expression for the disutility U (the one after the last equality sign) and minimize it with respect to t_1^* and t_2^*, that is, write down the first order conditions. Solve for t_1^* and t_2^*. (8 points)

7. Check that the second order conditions for a minimum of U are met at t_1^* and t_2^*. [minimum, not maximum! If you could not solve for t_1^* and t_2^* in the previous point, state in general the second order conditions for a minimum] (4 points)

Solution to Problem 4.

1. If $t_1 < t_2$, the voters vote as follows. All the voters to the left of t_1, that is, a share t_1 of the vote, vote for politician 1. Politician 1 in addition gets half ot the voters that are in between t_1 and t_2, that is, she gets $(t_1 - t_2)/2$ votes. The total share is $t_1 + (t_2 - t_1)/2$. If $t_1 = t_2$ all voters are indifferent, and each party gets 1/2 of the vote. Finally, if $t_1 > t_2$ party 1 gets all the voters to the right of t_1 (a share $1 - t_1$) plus half of the voters between t_1 and t_2, for a total share of $1 - t_1 + (t_1 - t_2)/2$.

2. Note that, for given t_1, party 2 can always get $1/2$ of the votes by choosing $t_2^* = t_1$. We now study whether it can get more. Assume first $t_1 < 1/2$. By choosing $t_2 > t_1$, party 2 gets share $1 - (t_1 + (t_2 - t_1)/2) = 1 - (t_1 + t_2)/2$. By pushing t_2 infinitely close to t_1, party 2 can get a share of the votes that is almost equal to $1 - t_1 > 1/2$. By choosing $t_2 < t_1$, party 2 gets share $t_2 + (t_1 - t_2)/2 = (t_1 + t_2)/2$. Notice that, since $t_1 < 1/2$, $(t_1 + t_2)/2 < 1/2$. therefore, if $t_1 < 1/2$, the optimal stategy for firm 2 is to choose $t_2^* = t_1 + \varepsilon$ with $\varepsilon > 0$ very small. It is easy to show with a similar argument that if $t_1 > 1/2$, firm 2's optimal decision is to choose $t_2^* = t_1 - \varepsilon$ with $\varepsilon > 0$ very small. Finally, for $t_1 = 1/2$, the optimal choice is $t_2^* = t_1 = 1/2$.

3. Party 1 anticipates the choice of party 2. It knows that if it chooses $t_1 < 1/2$ party 2 will choose $t_2^* = t_1 + \varepsilon$. The resulting share of votes for player 1 is $t_1 + \varepsilon/2 < 1/2$. If party 1 chooses $t_1 = 1/2$ party 2 chooses $t_2^* = 1/2$ and party 1 receives $1/2$ of the vote. Finally, if party chooses $t_1 > 1/2$ party 2 will choose $t_2^* = t_1 - \varepsilon$ and party 1 gets share $1 - t_1 + \varepsilon/2 < 1/2$. Therefore the optimal choice for party 1 is to choose $t_1^* = 1/2$. In the subgame perfectequilibrium

4. In the subgame-perfect equilibrium of the Hotelling game both parties locate in the middle of the policy arena. This is a strange outcome. What's the point of having two parties if they choose the same policies? In American Politics, with two main parties in the political arena, we indeed see quite a bit of policy convergence to the middle.

5. In order to show that $t_1^* = t_2^* = 1/2$ is a Nash Equilibrium we need to show that no player can strictly benefit from deviating. In equilibrium both players gain a share $1/2$ of the vote. Consider now the possible deviations by player 1. If party 1 deviates to $t_1 < t_1^* = t_2^* = 1/2$, it gets the payoff $t_1 + (1/2 - t_1)/2 = (1/2 + t_1)/2 < 1/2$. Therefore, it is not profitable for player 1 to deviate to $t_1 < 1/2$. Consider now the other possible deviation. If party 1 deviates to $t_1 > t_1^* = t_2^* = 1/2$, it gets the payoff $1 - t_1 + (t_1 - 1/2)/2 = 1 - (1/2 + t_1)/2 < 1/2$. Therefore, it is not profitable for player 1 to deviate to $t_1 > 1/2$. There is no strategy for party 1 that increases the payoff beyond the equilibrium payoff of $1/2$. Since the game is symmetric, the same argument holds for deviations of player 2. Therefore, $t_1^* = t_2^* = 1/2$ is a Nash Equilibrium.

6. If we take first order conditions with respect to t_1^* and t_2^* of the disutility function ,we get
$$\frac{\partial U}{\partial t_1^*} = \frac{1}{3}\left\{3\left(t_1^*\right)^2 - \frac{3}{4}\left(t_2^* - t_1^*\right)^2\right\} = 0 \tag{1}$$
and
$$\frac{\partial U}{\partial t_2^*} = \frac{1}{3}\left\{-3\left(1 - t_2^*\right)^2 + \frac{3}{4}\left(t_2^* - t_1^*\right)^2\right\} = 0.$$
This implies
$$\frac{3}{4}\left(t_2^* - t_1^*\right)^2 = 3\left(t_1^*\right)^2 = 3\left(1 - t_2^*\right)^2.$$
The last equality implies $t_1^* = 1 - t_2^*$, that is, the two parties have to be equally distant from the extreme positions. If we plug back this condition into the f.o.c. (1), we get
$$\frac{1}{3}\left\{3\left(t_1^*\right)^2 - \frac{3}{4}\left(1 - t_1^* - t_1^*\right)^2\right\} = 0$$
or
$$0 = 4t_1^{*2} - \left(1 - 2t_1^*\right)^2 = 4t_1^{*2} - 1 + 4t_1^* - 4t_1^{*2} = -1 + 4t_1^*,$$
which implies $t_1^* = 1/4$ and, as a consequence of $t_1^* = 1 - t_2^*$, $t_2^* = 1 - 1/4 = 3/4$. The socially optimal policy allocation of the parties minimizes the distance between the voters and the parties.

7. The second order conditions are
$$\begin{aligned} H &= \frac{6}{3}\begin{bmatrix} t_1^* + \frac{1}{4}\left(t_2^* - t_1^*\right) & -\frac{1}{4}\left(t_2^* - t_1^*\right) \\ -\frac{1}{4}\left(t_2^* - t_1^*\right) & \left(1 - t_2^*\right) + \frac{1}{4}\left(t_2^* - t_1^*\right) \end{bmatrix} = \\ & \frac{6}{3}\begin{bmatrix} \frac{1}{4} + \frac{1}{4}\frac{1}{2} & -\frac{1}{4}\frac{1}{2} \\ -\frac{1}{4}\frac{1}{2} & \frac{1}{4} + \frac{1}{4}\frac{1}{2} \end{bmatrix} \end{aligned}$$
The term $t_1^* + \frac{1}{4}\left(t_2^* - t_1^*\right)$ should be positive, and it is. As for the determinant, it equals $\left(\frac{1}{4} + \frac{1}{4}\frac{1}{2}\right)^2 - \left(\frac{1}{4}\frac{1}{2}\right)^2 > 0$. Therefore the necessary conditions for a minimum of U are satisfied.

Problem 5. Wonderport economics. (28 points) You are in Wonderland and you just landed at the local airport Wonderport. In Wonderport shops sell exclusively toys. Each toy is produced at a constant marginal cost c. In the population there are kids and adults, with K being the number of kids and A the number of adults. People go to Wonderport once a year. That is, K kids and A adults visit Wonderport per year. In Wonderland both kids and adults earn money and can afford to shop. The value that a kid assigns to a toy is v_k. That is, a kid will buy one toy if the price is lower than or equal to v_k, and not buy otherwise. Similarly, each adult values one toy v_a, with $v_k > v_a > c$. The value of any toy purchase in a year beyond the first is zero, both for kids and adults.

Unfortunately, we cannot offer you a free trip to Wonderland. However, as an apprentice economist, you get to guess the pricing at Wonderport. Have a safe journey of Wonderlanomics!

1. Once upon a time in Wonderport there used to be many independently-owned perfectly-competing shops selling toys. Assume that these shops were in the long-run equilibrium. What was the price of a toy back then? Did both kids and adults purchase toys? What were the profits of the firms? What about the surplus of the consumers (measured as willingness to pay minus price paid)? (5 points)

2. In 5,670 W.T. (Wonder Time) the government decided to introduce a per-unit tax t on addictive goods like toys. How much did the price charged to consumers change between years 5,669 and 5,670? Who bore the burden of a tax? Did both kids and adults puchase toys? What were the profits of the firms? What about the surplus of the consumers? (4 points)

3. In 6,200 W.T. a large company consolidated the shop industry. Since then, a monopolist owns the toy shops in Wonderport. (assume no tax) The monopolist can price discriminate by designing fully separate kid-shops and adult-shops. Resale of toys carries the death penalty. What will the price be in kid-shops? And in adult-shops? Do both groups buy? What are the profits of the firm? What about the surplus of the consumers? (6 points)

4. In 6,500 W.T. the government imposed once again a tax t on toys. What is the price in kid-shops? And in adult-shops? Do both groups buy? What are the profits of the firm? What about the surplus of the consumers? Who bears the burden of the tax? (5 points)

5. Finally, in 7,000 W.T. the government decides to remove the tax under the condition that the monopolist stops unfairly discriminating against kids. Now that the monopolist charges one price for toys, what is it as a function of v_k, v_a, K, and A? Do both groups buy? What are the profits of the firm? How do they compare to the case of price discrimination (and no tax)? What about the surplus of the consumers? (8 points)

Solution to Problem 5.

1. The case with many firms corresponds to the case of perfect competition. under perfect competition, firms charge price equal to marginal cost. therefore, p equals c. Since $v_k > v_a > c$, both kids and adults purchase. The profits of the firms equal zero and the surplus of the consumers is $A(v_a - c) + K(v_k - c)$.

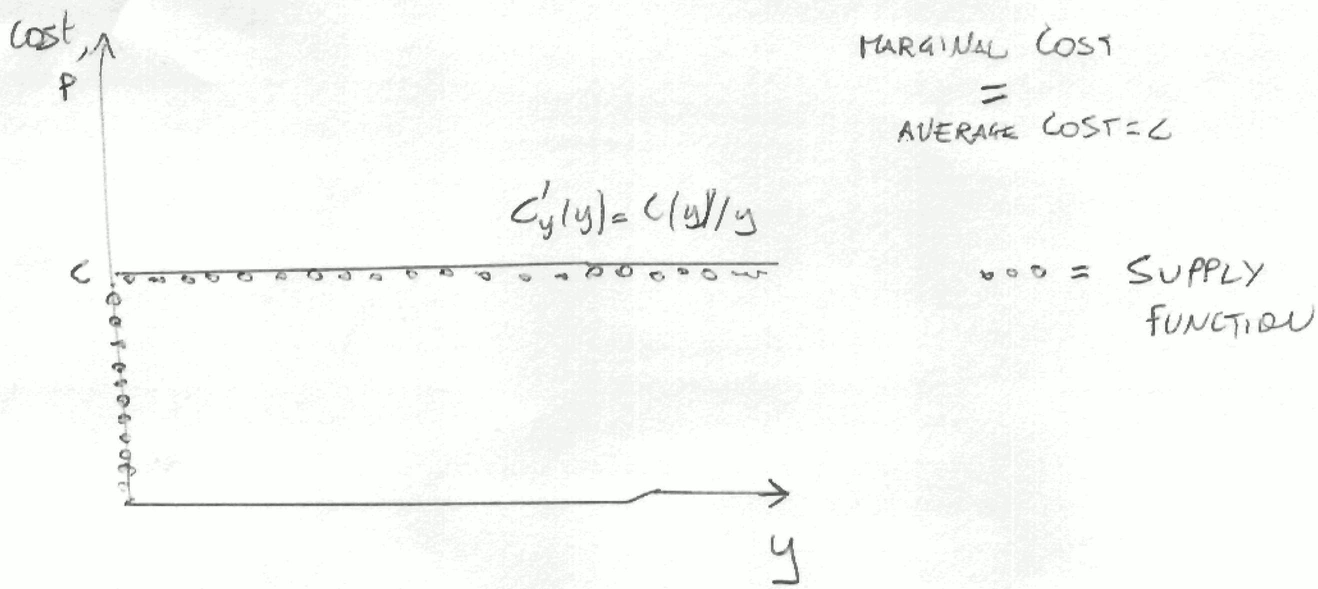
2. Under perfect competition with constant marginal cost, the burden of the tax falls completely on the consumers. The producers cannot charge anything less than $c + t$ for a toy, nor will they charge more since the presence of perfect competition imposes that prices must equal marginal cost of production. Kids purchase if $v_k > c + t$ and adults purchase if $v_a > c + t$. The profits are again zero and the surplus of the consumers is $A(v_a - c - t)\,\mathbf{1}(v_k - c - t \geq 0) + K(v_k - c - t)\,\mathbf{1}(v_k - c - t \geq 0)$ where $\mathbf{1}$ is the indicator function, that is it equals zero whenever the expression in parenthesis is true. and zero otherwise.

3. The monopolist will charge the maximum possible price to kids and to adults separately. Therefore the price of the toy in kid-shops will be v_k and the price of the toy in adult-shops will be v_a. Both groups buy. The profits for the firm equal $(v_a - c)A + (v_k - c)K$ and the consumer surplus is zero.

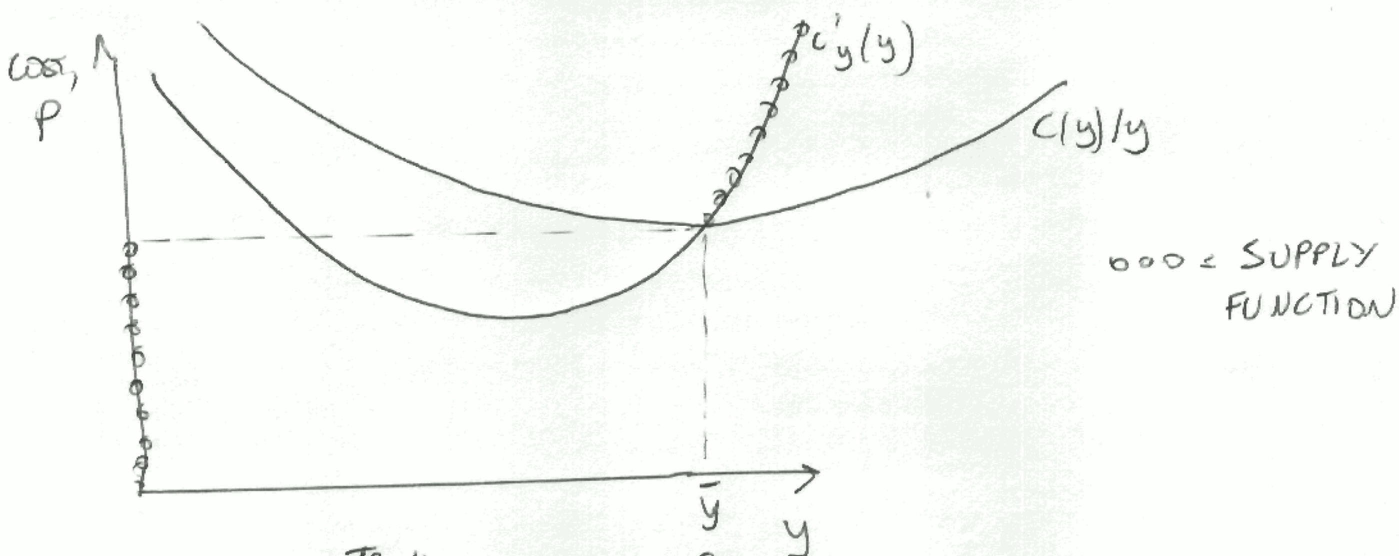
4. After the imposition of the tax, the firm charges v_a in adult shops if $v_a > c+t$ and $c+t$ otherwise (or any other price so that noone will buy). Similarly, the firm charges v_k in kid shops if $v_k > c+t$ and $c+t$ otherwise. If $v_k > v_a > c+t$ both kids and adults buy and firm profits are $(v_a - c - t)\,A + (v_k - c - t)\,K$. If $v_k > c+t > v_a$ only kids buy and firm profits are $(v_k - c - t)\,K$. If $c+t > v_k > v_a$ neither kids nor adults buy and firm profits are 0. As for consumer surplus, it is always zero.

5. Now the monopolist can charge only one price. If the price is smaller than v_a, both consumers will buy. If the price p is such that $v_k > p > v_a$, only the children buy. Finally, if $p > v_k$, noone buys. Clearly, a price different from v_a or v_k cannot be optimal. The firm can increase its profits by setting the price at either v_a or v_k. At a price of v_a both kids and adults purchase and the profits are $(v_a - c)\,(A + K)$ .At the price of v_k only the kids purchase and the profits are $(v_k - c)\,K$. The firm prefers the price v_a if $(v_a - c)\,(A + K) \geq (v_k - c)\,K$. In this case, consumer surplus if $K\,(v_k - v_a)$. Otherwise, consumer surplus is zero. In either case, the profits are clearly lower than the profits under perfect price discrimination.

From Figure 1a



From Figure 1b



If there are 3 firms, the industry-level supply curve will be