Econ 101A – Final exam
F 12 December.

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Problem 1. Nash equilibrium in 2X2 games (24 points) Consider the following game:

1\2	Left	Right
Up	$0, 0$	$1, 1$
Down	$4, \alpha$	$-2, -2$

1. Find all the pure *and mixed* strategy Nash Equilibria of the above game for $\alpha = 4$. You may find it useful to call u the probability that player 1 plays Up, $1 - u$ the probability that player 1 plays Down, l the probability that Player 2 plays Left, and $1 - l$ the probability that Player 2 plays Right. (10 points)
2. How does your answer for the mixed stategy change if $\alpha = 2$? (6 points)
3. Notice that we changed a payoff of player 2 from 4 to 2. Whose player equilibrium strategies change? Does this surprise you? Can you provide some explanation? (8 points)

Solution to Problem 1.

1. We determine the best response of player 1 to an action of player 2. Player 1 prefers to play Up if

$$EU\left(D\right) = l * 0 + \left(1 - l\right) * 1 \geq l * 4 + \left(1 - l\right) * \left(-2\right)$$

or

$$\left(1 - l\right) \geq -2 + 6l$$

or

$$l \leq 3/7.$$

So the best response correspondence for player 1 is

$$BR_1\left(l\right) = \begin{cases} u = 1 & \text{if} \quad l < 3/7 \\ \text{any } u \in [0, 1] & \text{if} \quad l = 3/7 \\ u = 0 & \text{if} \quad l > 3/7 \end{cases}$$

Player 2 prefers to play Left if

$$EU\left(L\right) = u * 0 + \left(1 - u\right) * \alpha \geq u * 1 + \left(1 - u\right) * \left(-2\right)$$

or

$$\alpha\left(1 - u\right) \geq -2 + 3u$$

or

$$u \leq \left(2 + \alpha\right) / \left(3 + \alpha\right).$$

So the best response correspondence for player 1 is

$$BR_2\left(u\right) = \begin{cases} l = 1 & \text{if} \quad u < \left(2 + \alpha\right) / \left(3 + \alpha\right) \\ \text{any } l \in [0, 1] & \text{if} \quad u = \left(2 + \alpha\right) / \left(3 + \alpha\right) \\ l = 0 & \text{if} \quad u > \left(2 + \alpha\right) / \left(3 + \alpha\right) \end{cases}$$

. We plot the best responses in Figure 1. The three points of intercection are the three Nash Equilibria. We can write them in therms of the values of u and l :

$$\left(u^*, l^*\right) = \left(\left(0, 1\right), \left(1, 0\right), \left(\frac{2 + \alpha}{3 + \alpha}, \frac{3}{7}\right)\right).$$

The first two equilibria are in pure strategies, the last in mixed strategies. For $\alpha = 4$ the mixed strategy equilibrium is $\left(u^* = \frac{6}{7}, l^* = \frac{3}{7}\right)$

2. In case $\alpha = 2$, the mixed stategy equilibrium becomes $\left(u^* = \frac{4}{5}, l^* = \frac{3}{7}\right)$.

3. We changed the payoffs for player 2 and the equilibrium strategy $u^*$of player 1 changes, with the strategy l^* of player 2 remaining constant. This is surprising, because one would expect that in response to a change in the payoffs of player 2 the strategy of player 2 changes. The reason for this result is that in a mixed strategy equilibrium

Problem 2. Homeworks and Altruism. (22 points) Student i has $H > 0$ hours of time and can either work on own homework h_1, or on homework of student 2, h_2. We denote the number of hours spent by student i on homework j as h_j^i,with $h_j^i \geq 0$. The time constraint for student i therefore is

$$h_1^i + h_2^i \leq H$$

The happiness of student i, $i = 1, 2$, is given by

$$U_i\left(h_i^1, h_i^2\right) = \left(h_i^1 + h_i^2\right)^{.5},$$

which captures the fact that the more hours are spent on homework h_i by the two students, the better the grade for i, and the happier student i. However, there are decreasing returns to spending time on the homework, captured by the exponent .5. Selfish student 2 maximizes $U_2\left(h_2^1, h_2^2\right)$, while altruistic student 1 maximizes $U_1\left(h_1^1, h_1^2\right) + \alpha U\left(h_2^1, h_2^2\right)$. Suppose that student 1 moves first, and student 2 moves second, after observing student 1's decision.

1. Solve utility maximization for student 2 subject to time constraint as a function of h_2^1, that is, of how much student 1 helps student 2. Find $h_2^{2*}\left(h_2^1\right)$. (5 points)

2. Set up the utility maximization for student 1 subject to time constraint. Student 1 anticipates $h_1^{2*}\left(h_2^1\right)$ and $h_2^{2*}\left(h_2^1\right)$. Write the first order condition. (4 points)

3. Obtain the solution for h_1^{1*} and h_2^{1*} if $\alpha = 0$ (selfishness). Show that $h_1^{1*} < H$ if $\alpha > 0$ (altruism) (7 points)

4. What is the solution for h_1^{1*} and h_2^{1*} if $\alpha < 0$ (spite)? (6 points)

Solution to Problem 2.

1. Utility maximization of student 2 is

$$\max_{h_1^2, h_2^2} \left(h_2^1 + h_2^2\right)^{.5}$$
$$s.t.\ h_1^2 + h_2^2 \leq H$$

Given that the objective function is increasing in h_2^2 but does not contain h_1^2, student 2 will spend all the hours doing own homework, that is, $h_2^{2*}\left(h_2^1\right) = H$ and $h_1^{2*}\left(h_2^1\right) = 0$. Notice that the solution does not depend on the strategy that student 1 previously chose. Student 2 does not return the favor if he/she received $h_2^1 > 0$.

2. Student 1 takes into account $h_2^{2*}\left(h_2^1\right) = H$ and $h_1^{2*}\left(h_2^1\right) = 0$. Therefore, the utility maximization of student 1 is

$$\max_{h_1^1, h_2^1} \left(h_1^1\right)^{.5} + \alpha\left(h_2^1 + H\right)^{.5}$$
$$s.t.\ h_1^1 + h_2^1 \leq H$$

Given that the objective function is increasing in both h_1^1 and h_2^1, the time constraint will be binding. We can therefore substitute $.h_2^{1*} = H - h_2^1$. We obtain

$$\max_{h_1^1} \left(h_1^1\right)^{.5} + \alpha \left(2H - h_1^1\right)^{.5}.$$

The first order condition is

$$.5 \left(h_1^1\right)^{-.5} - .5\alpha \left(2H - h_1^1\right)^{-.5} = 0. \tag{1}$$

3. The derivative of the utility function of student 1 with respect to h_1^1 is positive if

$$.5 \left(h_1^1\right)^{-.5} \geq .5\alpha \left(2H - h_1^1\right)^{-.5}. \tag{2}$$

Notice that for any of the feasible values of h_1^1, $(0 < h_1^1 \leq H)$, we obtain $\left(h_1^1\right)^{-.5} \geq \left(2H - h_1^1\right)^{-.5}$. For $\alpha \leq 1$, this implies that inequality (2) always holds, and therefore the agent will push the value of h_1^{1*} all the way up to H. For the case $\alpha > 1$, instead, the solution will be such that $h_1^{1*} < H$. For h_1^1 equal to H, inequality (2) does not hold since $\left(h_1^1\right)^{-.5}$ equals $\left(2H - h_1^1\right)^{-.5}$. at H. The agent therefore sets $h_1^{1*} < H$.

4. If $\alpha < 0$, inequality (2) will always be satisfied. Therefore, the agent increases h_1^1 all the way up to H. Here, if agent 1 could, he/she would make h_2^1 negative! (this is also true for the case $\alpha < 1$).

Problem 3. Externalities in Production (38 points) In this exercise, we consider the problem of externalities generated by two firms. An externality occurs when a firm does not take into account in its maximization problem the effect of one of its decisions on other firms. The idea of this problem is that each of the firms produces a pollutant that increases the costs of production of the other firm. Assume that two firms compete á la Cournot. Firm i produces q_i units of the good at total cost cq_iq_{-i}, with $b > c > 0$. For example, the total production cost of firm 1 is cq_2q_1. This captures the fact that increased production by the competitor increases the production costs. For $c = 0$, we obtain a standard Cournot case with zero marginal cost of production. The inverse demand function is $P(Q) = a - bQ = a - b(q_1 + q_2)$.

1. Consider first the Cournot solution. Write down the profit function for firm i as a function of the production of the competitor, q_{-i}. Write down the first order condition of firm i and solve for q_i^* as a function of q_{-i}^*, that is, find the best response function for firm i, for $i = 1, 2$. (6 points)
2. Find the solution for the Cournot oligopoly by requiring that both best response functions hold. In other words, solve the system of two first order conditions for the two firms. Find the solutions for $q_1^* = q_2^*$ and for p^* (4 points)
3. In the plane (q_1, q_2) graph the best response function of both firms for $a = b = 1$ and $c = 0$. Indicate the Cournot Nash equilibrium in the graph. (6 points)
4. Now assume that the two firms merge together and become one monopolist. The monopolist maximizes the total profits from the two plants, that is $(a - bq_1 - bq_2)(q_1 + q_2) - 2cq_1q_2$. Write down the first order conditions with respect to q_1 and q_2. Compare these first order conditions to the ones of the Cournot case in point 1. There are two differences between the f.o.c.s. Where do they come from? What is the sense in which the monopolist internalizes externalities? (8 points)
5. Find the optimum for q_1^M and q_2^M, that is the quantity that the merged monopolist produces in each plant. (the monopolist will produce an equal quantity q_M in both plants) (4 points)
6. Compare the quantities produced in monopoly and duopoly. In particular, the ratio q^*/q^M should come out to be

$$\frac{q^*}{q_M} = \frac{4b + 2c}{3b + c}.$$

Show that this ratio is increasing in c. What is the intuition for the fact that the 'overproduction' of duopoly relative to monopoly is more accentuated when c is large? (10 points)

Solution to Problem 3.

1. Each firm maximizes
$$\max_{q_i} (a - bq_i - bq_{-i})\, q_i - cq_i q_{-i}$$
which yields the f.o.c.
$$a - 2bq_i^* - bq_{-i} - cq_{-i} = 0$$
or
$$q_i^* = \frac{a}{2b} - \frac{b+c}{2b} q_{-i}. \tag{3}$$

2. The system of first order conditions is
$$q_1^* = \frac{a}{2b} - \frac{b+c}{2b} q_2^*$$
and
$$q_2^* = \frac{a}{2b} - \frac{b+c}{2b} q_1^*$$
We can solve the system by adding the first order conditions
$$q_1^* + q_2^* = \frac{a}{b} - \frac{b+c}{2b}\,(q_1^* + q_2^*)$$
or
$$q_1^* + q_2^* = \frac{a}{b}\frac{2b}{3b+c} = \frac{2a}{3b+c}.$$
By simmetry, we know $q_1^* = q_2^* = q^*$, which implies
$$q^* = \frac{a}{3b+c}$$
. The solution for p^* is
$$p^* = a - 2bq^* = a - 2b\frac{a}{3b+c} = a\left(\frac{b+c}{3b+c}\right).$$

3. See Figure 1.

4. The monopolist maximizes
$$\max_{q_1,q_2} (a - bq_1 - bq_2)\,(q_1 + q_2) - 2cq_1 q_2$$
with f.o.c.s
$$a - 2bq_1^M - 2bq_2^M - 2cq_2^M = 0$$
and
$$a - 2bq_2^M - 2bq_1^M - 2cq_1^M = 0$$
In comparing these f.o.c.s to the ones of the Cournot case, two differences emerge: the coefficient on the competitor quantity is higher, as well as on the competitor's cost. There are two externalities that the firm is taking into account. The first is standard. The firm takes into account that raising quantity produced in one plant will reduce profits in the other plant. The second externality arises because of externalisties in costs ($c > 0$). The monopolist takes into account that additional production in each firm will increase the cost in the other firm as well.

5. The solution of the monopoly case is as follows. The monopolist will produce equally in both plants, that is, $q_1^M = q_2^M = q^M$. Adding the two f.o.c.s yields
$$2a - 4bq^M - 4bq^M - 4cq^M = 0$$
or
$$q^M = \frac{2a}{8b+4c} = \frac{a}{4b+2c}.$$

6. The ratio q^*/q_M equals

$$\frac{q^*}{q_M} = \frac{\frac{a}{3b+c}}{\frac{a}{4b+2c}} = \frac{4b+2c}{3b+c}.$$

The derivative with respect to c equals

$$\partial \frac{q^*}{q_M} / \partial c = \frac{2(3b+c) - (4b+2c)}{(3b+c)^2} = \frac{6b+2c-4b-2c}{(3b+c)^2} = \frac{2b}{(3b+c)^2} > 0.$$

Problem 4. The economics of brass. (70 points) In this problem we will consider the production of a good, brass, that needs two inputs, copper and zinc. Unlike what we did in class, we are going to explicitely model the market for the inputs, that is, the prodution of copper and zinc. We are going to assume that there is a monopolist in each of the copper and zinc makerts.

1. Write down the chemical formula for a molecule of brass. Just kidding! That formula is Cu_3Zn_2 where Cu indicates one molecule of copper and Zn indicates one molecule of Zinc. Argue that therefore the production function for brass is $\min(c/3, z/2)$, where c is the quantity of copper and z is the quantity of zinc. (3 points)

2. Plot an isoquant for the production of brass. Remember, the axes are the quantities of inputs c and z. (4 points)

3. A brass producer wants to produce b units of brass. Assume that the cost of one unit of Copper is $p_c > 0$ and the cost of one unit of Zinc in $p_z > 0$. Solve for the cost-minimizing combination of inputs $c^*(p_c, p_z, b)$ and $z^*(p_c, p_z, b)$ that produce quantity b of output. Write down the cost function $C(p_c, p_z, b) = p_c c^*(p_c, p_z, b) + p_z z^*(p_c, p_z, b)$. (Hint: Be careful about taking derivatives, the drawing of the isoquant should help you in guiding you to the solution) (I can give you the answer for 10 points if you get stuck here) (10 points)

4. Assume that the brass industry is perfectly competitive. Derive the supply function of a brass firm $b^*(p_b, p_c, p_z)$. Graph the firm and industry supply function assuming N firms. Does it make a difference for the supply functions if there is just 1 or 10 or 100 firms? (5 points)

5. The market demand function for brass is $p_b(b) = a - b$, where b is the total quantity of brass produced in the economy. Plot demand and supply assuming $a = 10$, $p_c = 1$ and $p_z = 1$. Helping yourself with the graph, solve for the total market production of brass $b^*(p_c, p_z)$, as well as for the equilibrium price $p_b^*(p_c, p_z)$. (6 points)

6. Write down also the derived demand for the inputs $c^*(p_c, p_z)$ and $z^*(p_c, p_z)$ in equilibrium using the expression for demand for inputs you found in point 3. Argue that these are also the demand functions that the copper and zinc monopolists face. (4 points)

7. We now model the market for the inputs. Assume that the market for zinc is controlled by a monopolist, and that the market for copper is controlled by a (different) monopolist. The two monopolists decide on production simultaneously. In both cases, cost of production are zero. Write down the profit maximization problem for the copper producer as a function of the price of zinc p_z: $\max_{p_c} p_c c^*(p_c, p_z)$ (Hint: you are better off maximizing profits of the copper monopolist with respect to price p_c, rather than with respect to quantity c, although both give the same solution). Write down the first order conditions and solve for $p_c^*(p_z)$ and $c^*(p_z)$. Check the second order conditions (with respect to p_c) (7 points)

8. Similarly, write down and solve for the problem of the monopolist in the zinc industry. Find the solution for $p_z^*(p_c)$ and $z^*(p_c)$. (3 points)

9. Finally, now that we have solved for the company's production function as a function of the price set by the other monopolist, look for a Nash equilibrium for p_c^* and p_z^*. Derive also $z^*$and c^*. In which sense this game is like Cournot, despite the fact that the two firms are monopolists producing different goods? Compute the implied price of brass p_b^* (10 points)

10. The final results that you get for point 9 should imply $p_b^* = (2/3)\,a$. Suppose now that the copper and zinc monopolists, instead of competing with each other, merged and maximized the sum of their joint profits. I am not asking you to solve for this, I will give you the solution. In this case, the final price of brass will be $p_b^{*\prime} = a/2$. In correspondence of these two values of the price of brass, compute the quantity of brass b^* and $b^{*\prime}$ demanded in both cases using the demand function $p_b\,(b) = a - b$. Draw the solution graphically for $a = 10$ and compute the consumer surplus (I suggest measuring the area of the appropriate triangle in the graph). (8 points)

11. (Hard) Why is it that having one monopolist control the whole market of the inputs yields lower price and higher surplus in the final goods market? In other words, the situation with two competing monopolists in the input market yields higher price for brass and lower consumer welfare than the situation with just one firm that controls all the inputs. Why is this the case? Usually, it is bad for consumers if two firms merge and/or collude. (this is a famous counterexample) (10 points)

Solution to Problem 4.

1. The chemical formula indicates that in order to form one molecule of brass one needs 3 molecules of copper and 2 of zinc, in a fixed proportion. For a given quantity of z zinc atoms, having more than $(3/2)\,z$ molecules of copper does not help to produce more brass. The assumption that the goods need to be in fixed proportion leads to a production function á la Leontieff, $\min(c/3, z/2)$. For example, one needs 6 atoms of copper and 4 of zinc to produce 2 molecules of brass. Having 6 atoms of each still leads to only 2 molecules of brass. (of course we are neglecting the fact that one cannot split atoms into fractions!)

2. See Figure 2.

3. The cost minimization is

$$\begin{aligned}&\min_{c,z} p_c c + p_z z\\&s.t.\;\; \min(c/3, z/2) \geq b.\end{aligned}$$

In order to minimize costs, the companies producing brass will want to adopt a ratio of copper to steel such that $c^*/3 = z^*/2$, or $c^* = (3/2)\,z^*$. Having a ratio of imputs which is different is suboptimal, since the company can reduce costs while still producing the same quantity. To prove this, suppose that the company employed $c^* > (3/2)\,z^*$. It could then cut purchases of copper to $(3/2)\,z^*$ and make savings of $p_c * (c^* - (3/2)\,z^*) > 0$, while still producing the same quantity. Similarly, if $c^* > (3/2)\,z^*$, then the firm could cut the zinc input quantity. We therefore know that in the cost minimizing solution $c^*\,(z) = (3/2)\,z$, and we can therefore rewrite the problem as

$$\begin{aligned}&\min_{z} p_c \frac{3}{2} z + p_z z = \left(\frac{3}{2} p_c + p_z\right) z\\&s.t.\;\; z/2 \geq b.\end{aligned}$$

At this point, it is easy to see that the constaint will be satisfied with equality, since the firm does not want to purchase more zink than is necessary to obtain b units of brass. The solution ,therefore, is $z^*/2 = b$ or $z^* = 2b$ and, using $c^*\,(z) = (3/2)\,z$, $c^* = (3/2)\,2b = 3b$. The cost function therefore is

$$c\,(p_c, p_z, b) = p_c 3b + p_z 2b = (3p_c + 2p_z)\,b$$

[ironically, to the people to whom I gave the solution, I gave the following wrong solution. Many apologies. You were not penalized, and this should not have impeded you in the calculations

$$c(p_c, p_z, b) = p_c 2b + p_z 3b = (2p_c + 3p_z)\, b$$

The 'wrong'solution below is in square brackets]

4. The supply function is given by the marginal cost function above the average cost. In this case, since the cost function $c(p_c, p_z, b)$ is linear in b, the marginal cost and average cost both equal $(2p_c + 3p_z)$. Therefore, the supply function of a brass firm is

$$b^*(p_b, p_c, p_z) = \begin{cases} \infty & \text{if} \quad p_b > (2p_c + 3p_z) \;\; [2p_c + 3p_z] \\ \text{any } b \in [0, \infty) & \text{if} \quad p_b = (2p_c + 3p_z) \;\; [2p_c + 3p_z] \\ 0 & \text{if} \quad p_b < (2p_c + 3p_z) \;\; [2p_c + 3p_z] \end{cases}$$

The industry supply function has the same shape as the individual firm supply function (see Figure 3).

5. The equilibrium in the brass market is determined by the intersection of demand and supply. Given that supply is horizontal, supply determines the equilibrium price p_b^*, with $p_b^*(p_c, p_z) = (3p_c + 2p_z)$ $[2p_c + 3p_z]$ Given this, we can find the indeustry supply of brass on the demand curve for brass as $b^*(p_c, p_z) = a - (3p_c + 2p_z)$. $[a - (2p_c + 3p_z)$.]

6. The derived demands for the inputs are $c^*(p_c, p_z) = 3b^*(p_c, p_z) = 3(a - (3p_c + 2p_z)) = 3a - 9p_c - 6p_z$ and $z^*(p_c, p_z) = 2b^*(p_c, p_z) = 2(a - (3p_c + 2p_z)) = 2a - 6p_c - 4p_z$.[WRONG, ACCEPTED VERSION: $c^*(p_c, p_z) = 3b^*(p_c, p_z) = 3(a - (2p_c + 3p_z)) = 3a - 6p_c - 9p_z$ and $z^*(p_c, p_z) = 2b^*(p_c, p_z) = 2(a - (2p_c + 3p_z)) = 2a - 4p_c - 6p_z$.]

7. Now, the inputs market. The monopolist that produces copper realizes that its demand function is determined by the demand of brass, which itself depends also on the price of zinc. In other words, copper is an intermediate product in this case, and its demand function is given by the input demand function of the final good (brass) that uses it. The monopolist producer of copper solves:

$$\max_{p_c} p_c c^*(p_c, p_z) = p_c(3a - 9p_c - 6p_z)$$

with f.o.c.

$$3a - 18p_c^* - 6p_z = 0$$

which yields

$$p_c^*(p_z) = \frac{1}{6}a - \frac{1}{3}p_z$$

and

$$c^*(p_z) = \left(3a - 9\left(\frac{1}{6}a - \frac{1}{3}p_z\right) - 6p_z\right) = \frac{3}{2}a - 3p_z$$

The second order condition is $-8 < 0$, hence the condition is satisfied. [WRONG, ACCEPTED VERSION: The monopolist producer of copper solves:

$$\max_{p_c} p_c c^*(p_c, p_z) = p_c(3a - 6p_c - 9p_z)$$

with f.o.c.

$$3a - 12p_c^* - 9p_z = 0$$

which yields

$$p_c^*(p_z) = \frac{1}{4}a - \frac{3}{4}p_z$$

and

$$c^*(p_z) = \left(3a - 6\left(\frac{1}{4}a - \frac{3}{4}p_z\right) - 9p_z\right) = \frac{3}{2}a - \frac{9}{2}p_z$$

The second order condition is $-8 < 0$, hence the condition is satisfied.]

8. For zinc, we do a similar process:

$$\max_{p_z} p_z z^* (p_c, p_z) = p_z (2a - 6p_c - 4p_z)$$

with f.o.c.

$$2a - 6p_c - 8p_z^* = 0$$

whic yields

$$p_z^* (p_c) = \frac{1}{4}a - \frac{3}{4}p_c$$

and

$$z^* (p_c) = \left(2a - 6p_c - 4\left(\frac{1}{4}a - \frac{3}{4}p_c\right)\right) = a - 3p_c.$$

[WRONG, ACCEPTED VERSION: For zinc, we do a similar process:

$$\max_{p_z} p_z z^* (p_c, p_z) = p_z (2a - 4p_c - 6p_z)$$

with f.o.c.

$$2a - 4p_c - 12p_z^* = 0$$

whic yields

$$p_z^* (p_c) = \frac{1}{6}a - \frac{1}{3}p_c$$

and

$$z^* (p_c) = \left(2a - 4p_c - 6\left(\frac{1}{6}a - \frac{1}{3}p_c\right)\right) = a - 2p_c.$$

]

9. Since firms are deciding simultaneously on the production decision, the Nash Equilibrium can be obtained as the system of the two solutions:

$$p_c^* (p_z) = \frac{1}{6}a - \frac{1}{3}p_z$$

and

$$p_z^* (p_c) = \frac{1}{4}a - \frac{3}{4}p_c.$$

We obtain

$$p_c^* = \frac{1}{6}a - \frac{1}{3}\left(\frac{1}{4}a - \frac{3}{4}p_c^*\right)$$

or

$$p_c^* = \frac{4}{3}\frac{1}{12}a = \frac{1}{9}a$$

and

$$p_z^* (p_c) = \frac{1}{4}a - \frac{3}{4}\left(\frac{1}{9}a\right) = \frac{1}{6}a.$$

The quantities produced are

$$z^* = a - 3p_c^* = a - 3\left(\frac{1}{9}a\right) = \frac{2}{3}a$$

and

$$c^* = \frac{3}{2}a - 3p_z^* = \frac{3}{2}a - 3\left(\frac{1}{6}a\right) = a$$

This game is like a Cournot duopoly because the two monopolists ultimately set prices in the brass market, and there they compete with each other. The price of brass p_b^* equals

$$p_b^* = 3p_c^* + 2p_z^* = 3\left(\frac{1}{9}a\right) + 2\left(\frac{1}{6}a\right) = \frac{2}{3}a.$$

[WRONG, ACCEPTED VERSION: Since firms are deciding simultaneously on the production decision, the Nash Equilibrium can be obtained as the system of the two solutions:

$$p_c^*(p_z) = \frac{1}{4}a - \frac{3}{4}p_z$$

and

$$p_z^*(p_c) = \frac{1}{6}a - \frac{1}{3}p_c.$$

We obtain

$$p_c^* = \frac{1}{4}a - \frac{3}{4}\left(\frac{1}{6}a - \frac{1}{3}p_c^*\right)$$

or

$$p_c^* = \frac{4}{3}\frac{1}{8}a = \frac{1}{6}a$$

and

$$p_z^*(p_c) = \frac{1}{6}a - \frac{1}{3}\left(\frac{1}{6}a\right) = \frac{1}{9}a.$$

The quantities produced are

$$z^* = a - 2p_c^* = a - 2\left(\frac{1}{6}a\right) = \frac{2}{3}a$$

and

$$c^* = \frac{3}{2}a - \frac{9}{2}p_z^* = \frac{3}{2}a - \frac{9}{2}\left(\frac{1}{9}a\right) = a$$

This game is like a Cournot duopoly because the two monopolists ultimately set prices in the brass market, and there they compete with each other. The price of brass p_b^* equals

$$p_b^* = 2p_c^* + 3p_z^* = 2\left(\frac{1}{6}a\right) + 3\left(\frac{1}{9}a\right) = \frac{2}{3}a.$$

]

10. The total quantity demanded of brass will equal $a - p_b$. Therefore, we obtain

$$b^* = a - p_b^* = a - \frac{2}{3}a = \frac{1}{3}a$$

and

$$b^{*\prime} = a - p_b^{*\prime} = a - \frac{1}{2}a = \frac{1}{2}a.$$

Consumer surplus will equal the area in the Figure below the demand function and above the price:

$$S^* = \frac{1}{3}a\left(a - \frac{2}{3}a\right)/2 = \frac{1}{18}a^2$$

and

$$S^{*\prime} = \frac{1}{2}a\left(a - \frac{1}{2}a\right)/2 = \frac{1}{4}a^2.$$