Economics 101A
(Lecture 9, Revised)

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Outline

1. Expenditure Minimization II
2. Expenditure Min: First order conditions
3. Slutzky equation
4. Complements and substitutes
5. Do utility functions exist?

1 Expenditure minimization II

- Nicholson, Ch. 4, pp. 105–108.

- Solve problem **EMIN** (minimize expenditure):

$$\min p_1 x_1 + p_2 x_2$$
$$s.t.\ u(x_1, x_2) \geq \bar{u}$$

- $h_i(p_1, p_2, \bar{u})$ is *Hicksian or compensated demand*

- Optimum coincides with optimum of Utility Maximization!

- Formally:

$$h_i(p_1, p_2, \bar{u}) = x_i^*(p_1, p_2, e(p_1, p_2, \bar{u}))$$

- Expenditure function is expenditure at optimum

- $e(p_1, p_2, \bar{u}) = p_1 h_1(p_1, p_2, \bar{u}) + p_2 h_2(p_1, p_2, \bar{u})$

- $h_i(p_i)$ is *Hicksian or compensated demand* function

- Is h_i always decreasing in p_i? Yes!

- Graphical proof: moving along a convex indifference curve

- (For non-convex indifferent curves, still true)

- Now: go back to case where p_2 increases to $p_2' > p_2$

- What is $\partial x_2^* / \partial p_2$? Decompose effect:

  1. Substitution effect of an increase in p_i

     – $\partial h_2^* / \partial p_2$, that is change in EMIN point as p_2 descreases

     – Moving along an indifference curve

     – Certainly $\partial h_2^* / \partial p_2 < 0$

2. Income effect of an increase in p_i

   - $\partial x_2^*/\partial M$, increase in consumption of good 2 due to increased income

- * Shift out a budget line

  * $\partial x_2^*/\partial M > 0$ for normal goods, $\partial x_2^*/\partial M < 0$ for inferior goods

1.1 EMin: First Order Conditions

- Using first order conditions:

$$L(x_1, x_2, \lambda) = p_1 x_1 + p_2 x_2 - \lambda \left(u(x_1, x_2) - \bar{u} \right)$$

$$\frac{\partial L}{\partial x_i} = p_i - \lambda u_i'(x_1, x_2) = 0$$

- Write as ratios:

$$\frac{u_1'(x_1, x_2)}{u_2'(x_1, x_2)} = \frac{p_1}{p_2}$$

- $MRS =$ ratio of prices as in utility maximization!

- However: different constraint $\Longrightarrow \lambda$ is different

- Example 1: Cobb-Douglas utility

$$\min p_1 x_1 + p_2 x_2$$
$$s.t.\ x_1^{\alpha} x_2^{1-\alpha} \geq \bar{u}$$

- Lagrangean $=$

- F.o.c.:

- Solution: $h_1^* =$ $\quad , h_2^* =$

- $\partial h_i^* / \partial p_i < 0$, $\partial h_i^* / \partial p_j > 0$, $j \neq i$

2 Slutsky equation

- Nicholson, Ch. 5, pp. 131–136.

- $h_i(p_1, p_2, \bar{u}) = x_i^*(p_1, p_2, e(p_1, p_2, \bar{u}))$

- How does the Hicksian demand change if price p_i changes?

$$\frac{dh_i}{dp_i} = \frac{\partial x_i^*(\mathbf{p}, e)}{\partial p_i} + \frac{\partial x_i^*(\mathbf{p}, e)}{\partial M}\frac{\partial e(\mathbf{p}, \bar{u})}{\partial p_i}$$

- What is $\frac{\partial e(\mathbf{p},\bar{u})}{\partial p_i}$? Envelope theorem:

$$\begin{aligned}\frac{\partial e(\mathbf{p}, \bar{u})}{\partial p_i} &= \frac{\partial}{\partial p_i}\left[p_1 h_1^* + p_2 h_2^* - \lambda(u(h_1^*, h_2^*, \bar{u}) - \bar{u})\right] \\ &= h_i^*(p_1, p_2, \bar{u}) = x_i^*(p_1, p_2, e(p, \bar{u}))\end{aligned}$$

- Therefore

$$\frac{\partial h_i\left(\mathbf{p},\bar{u}\right)}{\partial p_i}=\frac{\partial x_i^*\left(\mathbf{p},e\right)}{\partial p_i}+\frac{\partial x_i^*\left(\mathbf{p},e\right)}{\partial M}x_1^*(p_1,p_2,e)$$

- Rewrite as

$$\begin{aligned}\frac{\partial x_i^*\left(\mathbf{p},M\right)}{\partial p_i} &= \frac{\partial h_i\left(\mathbf{p},v(\mathbf{p},M)\right)}{\partial p_i} \\ &\quad -x_1^*(p_1,p_2,M)\frac{\partial x_i^*\left(\mathbf{p},M\right)}{\partial M}\end{aligned}$$

- Important result! Allows decomposition into substitution and income effect

- Two effects of change in price:

  1. Substitution effect negative: $\frac{\partial h_i(\mathbf{p}, v(\mathbf{p}, M))}{\partial p_i} < 0$

  2. Income effect: $-x_1^*(p_1, p_2, M)\frac{\partial x_i^*(\mathbf{p}, M)}{\partial M}$

  – * negative if good i is normal ($\frac{\partial x_i^*(\mathbf{p}, M)}{\partial M} > 0$)

    * positive if good i is inferior ($\frac{\partial x_i^*(\mathbf{p}, M)}{\partial M} < 0$)

- Overall, sign of $\frac{\partial x_i^*(\mathbf{p}, M)}{\partial p_i}$?

  – negative if good i is normal

  – it depends if good i is inferior

- Example 1 (ctd.). Apply Slutsky equation

- $x_i^* = \alpha M / p_i$

- $h_i^* =$

- Derivative of Hicksian demand with respect to price:

$$\frac{\partial h_i\left(\mathbf{p}, \overline{u}\right)}{\partial p_i} =$$

- Rewrite h_i^* as function of m: $h_i\left(\mathbf{p}, v(\mathbf{p}, M)\right)$

- Compute $v(\mathbf{p}, M) =$

- Substitution effect:

$$\frac{\partial h_i\left(\mathbf{p}, v(\mathbf{p}, M)\right)}{\partial p_i} =$$

- Income effect:

$$-x_i^*(p_1, p_2, M)\frac{\partial x_i^*\left(\mathbf{p}, M\right)}{\partial M} =$$

- Sum them up to get

$$\frac{\partial x_i^*\left(\mathbf{p}, M\right)}{\partial p_i} =$$

- It works!

3 Complements and substitutes

- Nicholson, Ch. 6, pp. 152–158.

- How about if price of another good changes?

- Generalize Slutsky equation

- Slutsky Equation:

$$
\begin{aligned}
\frac{\partial x_i^*(\mathbf{p}, M)}{\partial p_j} &= \frac{\partial h_i(\mathbf{p}, v(\mathbf{p}, M))}{\partial p_j} \\
&\quad - x_j^*(p_1, p_2, M)\frac{\partial x_i^*(\mathbf{p}, M)}{\partial M}
\end{aligned}
$$

- Substitution effect

$$\frac{\partial h_i\left(\mathbf{p}, v(\mathbf{p}, M)\right)}{\partial p_j} > 0$$

for $n = 2$ (two goods). Ambiguous for $n > 2$.

- Income effect:

$$-x_j^*(p_1, p_2, M)\frac{\partial x_i^*\left(\mathbf{p}, M\right)}{\partial M}$$

  - negative if good i is normal
  - positive if good i is inferior

- How do we define complements and substitutes?

- Def. 1. Goods i and j are **gross substitutes** at price $\mathbf{p}$ and income M if

$$\frac{\partial x_i^* (\mathbf{p}, M)}{\partial p_j} > 0$$

- Def. 2. Goods i and j are **gross complements** at price $\mathbf{p}$ and income M if

$$\frac{\partial x_i^* (\mathbf{p}, M)}{\partial p_j} < 0$$

- Example 1 (ctd.): $x_1^* = \alpha M/p_1$, $x_2^* = \beta M/p_2$.

- Gross complements or gross substitutes? Neither!

- Notice: $\frac{\partial x_i^*(\mathbf{p},M)}{\partial p_j}$ is usually different from $\frac{\partial x_j^*(\mathbf{p},M)}{\partial p_i}$

- Better definition.

- Def. 3. Goods i and j are **net substitutes** at price $\mathbf{p}$ and income M if

$$\frac{\partial h_i^* \left(\mathbf{p}, v(\mathbf{p}, M)\right)}{\partial p_j} = \frac{\partial h_j^* \left(\mathbf{p}, v(\mathbf{p}, M)\right)}{\partial p_i} > 0$$

- Def. 4. Goods i and j are **net complements** at price $\mathbf{p}$ and income M if

$$\frac{\partial h_i^* \left(\mathbf{p}, v(\mathbf{p}, M)\right)}{\partial p_j} = \frac{\partial h_j^* \left(\mathbf{p}, v(\mathbf{p}, M)\right)}{\partial p_i} < 0$$

- Example 1 (ctd.): $h_1^* = \overline{u} \left(\frac{\alpha}{1-\alpha} \frac{p_2}{p_1}\right)^{1-\alpha}$

- Net complements or net substitutes? Net substitutes!

4 Do utility functions exist?

- Preferences and utilities are theoretical objects

- Many different ways to write them

- How do we tie them to the world?

- Use actual choices – revealed preferences approach

- Typical economists' approach. Compromise of:
  - realism
  - simplicity

- Assume a class of utility functions (CES, Cobb-Douglas...) with free parameters

- Estimate the parameters using the data