Economics 101A
(Lecture 16, Revised)

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Outline

1. Addenda to isoquants
2. Cost Minimization: Summary
3. Cost Minimization: Example
4. Geometry of Cost Curves

1 Addenda to isoquants

- Production function $f(L, K)$

- When are isoquants convex? When $d^2K/d^2L > 0$

- Mathematically,

$$\frac{dK}{dL}|_{\text{isoquant}} = -\frac{f_L'(L, K(L))}{f_K'(L, K(L))}$$

- What is

$$\frac{d^2K}{d^2L}|_{\text{isoquant}} = ?$$

Good exercise!

- The steps are as follows:

- d^2K/d^2L is the second derivative with respect to L of dK/dL. It follows $\frac{d^2K}{d^2L}|_{\text{isoquant}} =$

$$-\frac{\left[f''_{L,L}(L,K) + f''_{L,K}(L,K)\frac{\partial K(L)}{\partial L}\right] f'_K(L,K)}{\left(f'_K(K,L)\right)^2}$$
$$-\frac{-\left[f''_{K,L}(L,K) + f''_{K,K}(L,K)\frac{\partial K(L)}{\partial L}\right] f'_L(L,K)}{\left(f'_K(K,L)\right)^2}$$

- Substitute in

$$\frac{\partial K(L)}{\partial L} = -\frac{f'_L(L,K(L))}{f'_K(L,K(L))}$$

- Simplify and get

- $$\frac{d^2K}{d^2L}\Big|_{\text{isoquant}} = -\frac{f''_{L,L}(L,K(L))\, f'_K(L,K(L))}{\left(f'_K(K(L),L)\right)^2} + \frac{2f''_{K,L}(L,K(L))\, f'_L(L,K(L))}{\left(f'_K(K(L),L)\right)^2} - \frac{f''_{K,K}(L,K(L))\, \frac{\left[f'_L(L,K(L))\right]^2}{f'_K(L,K(L))}}{\left(f'_K(K(L),L)\right)^2}$$

- Terms 1 and 3 are always positive.

- Term 2 is positive if $f''_{K,L}(L,K(L)) \geq 0$

- Conclusion: $f''_{K,L}(L,K(L)) \geq 0$ is the only additional assumption we need to guarantee convex isoquants ($d^2K/d^2L > 0$)

2 Cost Minimization: Summary

- First stage. Firm's objective function:

$$\min_{L,K} wL + rK$$
$$s.t. f(L, K) \geq y$$

- Equality in constraint holds if:

  1. $w > 0$, $r > 0$;

  2. f strictly increasing in at least L or K.

- Counterexample if ass. 1 is not satisfied

- Counterexample if ass. 2 is not satisfied

- Second stage. Firm's objective function:

$$\max_y py - c(w, r, y)$$

- First order condition:

$$p - c'_y(w, r, y^*) = 0$$

- Second order condition:

$$-c''_{y,y}(w, r, y^*) < 0$$

- For maximum, need increasing marginal cost curve.

3 Cost Minimization: Example

- [HEAVILY REVISED BELOW]

- Continue example above: $y = f(L, K) = AK^{\alpha}L^{\beta}$

- Cost minimization:

$$\begin{aligned} &\min wL + rK \\ &s.t. AK^{\alpha}L^{\beta} = y \end{aligned}$$

- Solutions:

  – Optimal amount of labor:

$$L^*(r, w, y) = \left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}}$$

- Optimal amount of capital:

$$K^*(r,w,y) = \frac{w}{r}\frac{\alpha}{\beta}\left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}}\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} =$$
$$= \left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}}\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}$$

- Check various comparative statics:
  - $\partial L^*/\partial A < 0$ (technological progress and unemployment)
  - $\partial L^*/\partial y > 0$ (more workers needed to produce more output)
  - $\partial L^*/\partial w < 0$, $\partial L^*/\partial r > 0$ (substitute away from more expensive inputs)

- Parallel comparative statics for K^*

- Cost function

$$\begin{aligned} c(w,r,y) &= wL^*(r,w,y) + rK^*(r,w,y) = \\ &= \left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}} \left[\begin{array}{c} w\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} + \\ +r\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \end{array} \right] \end{aligned}$$

- Define $B := w\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} + r\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}$

- Cost-minimizing output choice:

$$\max py - B\left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}}$$

- First order condition:

$$p - \frac{1}{\alpha+\beta}\frac{B}{A}\left(\frac{y}{A}\right)^{\frac{1-(\alpha+\beta)}{\alpha+\beta}} = 0$$

- Second order condition:

$$-\frac{1}{\alpha+\beta}\frac{1-(\alpha+\beta)}{\alpha+\beta}\frac{B}{A^2}\left(\frac{y}{A}\right)^{\frac{1-2(\alpha+\beta)}{\alpha+\beta}}$$

- When is the second order condition satisfied?

- Solution:
  - $\alpha + \beta = 1$ (CRS):
    * S.o.c. equal to 0
    * Solution depends on p
    * For $p > \frac{1}{\alpha+\beta}\frac{B}{A}$, produce $y^* \to \infty$
    * For $p = \frac{1}{\alpha+\beta}\frac{B}{A}$, produce any $y^* \in [0, \infty)$
    * For $p < \frac{1}{\alpha+\beta}\frac{B}{A}$, produce $y^* = 0$

- $\alpha + \beta > 1$ (IRS):
  * S.o.c. positive
  * Solution of f.o.c. is a minimum!
  * Solution is $y^* \rightarrow \infty$.
  * Keep increasing production since higher production is associated iwth higher returns

- $\alpha + \beta < 1$ (DRS):
  * s.o.c. negative. OK!
  * Solution of f.o.c. is an interior optimum
  * This is the only "well-behaved" case under perfect competition
  * Here can define a supply function

4 Geometry of cost curves

- Nicholson, Ch. 12, pp. 307–312 and Ch. 13, pp. 342–346.

- Marginal costs $MC = \partial c/\partial y \rightarrow$ Cost minimization

$$p = MC = \partial c\left(w, r, y\right)/\partial y$$

- Average costs $AC = c/y \rightarrow$ Does firm break even?

$$\begin{aligned} \pi &= py - c\left(w, r, y\right) > 0 \text{ iff} \\ \pi/y &= p - c\left(w, r, y\right)/y > 0 \text{ iff} \\ c\left(w, r, y\right)/y &= AC < p \end{aligned}$$

- **Supply function.** Portion of marginal cost MC above average costs.(price equals marginal cost)

- Assume only 1 input (expenditure minimization is trivial)

- **Case 1.** Production function. $y = L^{\alpha}$

  - Cost function? (cost of input is w):

$$c(w, y) = wL^{*}(w, y) = wy^{1/\alpha}$$

  - Marginal cost?

$$\frac{\partial c(w, y)}{\partial y} = \frac{1}{\alpha} wy^{(1-\alpha)/\alpha}$$

  - Average cost $c(w, y)/y$?

$$\frac{c(w, y)}{y} = \frac{wy^{1/\alpha}}{y} = wy^{(1-\alpha)/\alpha}$$

- **Case 1a.** $\alpha > 1$. Plot production function, total cost, average and marginal. Supply function?

- **Case 1b.** $\alpha = 1$. Plot production function, total cost, average and marginal. Supply function?

- **Case 1c.** $\alpha < 1$. Plot production function, total cost, average and marginal. Supply function?

- **Case 2.** *Non-convex technology.* Plot production function, total cost, average and marginal. Supply function?

- **Case 3.** *Technology with setup cost.* Plot production function, total cost, average and marginal. Supply function?