Econ 101A – Final exam
Th 16 December.

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Please solve Problem 1 and 2 in the first blue book and Problems 3 and 4 in the second Blue Book. Good luck!

Problem 1. Car production (34 points). Consider a market for cars with just one firm. The firm has a linear cost function $C(q) = 2q$. The market inverse demand function is $P(Q) = 9 - Q$, where Q is the total quantity produced. Since initially there is just one firm, $q = Q$.

1. Set up the maximization problem for the monopolist and determine the optimal price and quantity of cars produced (6 points)
2. How much profit does the firm make? (4 points)
3. Consider now the case of a second firm entering the market. The two firms choose quantities simultaneously, that is, they compete á la Cournot. Set up the maximization problem. Determine the optimal price and quantity of cars produced. (6 points)
4. Compare the quantities and prices produced to the monopoly case. Provide intuition on the result. (4 points)
5. Compare total profits in Cournot and in monopoly (3 points).
6. Draw a graph with price on the y axis and quantity on the x axis. Locate the Cournot and monopoly outcomes. Compute the consumer surplus for the Cournot and the monopoly cases. Which market do consumers prefer? Provide intuition for the answer (7 points)
7. On the graph, identify the deadweight loss of going from Cournot to monopoly. (4 points)

Solution to Problem 1.

1. The monopolist maximizes
$$\max_q P(q)\,q - C(q) = (9 - q)\,q - 2q$$
which yields the f.o.c.
$$9 - 2q^* - 2 = 0$$
or
$$q^* = 7/2. \tag{1}$$
Given $q^* = 7/2$, the monopolist will charge price
$$p^* = 9 - q^* = 11/2.$$
2. The profit of the monopolist is
$$(9 - q^*)\,q^* - 2q^* = (11/2) * 7/2 - 7 = (77 - 28)/4 = 49/4.$$

3. In the Cournot case each firm i solves

$$\max_{q_i} P(q_i + q_j)\, q_i - C(q_i) = (9 - q_i - q_j)\, q_i - 2q_i$$

which yields the f.o.c.

$$9 - 2q_i^* - q_j - 2 = 0$$

or a reaction function

$$q_i^* = \frac{7 - q_j}{2}$$

Solving the system of two equations ($i = 1, 2$) gives

$$q_1^* = q_2^* = q_C^* = \frac{7}{3}.$$

The price is

$$p(Q) = 9 - 2 * \frac{7}{3} = \frac{27 - 14}{3} = \frac{13}{3}.$$

4. The total quantity produced in Cournot (14/3) is higher that the quantity produced in monopoly, while the price is lower ($13/3 < 11/2$). In Cournot firms produce more because they do not take into account the negative externality on the profits of the other firm induced by higher production.

5. The profit of each Cournot duopolist is

$$(9 - 2q_C^*)\, q_C^* - 2q_C^* = (13/3 * 7/3) - 14/3 = (91 - 42)/9 = 49/9$$

and the total profits equal 98/9, which is less than 49/4, which is the profit of the monopolist.

6. The consumer surplus is the triangle below the demand function and above the price charged in equilibrium. It equals $(9 - p) * q/2$. For the monopoly case, the surplus is $(9 - 11/2) * 7/4 = (7/2) * 7/4 = 49/8$. For the duopoly case, the surplus is $(9 - 13/3) * 14/6 = (14/3) * 7/3 = 98/9$. Cournot yields almost twice as much consumer surplus as monopoly. The increase in consumer surplus comes about because of both lower prices and higher quantities produced.

7. Not all the consumer surplus lost from monopoly goes 'wasted'. Some of it goes to the producer in the form of higher profits. However, a part of it is just lost. See graph.

Problem 2. Car Driving. (52 points) Two agents ($i = 1, 2$) are deciding how fast to drive and how much to consume. Each individual chooses speed x_i and get utility $u(x_i)$ from the choice of speed, with $u'(x) > 0$ and $u''(x) < 0$ or all x. (that is, each agent like faster speed because it allows her to get to more places in less time; in addition, there are diminishing gains to higher speed). The cost of driving faster is that it increases the probability of an accident. The probability of an accident for agent i is $\pi(x_i) + \pi(x_j)$, with $\pi'(x) > 0$ and $\pi''(x) > 0$. [Notation: x_j denotes the speed chosen by the other driver] The faster any of the agents drives, the higher the probability of accident for both. Furthermore, the probability of accident is convex in driving speed. The cost of an accident is c. The quantity of consumption is y_i, with price normalized to 1. The overall utility $v(x_i, y_i)$ of agent i is

$$v(x_i, y_i) = u(x_i) + y_i,$$

where y_i is the amount consumed of good i. The budget constraint is

$$(\pi(x_i) + \pi(x_j))\, c + y_i = M_i,$$

where M_i is the income of agent i.

1. Write out the maximization problem. Obtain the first-order condition for agent i with respect to x_i^* and write the expression for y_i^* [You are better off substituting the constraint into the utility function] (5 points)

2. Check the second order conditions. (4 points)

3. Use the implicit function theorem to obtain an expression for $\partial x_i^*/\partial c$ (speed of driving and cost of accident) What is the sign? Provide intuition (5 points)

4. Use the implicit function theorem to obtain an expression for $\partial x_i^*/\partial M$? (speed of driving and income) Provide intuition, in particular on the specific assumptions driving this result. (8 points)

5. Now assume that the decisions on speed ($x_{1,}$ x_2) and consumption (y_1, y_2) are taken by a central planner. The central planner maximizes the sum of the utilities of the two agents subject to the two budget constraints. Write the maximization problem. [Recommended substitution of the budget constraints into the objective function] (3 points)

6. Obtain the first order conditions of the problem of the planner. Do the solutions for x_1^P (planner problem) differ from x_1^* (individual problem)? In which direction? Provide intuition and try to characterize the general problem surfacing here. (10 points)

7. Go back now to the individual optimization problem. Assume now that agent i pays a fine t for each accident that involves her (no matter who caused it). (for example, the insurance premium increases in subsequent years) Solve the new problem for the individual. (6 points)

8. What is the level of fine t such that the solution to the individual problem coincides with the social optimum? Comment on the magnitude you find (7 points)

9. Does this problem suggest also a justification for speed limits? (4 points)

Solution to Problem 2.

1. Utility maximization of agent i is

$$\max_{x_i, y_i} u(x_i) + y_i$$
$$s.t.\ \ (\pi(x_1) + \pi(x_2))\, c + y_1 = M_i$$

We can substitute y_i into the objective function to transform this into

$$\max_{x_i, y_i} u(x_i) + M_i - (\pi(x_1) + \pi(x_2))\, c.$$

This leads to the first-order condition

$$u'(x_i) - \pi'(x_i)\, c = 0$$

The consumption level is defined by

$$y_1^* = M_i - (\pi(x_1) + \pi(x_2))\, c.$$

2. The second order conditions are

$$u''_{x_i}(x_i) - \pi''(x_i)\, c < 0$$

Since $u''_{x_i}(x_i) < 0$ for all x and $\pi''(x_1, x_2) > 0$ for all x, the second order conditions are satisfied.

3. Using the implicit function theorem,

$$\frac{\partial x_i^*}{\partial c} = -\frac{-\pi'(x_i)}{u''(x_i) - \pi''(x_i)\,c} < 0.$$

The higher the cost of an accident, the less fast people drive. It's a classical substitution effect, speed has become more expensive.

4. Using the implicit function theorem,

$$\frac{\partial x_i^*}{\partial M} = -\frac{0}{u''(x_i) - \pi''(x_1)\,c} = 0.$$

An increase in income does not affect optimal driving speed, in other words there is no income effect. This depends on the quasi-linearity in the utility function, that is, the fact that the utility depends linearly on consumption of good y. This leads to the absence of income effects.

5. The problem of the planner is

$$\begin{aligned} &\max_{x_1,y_1,x_2,y_2} u(x_1) + y_1 + u(x_2) + y_2 \\ &s.t.\ \ (\pi(x_1,x_2))\,c + y_1 = M_1 \\ &\text{and}\ \ (\pi(x_1,x_2))\,c + y_2 = M_2. \end{aligned}$$

We can substitute for y_1 and y_2 and obtain

$$\max_{x_1,x_2} u(x_1) + u(x_2) + M_1 + M_2 - 2(\pi(x_1,x_2))\,c.$$

6. The first order conditions are

$$u'(x_1^*) - 2\pi'(x_1)\,c = 0$$
$$u'(x_2^*) - 2\pi'(x_2)\,c = 0$$

The first order conditions for the social planner differ from the conditions for the individuals because of a 2 multiplies the term on the probability of an accident. This is going to imply that $x_i^P < x_i^*$, that is, that the social planner chooses slower speeds. A formal way to show this is to do the comparative statics of

$$u'(x_i^*) - \alpha\pi'(x_i)\,c = 0$$

with respect to α. We get

$$\frac{\partial x_i^*}{\partial \alpha} = -\frac{-\pi'(x_i)\,c}{u''(x_i) - \pi''(x_i)\,c} < 0,$$

as we expected. The intuition here is that individuals neglect the negative externality that they have on others by driving too fast. The central planner takes it into account and therefore chooses lower speeds.

7. The new utility maximization is

$$\begin{aligned} &\max_{x_i,y_i} u(x_i) + y_i \\ &s.t.\ \ (\pi(x_1) + \pi(x_2))\,(c+t) + y_1 = M_i \end{aligned}$$

We can substitute y_i into the objective function to transform this into

$$\max_{x_i,y_i} u(x_i) + M_i - (\pi(x_1) + \pi(x_2))\,(c+t)\,.$$

The first order condition is now

$$u'(x_i) - \pi'(x_i)\,(c+t) = 0$$

8. If the fine t is set equal to c, the new problem becomes

$$\max_{x_i, y_i} u(x_i) + y_i$$
$$s.t.\ (\pi(x_1) + \pi(x_2))(c + t) + y_1 = M_i$$

We can substitute y_i into the objective function to transform this into

$$\max_{x_i, y_i} u(x_i) + M_i - (\pi(x_1) + \pi(x_2))(c + t).$$

This leads to the first-order condition

$$u'(x_i) - \pi'(x_i)(c + t) = 0.$$

By comparing these first order conditions and the f.o.c.s of the social planner it is easy to see that the two coincide for $t = c$. Intuitively, it is necessary to charge the individual also for the damage done, in case of accident, to the other driver involved in the accident. This leads the individual to internalize the externality, to take into account the damage done to others that he otherwise neglects.

9. An alternative policy, rather than imposing fines, is to limit the speed at the level that the social planner would optimally choose. This is a reasonable justification for speed limits observed in almost all societies. The problem of this solution, as well as of the fine, is that we need to know the utility function of the individual in order to implement it. With the fine, it is enough to know the damage in case of accident.

Problem 3. Altruism and dictator games. (41 points) In an experiment called the dictator game, a dictator (Player D) decides how to share \$10 dollars with the recipient (Player R). Label g (for gift) the transfer from the dictator to the recipient, with $0 \leq g \leq 10$. The monetary payoff π_D for the dictator is $(10 - g)$ and the monetary payoff π_R for the recipient is g. The typical outcome of this game is that 50 percent of subjects chooses $g = \$5$, and 50 percent chooses $g = \$0$. We now consider different models to see if they can rationalize this behavior.

1. Consider the case of dictator A, a selfish dictator. His utility function is $u_D = \pi_D$. What g does a selfish dictator choose in this case to maximize utility? (4 points)

2. Consider the case of dictator B that has the following utility function: $u_D = (1-\rho)\pi_D + \rho\pi_R$. Provide an interpretation for this utility function and for parameter ρ, with $0 \leq \rho < 1$. How do you interpret the special cases $\rho = 0$ and $\rho = .5$? (6 points)

3. How do you interpret the case $\rho < 0$? (3 points)

4. Keep assuming $u_D = (1-\rho)\pi_D + \rho\pi_R$, and $\rho < 1$. Solve for the optimal gift $g^*(\rho)$. (The notation reminds you that g^* is a function of ρ.) (7 points)

5. How well can dictators of type A or B explain observed play in the dictator game? (3 points)

6. Consider now the case of dictator C, with utility function

$$u_D = \begin{cases} (1-\rho)\pi_D + \rho\pi_R & \text{if} \quad \pi_D \geq \pi_R \\ (1-\sigma)\pi_D + \sigma\pi_R & \text{if} \quad \pi_D < \pi_R. \end{cases}$$

Assume $\rho \in [0, 1]$, $\sigma < \rho$. Provide an interpretation of this utility function. What psychological intuition does it capture? (6 points)

7. Solve for the optimal g^* for the case $.5 < \rho < 1$ and $\sigma < .5$. (8 points)

8. Provide intuition for the solution. Does the behavior of dictator C help to explain the data? (4 points)

Solution to Problem 3.

1. A selfish dictator maximizes $\pi_D = g$ subject to the constraint $0 \leq g \leq 10$. Since the utility is increasing in g, the dictator chooses $g^* = 10$, that is, he keeps everything for himself.

2. This second type of dictator is altruistic for $0 \leq \rho < 1$. The parameter ρ denotes how much weight is put on the utility of the recipient The case $\rho = 0$ corresponds to the case of selfish players (as in point 1), the case $\rho = .5$ corresponds to the case in which the dictator gives the same weight to the utility of the recipient as to his own utility.

3. The case $\rho < 0$ corresponds to the case of spiteful (envious) players. A spiteful player is happier, the lower the payoff of the recipient.

4. The altruistic dictator maximizes

$$(1-\rho)\pi_D + \rho\pi_R = (1-\rho)(10-g) + \rho g = 10(1-\rho) + g(2\rho - 1)$$

subject to the constraint $0 \leq g \leq 10$. The utility function of the dictator is increasing in g if $(2\rho - 1) > 0$, or $\rho > 1/2$. For $1/2 < \rho < 1$, therefore, the agent chooses $g^* = 1$, that is, transfers everything to the recipient. For $\rho < 1/2$, the utility function is decreasing in g. Therefore, for $\rho < 1/2$, the dictator transfers $g^* = 0$. For $\rho = 1/2$, the utility is independent of g and any $g \in [0, 10]$ is optimal.

5. We can easily explain the 50 percent of subjects that chooses $g = 0$. This subjects may be dictators of type A or dictators of type B with $\rho \leq 1/2$. We cannot explain, instead, the fact that half of the subjects give $g = \$5$, unless we assume the knife-edge case $\rho = .5$, in which case anything can happen.

6. Dictator C is altruistic toward the recipient as long as his payoff π_D is higher than the payoff of the recipient. If the recipient is ahead of the dictator, the dictator is less altruistic or even spiteful. It is easier to be generous toward others if the others are less well-off than us. Conversely, it is easier to be envious of people that are better off than us.

7. For $g \leq 5$, Dictator C maximizes

$$(1-\rho)\pi_D + \rho\pi_R = (1-\rho)(10-g) + \rho g = 10(1-\rho) + g(2\rho - 1).$$

Given that $.5 < \rho < 1$, the optimal choice is $g^* = 5$. For $10 > g > 5$, Dictator C maximizes

$$(1-\sigma)\pi_D + \sigma\pi_R = (1-\sigma)(10-g) + \sigma g = 10(1-\sigma) + g(2\sigma - 1).$$

Given that $\sigma < .5$, the optimal choice is to lower g as much as possible, that, it is optimal to choose $g = 5$. Therefore the utility function is first increasing in g up to $g = 5$, and then decreasing in g. The optimal choice therefore is $g^* = 5$.

8. Dictator C can help explain the behavior of 50 percent of the people that choose $g = 5$. Intuitively, dictator C is very altruistic as long as he is ahead. Therefore, he is happy to give up to $g = 5$ to the recipient. When g is higher than 5, though, the dictator is behind the recipient in payoffs. Under this situation, the dictator is less generous, and prefers to keep things for himself. A donation of 5 comes out of this combination of altruism (when ahead) and (relative) selfishness (when behind).

Problem 4. Bertrand Competition in discrete increments (46 points) Consider a variant of the Bertrand model of competition with two firms that we covered in class. The difference from the model in class is that prices are not a continuous variable, but rather a discrete variable. Prices vary in multiples of 1 cent. Firms can charge prices of 0, .01, .02, .03,... etc. The profits of firm i are

$$\pi_i(p_i, p_j) = \begin{cases} (p_i - c_i) D(p_i) & \text{if} \quad p_i < p_j \\ (p_i - c_i) D(p_i)/2 & \text{if} \quad p_i = p_j. \\ 0 & \text{if} \quad p_i > p_j. \end{cases}$$

The demand function $D(p)$ is strictly decreasing in p, that is, $D'(p) < 0$. Assume first that both firms have the same marginal cost $c_1 = c_2 = c$, and that the marginal cost c is a multiple of 1 cent. (The firm can charge $c - .01$, c, $c + .01$, $c + .02$, etc.)

1. Write down the definition of Nash Equilibrium as it applies to this game, that is, with p_i as the strategy of player i and $\pi_i(p_i, p_j)$ as the function that player 1 maximizes. Provide both the formal definition and the intuition. Do not substitute in the expression for π_i. (7 points)

2. Show that $p_1^* = p_2^* = c$ (that is, marginal cost pricing) is a Nash Equilibrium. (5 points)

3. Unlike in the case in which prices are continuous, this is not the only Nash Equilibrium. Find one other Nash Equilibrium. (you need to prove that it is a Nash Equilibrium) [Hint: The peculiar feature of this setup is that the firm can only charge prices that are multiples of 1 cent] (8 points)

4. (Harder, and long) Characterize all the (pure strategy) Nash Equilibria of the game. Show that there are no other Nash Equilibria. (18 points)

5. Now consider the case of two firms with different marginal costs, both multiples of 1 cent. Firm 1 has marginal cost c_1, with $c_1 < c_2$, the marginal cost of firm 2. For simplicity, assume $c_2 - c_1 > .01$, that is, the difference in marginal costs is more than one cent. Is $p_1^* = c_2 - .01$, $p_2^* = c_2$ an equilibrium? Explain intuitively as well. (8 points)

Solution to Problem 4.

1. A set of price $p^* = (p_1^*, p_2^*)$ is a Nash Equilibrium if

$$\pi_i(p_i^*, p_j^*) \geq \pi_i(p_i, p_j^*) \text{ for all } p_i \text{ and for all players } i = 1, 2.$$

   (We denote by p_j the payoff of the other player when considering player i).

2. We apply the definition at point 1 for player 1 first. It has to be the case that $\pi_1(c, c) \geq \pi_1(p, c)$ for all prices p. Since $\pi_1(c, c) = (c - c) D(c)/2 = 0$, we need to show $0 \geq \pi_1(p, c)$ for all p. Consider a p higher than c. In this case, $\pi_1(p, c) = 0$, which satisfies the Nash Equilibrium definition. Consider now a p lower than c. In this case, $\pi_1(p, c) = (p - c) D(p) < 0$. Again, $0 < \pi_1(p, c)$. We have shown that it is optimal for player 1 to play $p_1 = c$. We can repeat the same proof for player 2 just substituting 2 for 1. Therefore $p_1^* = c, p_2^* = c$ is a Nash Equilibrium.

3. Another equilibrium is $p_1^* = p_2^* = c + .01$, that is, both firms charge one cent above marginal cost. (see other equilibria below) We apply the definition of Nash equilibrium for player 1 first. It has to be the case that $\pi_1(c + .01, c + .01) \geq \pi_1(p, c + .01)$ for all prices p. Since $\pi_1(c + .01, c + .01) = (c + .01 - c) D(c + .01)/2 = .005D(c + .01) > 0$, we need to show $.005D(c + .01) \geq \pi_1(p, c + .01)$ for all p. Consider a p higher than $c + .01$. In this case, $\pi_1(p, c) = 0 < .005D(c + .01)$. This is not a profitable deviation. Consider now a p lower than $c + .01$. In this case, $\pi_1(p, c) = (p - c) D(p)$. This amount is 0 for $p = c$ and is negative for $p < c$. In both cases, the price yields lower profits than charging $c + .01$. We have shown that it is optimal for player 1 to play $p_1 = c + .01$. We can repeat the same proof for player 2 just substituting 2 for 1. Therefore $p_1^* = c, p_2^* = c + .01$ is a Nash Equilibrium.

4. Sets of equilibria.

(a) Consider equilibria where $(p_1^*, p_2^*) = (p_1, p_2)$ with $p_1 \le p_2 < c$ or $p_1 < p_2 = c$ or $p_1 < c \le p_2$. These clearly are not equilibria because the lower-price firm (firm 1) has incentive to increase the price, for example to $p_1' = c$. This increases profits from $(p_1 - c) D(p_1) < 0$ to $(p_1' - c) D(p_1') = 0$.

(b) Consider then equilibria where $(p_1^*, p_2^*) = (p_1, p_2)$ with $c < p_1 < p_2$, that is, one firm charges more than the other and both are (at least weakly) above marginal cost. These are not equilibria because firm 2 can deviate to p_1 and make positive profits, that is, $(p_1 - c) D(p_1)/2 > 0$, instead of zero profits.

(c) Finally, consider equilibria where $(p_1^*, p_2^*) = (p_1, p_2)$ with $c = p_1 < p_2$, that is, one firm charges at marginal cost and the other charges above. These are not equilibria because firm 1 can deviate to p_2 and make positive profits, that is, $(p_2 - c) D(p_2)/2 > 0$, instead of zero profits.

(d) The final set of potential equilibria to be considered have the form $(p_1^*, p_2^*) = (p, p)$ with $p > c + .01$, that is, both firms are charging the same price, higher than $c + .01$. Both firms are making $(p - c) D(p)/2$ profits. Consider a deviation to $p - .01$, that is, to a price one cent lower. By deviating to $p - .01$, firm 1 makes $(p - c - .01) D(p - .01)$. The deviation is profitable if $(p - c) D(p)/2 < (p - c - .01) D(p - .01)$ or

$$\frac{1}{2}\frac{(p-c)}{(p-c-.01)} < \frac{D(p-.01)}{D(p)}.$$

The left-hand side is equal to 1 if $p = c + .02$ and smaller than 1 for $p > c + .02$. The right-hand side is always larger than 1, since $D(p)$ is decreasing in price. It follows that the righ-hand side is bigger than the left-hand side, and therefore that the deviation is profitable. We can therefore say that $(p_1^*, p_2^*) = (p, p)$, with $p > c + .01$, is not an equilibrium.

(e) To summarize: the only two Nash equilibria in pure strategies are $(p_1^*, p_2^*) = (c, c)$ and $(p_1^*, p_2^*) = (c + .01, c + .01)$.

5. We have to show that no player has an incentive to deviate from $p_1^* = c_2 - .01$, $p_2^* = c_2$. In equilibrium, $\pi_1(c_2 - .01, c_2) = (c_2 - c_1 - .01) D(c_2 - .01) > 0$ and $\pi_2(c_2 - .01, c_2) = 0$.

(a) We start from Player 2. Player 2 has no incentive to increase p_2, since this would also yield zero profits. Decreasing price below c_2 will yield zero or negative profits. Therefore, player 2 has no incentive to deviate.

(b) On to Player 1. Player 1 can increase price to $p_1 > p_2^*$. In this case, it earn zero profit, not a profitable deviation. Alternatively, Player 1 can increase price to p_2^*. The deviation is profitable if $(c_2 - c_1) D(c_2)/2 > (c_2 - c_1 - .01) D(c_2 - .01)$ or

$$\frac{1}{2}\frac{(c_2-c_1)}{(c_2-c_1-.01)} > \frac{D(c_2-.01)}{D(c_2)}.$$

By the same argument above, the left-hand side is at most one, while the right-hand side is larger than 1. It follows that this deviation is not profitable.

(c) There is one deviation left to be considered. Player 1 can decrease the price from $p_1^* = c_2 - .01$ to a lower price p', with $c_1 < p' < p_1^*$. In this case, the profit for firm 1 changes from $(p_1^* - c_1) D(p_1^*)$ to $(p' - c_1) D(p')$. The change in profit is associated with two forces. On the one hand, the per-unit profit $(p_1 - c_1)$ goes down. On the other hand, the number of units sold $D(p_1)$ increases. Whether the change increases profits depends on how much demand increases. In general, one cannot say.