

# Economics 101A

## (Lecture 4)

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## Outline

1. Constrained Maximization (from last lecture)
2. Envelope Theorem II
3. Preferences
4. Properties of Preferences
5. From Preferences to Utility

# 1 Constrained Maximization (ctnd)

- **Lagrange Multiplier Theorem, necessary condition.** Consider a problem of the type

$$\begin{array}{l} \max_{x_1, \dots, x_n} f(x_1, x_2, \dots, x_n; \mathbf{p}) \\ \text{s.t.} \quad \left\{ \begin{array}{l} h_1(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \\ h_2(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \\ \dots \\ h_m(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \end{array} \right. \end{array}$$

with  $n > m$ . Let  $\mathbf{x}^* = \mathbf{x}^*(\mathbf{p})$  be a local solution to this problem.

- Assume:
  - $f$  and  $h$  differentiable at  $x^*$
  - the following Jacobian matrix at  $\mathbf{x}^*$  has maximal rank

$$J = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}^*) \\ \dots & \dots & \dots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

- Then, there exists a vector  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$  such that  $(\mathbf{x}^*, \boldsymbol{\lambda})$  maximize the Lagrangean function

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}; \mathbf{p}) - \sum_{j=0}^m \lambda_j h_j(\mathbf{x}; \mathbf{p})$$

- Case  $n = 2, m = 1$ .
- First order conditions are

$$\frac{\partial f(\mathbf{x}; \mathbf{p})}{\partial x_i} - \lambda \frac{\partial h(\mathbf{x}; \mathbf{p})}{\partial x_i} = 0$$

for  $i = 1, 2$

- Rewrite as

$$\frac{f'_{x_1}}{f'_{x_2}} = \frac{h'_{x_1}}{h'_{x_2}}$$

- **Constrained Maximization, Sufficient condition for the case  $n = 2, m = 1$ .**

- If  $\mathbf{x}^*$  satisfies the Lagrangean condition, and the determinant of the bordered Hessian

$$H = \begin{pmatrix} 0 & -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1^2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2 \partial x_1}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2^2}(\mathbf{x}^*) \end{pmatrix}$$

is positive, then  $\mathbf{x}^*$  is a constrained maximum.

- If it is negative, then  $\mathbf{x}^*$  is a constrained minimum.
- Why? This is just the Hessian of the Lagrangean  $L$  with respect to  $\lambda$ ,  $x_1$ , and  $x_2$

- Example 4:  $\max_{x,y} x^2 - xy + y^2$  s.t.  $x^2 + y^2 - p = 0$

- $\max_{x,y,\lambda} x^2 - xy + y^2 - \lambda(x^2 + y^2 - p)$

- F.o.c. with respect to  $x$ :

- F.o.c. with respect to  $y$ :

- F.o.c. with respect to  $\lambda$ :

- Candidates to solution?

- Maxima and minima?

## 2 Envelope Theorem II

- Nicholson, Ch. 2, pp. 46-47.
- **Envelope Theorem for Constrained Maximization.** In problem above consider  $F(p) \equiv f(\mathbf{x}^*(\mathbf{p}); \mathbf{p})$ . We are interested in  $dF(p)/dp$ . We can neglect indirect effects:

$$\frac{dF}{dp_i} = \frac{\partial f(\mathbf{x}^*(\mathbf{p}); \mathbf{p})}{\partial p_i} - \sum_{j=0}^m \lambda_j \frac{\partial h_j(\mathbf{x}^*(\mathbf{p}); \mathbf{p})}{\partial p_i}$$

- Example 4 (continued).  $\max_{x,y} x^2 - xy + y^2$  s.t.  
 $x^2 + y^2 - p = 0$
- $df(x^*(p), y^*(p))/dp?$
- Envelope Theorem.

# 3 Preferences

- Part 1 of our journey in microeconomics: *Consumer Theory*
- Choice of consumption bundle:
  1. Consumption today or tomorrow
  2. work, study, and leisure
  3. choice of government policy
- Starting point: preferences.
  1. 1 egg today  $\succ$  1 chicken tomorrow
  2. 1 hour doing problem set  $\succ$  1 hour in class  $\succ$  ...  $\succ$  1 hour out with friends
  3. War on Iraq  $\succ$  Sanctions on Iraq



## 4 Properties of Preferences

- Nicholson, Ch.3, p. 66.
- Commodity set  $X$  (apples vs. strawberries, work vs. leisure, consume today vs. tomorrow)
- Preference relation  $\succeq$  over  $X$
- A preference relation  $\succeq$  is *rational* if
  1. It is *complete*: For all  $x$  and  $y$  in  $X$ , either  $x \succeq y$ , or  $y \succeq x$  or both
  2. It is *transitive*: For all  $x$ ,  $y$ , and  $z$ ,  $x \succeq y$  and  $y \succeq z$  implies  $x \succeq z$
- Preference relation  $\succeq$  is *continuous* if for all  $y$  in  $X$ , the sets  $\{x : x \succeq y\}$  and  $\{x : y \succeq x\}$  are closed sets.

- Example:  $X = \mathbb{R}^2$  with map of indifference curves

- Counterexamples:

1. Incomplete preferences. Dominance rule.

2. Intransitive preferences. Quasi-discernible differences.

3. Discontinuous preferences. Lexicographic order

- Indifference relation  $\sim$ :  $x \sim y$  if  $x \succeq y$  and  $y \succeq x$
- Strict preference:  $x \succ y$  if  $x \succeq y$  and not  $y \succeq x$
- Exercise. If  $\succeq$  is rational,
  - $\succ$  is transitive
  - $\sim$  is transitive
  - Reflexive property of  $\succeq$ . For all  $x$ ,  $x \succeq x$ .

- Other features of preferences
  
- Preference relation  $\succsim$  is:
  - *monotonic* if  $x \succeq y$  implies  $x \succ y$ .
  
  - *strictly monotonic* if  $x \succeq y$  and  $x_j > y_j$  for some  $j$  implies  $x \succ y$ .
  
  - *convex* if for all  $x, y$ , and  $z$  in  $X$  such that  $x \succ z$  and  $y \succeq z$ , then  $tx + (1 - t)y \succ z$  for all  $t$  in  $[0, 1]$

## 5 From preferences to utility

- Nicholson, Ch. 3
- Economists like to use utility functions  $u : X \rightarrow R$
- $u(x)$  is 'liking' of good  $x$
- $u(a) > u(b)$  means: I prefer  $a$  to  $b$ .
- **Def.** Utility function  $u$  represents preferences  $\succeq$  if, for all  $x$  and  $y$  in  $X$ ,  $x \succeq y$  if and only if  $u(x) \geq u(y)$ .
- **Theorem.** If preference relation  $\succeq$  is rational and continuous, there exists a continuous utility function  $u : X \rightarrow R$  that represents it.

- [Skip proof]

- Example:

$$(x_1, x_2) \succeq (y_1, y_2) \text{ iff } x_1 + x_2 \geq y_1 + y_2$$

- Draw:

- Utility function that represents it:  $u(x) = x_1 + x_2$

- But... Utility function representing  $\succeq$  is not unique

- Take  $\exp(u(x))$

- $u(a) > u(b) \iff \exp(u(a)) > \exp(u(b))$

- If  $u(x)$  represents preferences  $\succeq$  and  $f$  is a strictly increasing function, then  $f(u(x))$  represents  $\succeq$  as well.
  
- If preferences are represented from a utility function, are they rational?
  - completeness
  - transitivity

- Indifference curves:  $u(x_1, x_2) = \bar{u}$
- They are just implicit functions!  $u(x_1, x_2) - \bar{u} = 0$

$$\frac{dx_2}{dx_1} = -\frac{U'_{x_1}}{U'_{x_2}} = MRS$$

- Indifference curves for:
  - monotonic preferences;
  - strictly monotonic preferences;
  - convex preferences



## 6 Next Class

- Common Utility Functions
- Utility Maximization