Econ 101A – Solution to Midterm 1

Problem 1. Utility maximization. (52 points) In this exercise, we consider a standard maximization problem with an unusual utility function. The utility function is

$$u(x, y) = \sqrt{x} + \sqrt{y}.$$

The price of good x is p_x and the price of good y is p_y. We denote income by M, as usual, with $M > 0$. This function is well-defined for $x > 0$ and for $y > 0$. From now on, assume $x > 0$ and $y > 0$ unless otherwise stated.

1. Compute $\partial u/\partial x$ and $\partial^2 u/\partial^2 x$. Is the utility function increasing in x? Is the utility function concave in x? (3 points)

2. The consumer maximizes utility subject to a budget constraint. Write down the maximization problem of the consumer with respect to x and y. Explain *briefly* why the budget constraint is satisfied with equality. (Hint: you can use the answer in point 1) (5 points)

3. Write down the Lagrangean function. (2 points)

4. Write down the first order conditions for this problem with respect to x, y, and λ. (4 points)

5. Solve explicitly for x^* and y^* as a function of p_x, p_y, and M. (8 points)

6. Do the solutions for x^* and y^* satisfy the positivity constraint, that is, $x^* > 0$ and $y^* > 0$?.(2 points)

7. Are these points maxima of the problem above? Check that the determinant of the bordered Hessian is positive at x^*, y^*, and λ^*. (8 points)

8. Use the expression for x^* that you obtained in point 6. Differentiate it with respect to M, that is, compute $\partial x^*/\partial M$. Is this good a normal good? (4 points)

9. We are now interested in the sign of $\partial x^*/\partial p_x$. That is, we would like to know if the demand function is downward sloping. (at higher prices, a lower quantity of the good is demanded). Argue, using the answer that you gave in point 9 and something you learnt in class, that we know the sign of $\partial x^*/\partial p_x$. What is this sign? (8 points)

10. Can you guess the solutions for x^* and y^* for the following maximization problem? Develop your argument. (8 point)

$$\max \left(\sqrt{x} + \sqrt{y} \right)^2 \tag{1}$$
$$s.t. p_x x + p_y y \leq M$$

Solution to Problem 1.

1. $\partial u(x, y)/\partial x = (1/2) * x^{-1/2} > 0$ and $\partial^2 u(x, y)/\partial^2 x = -(1/4) * x^{-3/2} < 0$ for $x > 0$. The first derivative tells us that the utility function is increasing in x for all positive x. The second derivative tell us that the utility function is concave in x, that is, the marginal utility from consumption of good x decreases with the consumption of x.

2. The budget constraint is $p_x x + p_y y \leq M$. The maximization problem is

$$\max_{x,y} \sqrt{x} + \sqrt{y}$$
$$\text{s.t. } p_x x + p_y y \leq M.$$

We can write down the budget constraint with equality because the utility function is strictly increasing both in x and y. (Now follows the long explanation, you were not required to give this) As we proved in point 1, $\partial u(x, y)/\partial x > 0$, and we can by symmetry show $\partial u(x, y)/\partial y > 0$. Given that utility is strictly increasing in both goods, the consumer will never choose a point on the interior of the budget set, i.e., a point $(\hat{x}, \hat{y})$ such that $p_x \hat{x} + p_y \hat{y} < M$. The reason is that the consumer could choose a point $(\hat{x}', \hat{y}')$

with $\hat{x}' > \hat{x}$ and $\hat{y}' > \hat{y}$ that still satisfies the budget constraint, i.e., such that $p_x\hat{x}' + p_y\hat{y}' \leq M$. (just pick $(\hat{x}', \hat{y}')$ sufficiently close to $(\hat{x}, \hat{y})$) But, given the monotonicity of u, the bundle $(\hat{x}', \hat{y}')$ provides a higher utility than the bundle $(\hat{x}, \hat{y})$. Therefore the consumer in the optimum will never choose a bundle $(\hat{x}, \hat{y})$ such that $p_x\hat{x}' + p_y\hat{y}' < M$. We can therefore limit ourselves to the points with $p_x x + p_y y = M$.

3. Lagrangean is $L(x, y, \lambda) = \sqrt{x} + \sqrt{y} - \lambda\left(p_x x + p_y y - M\right).$

4. First order conditions:

$$\frac{\partial L}{\partial x} = (1/2) * (x^*)^{-1/2} - \lambda^* p_x = 0$$

$$\frac{\partial L}{\partial y} = (1/2) * (y^*)^{-1/2} - \lambda^* p_y = 0$$

$$\frac{\partial L}{\partial \lambda} = -\left(p_x x + p_y y - M\right) = 0$$

5. Using the first two first order conditions, we find

$$\frac{(y^*)^{1/2}}{(x^*)^{1/2}} = \frac{p_x}{p_y}$$

or $(y^*)^{1/2} = (p_x/p_y)(x^*)^{1/2}$. We can raise this expression to the square power to get

$$y^* = \frac{p_x^2}{p_y^2} x^*. \tag{2}$$

We substitute this into the budget constraint to get $p_x x^* + p_y \frac{p_x^2}{p_y^2} x^* = M$ or $p_x x^* + p_x \frac{p_x}{p_y} x^* = M$ or $x^* p_x \left(1 + \frac{p_x}{p_y}\right) = M$ or

$$x^* = \frac{M}{p_x} \frac{1}{1 + \frac{p_x}{p_y}}. \tag{3}$$

We substitute (3) into (2) to get

$$y^* = \frac{p_x^2}{p_y^2} \frac{M}{p_x} \frac{1}{1 + \frac{p_x}{p_y}} = \frac{M}{p_y} \frac{p_x}{p_y} \frac{1}{1 + \frac{p_x}{p_y}} = \frac{M}{p_y} \frac{1}{1 + \frac{p_y}{p_x}}. \tag{4}$$

6. Using expressions (3) and (4) it is easy to see that x^* and y^* are both positive as long as income M is positive.

7. The determinant of the bordered Hessian is

$$\det H = \begin{vmatrix} 0 & -p_x & -p_y \\ -p_x & \frac{\partial^2 L}{\partial^2 x} & \frac{\partial^2 L}{\partial x \partial y} \\ -p_y & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial^2 y} \end{vmatrix} = \begin{vmatrix} 0 & -p_x & -p_y \\ -p_x & -\frac{1}{4}(x^*)^{-3/2} & 0 \\ -p_y & 0 & -\frac{1}{4}(y^*)^{-3/2} \end{vmatrix} =$$

$$= 0 - (-p_x)\left(\frac{1}{4}p_x(y^*)^{-3/2}\right) + (-p_y)\left(-\frac{1}{4}p_y(x^*)^{-3/2}\right) =$$

$$= \frac{1}{4}(p_x)^2(y^*)^{-3/2} + \frac{1}{4}(p_y)^2(x^*)^{-3/2}.$$

From $x^* > 0$ and $y^* > 0$ (see point 7) follows $\det H > 0$.

8. The derivative of x^* with respect to income M is

$$\frac{\partial x^*}{\partial M} = \frac{1}{p_x} \frac{1}{1 + \frac{p_x}{p_y}} > 0.$$

This good is a normal good, since its demand is increasing in income M.

9. In order to determine the sign of $\partial x^*/\partial p_x$, we do not need to do any derivation. We can use the Slutzky equation that we derived in class. In particular,

$$\frac{\partial x^*}{\partial p_x} = \frac{\partial h^*}{\partial p_x} - x^* \frac{\partial x^*}{\partial M}$$

where h^* is the Hicksian or compensated demand function. We know that $\partial h^*/\partial p_x < 0$, that is, compensated demand functions are always downward sloping. From point 8, we also know $\partial x^*/\partial M > 0$, that is, good x is a normal good. It follows, since both right-hand side terms are negative, that the left-hand side, $\partial x^*/\partial p_x$ is also negative. The intuitive explanation: We know that x is a normal good. For normal goods, the income and substitution effect go in the same direction: an increase in the price of good x makes good y more desirable. In addition, the consumer is poorer, which leads her to consume less of good x.

10. Problem (1) has one very important similarity to the initial problem: the utility function in the new problem is the square of the utility function in the old problem. Since the utility function in the old problem was always positive (for $x > 0$ and $y > 0$), it follows that the utility function in the new problem is an increasing function of the utility function in the old problem. (if the utility function in the old problem could take on negative values, this argument would not apply, since the square function would not be an increasing function over this larger range) Since the utility function is just an increasing function of the old one, it represents the same preferences. The solution to the new problem ,therefore, has to coincide with the solution of the old problem. One way to see this is as follows. Consider Problem 1

$$\max u\,(x, y)$$
$$s.t.\,p_x x + p_y y \leq M$$

and Problem 2:

$$\max f\,(u\,(x, y))$$
$$s.t.\,p_x x + p_y y \leq M$$

where $f : R \to R$ is an increasing function. Consider the f.o.c. of Problem 2:

$$f' * u'_x - \lambda p_x = 0$$
$$f' * u'_y - \lambda p_y = 0$$
$$-\,(p_x x + p_y y - M) = 0.$$

As we transform the first 2 f.o.c.s to get the usual condition on the MRS, we obtain

$$\frac{u'_x}{u'_y} = \frac{p_x}{p_y}$$

where the function f disappeared! This is the same condition that we would have obtained in Problem 1. Since the budget constraint is the same in both problems, it follows that the solutions are the same in the two problems. Notice that, in order to do the above simplification, we need $f' \neq 0$, that is, the function has to be strictly increasing, which was true for $x > 0$ and $y > 0$. [We should also check the second order conditions].

Short problems. (26 points) In this part, you are required to provide short answers to the problems below. Provide the steps in the derivation of your answer.

Short problem 1. (16 points) Consider the utility function

$$U(x) = \begin{cases} x & \text{if} \quad 0 \leq x \leq 10 \\ (20 - x) & \text{if} \quad 10 < x \leq 20 \end{cases}$$

defined for $0 \leq x \leq 20$.

1. Plot the utility function as a function of x. (1 point)

2. Is $U(10) > U(5)$? Is $U(20) > U(5)$? Are the preferences represented by this utility function monotonic (that is, if $y \geq x$, then $y \succeq x$)? (5 points)

3. (Harder) Write down the set of preferences over the numbers between 0 and 20 that this utility function represents. Try to be precise, but you can help yourself using words. [Hint: for each x, $y \in [0, 20]$, when is $x \succeq y$?] (10 points)

Short problem 2. (10 points)

1. Consider the implicit function $g(x, y) = y - 1 - \ln(x * y) = 0$ for $x > 0$, $y > 0$.

   (a) Use the implicit function theorem to write down $\partial y/\partial x$.(6 points)

   (b) Is this derivative well-defined for $(x = 1, y = 1)$? (4 points)

Solution to Short Problem 1.

1. See Figure.

2. $U(10) = 10 > U(5) = 5$. As for the second inequality, $U(20) = 0 < U(5) = 5$. The preferences represented by this utility function are not monotonic, since $20 > 5$, but it is not the case that $20 \succeq 5$.

3. The set of preferences that are represented by the utility function can be described as follows:

$$x \succeq y \text{ if } \begin{cases} y \leq 10 \text{ and } y \leq x \leq 20 - y, \text{ or} \\ y > 10 \text{ and } 20 - y \leq x \leq y. \end{cases}$$

Solution to Short Problem 2.

1. Use the implicit function theorem:

$$\partial y/\partial x = -\frac{\partial g(x, y)/\partial x}{\partial g(x, y)/\partial y} = -\frac{-y/(xy)}{1 - x/(xy)} = -\frac{1/x}{1 - 1/y} \tag{5}$$

2. In order to make sure that the derivative is well defined we need to make sure that the function is differentiable everywhere for $x > 0$, $y > 0$, which it is. In addition, we need to check $\partial g(x, y)/\partial y \neq 0$ at the point $(x = 1, y = 1)$. But the latter derivative is $1 - x/(xy)$, which is indeed 0 at the point above. Therefore, at this point we cannot apply the implicit function theorem. (If we did, notice that we would be dividing by zero in (5)).