Econ 101A – Midterm 2
Th 6 November 2003.

You have approximately 1 hour and 20 minutes to answer the questions in the midterm. I will collect the exams at 12.30 sharp. Show your work, and good luck!

Problem 1. Production. (43 points) In this exercise, we consider a firm producing a product using only one input, labor L. The production function f is as follows:

$$f(L) = \begin{cases} (L - \bar{L})^\alpha & \text{if} \quad L \geq \bar{L} \\ 0 & \text{if} \quad 0 \leq L < \bar{L} \end{cases} ,$$

that is, the firm produces a positive quantity of output only if there are at least $\bar{L}$ workers. Assume that the wage of a worker is w. Assume $\bar{L} > 0$ and $\alpha > 0$.

1. Draw a picture of the production function assuming $\alpha = .5$ and $\bar{L} = 1$. (1 point)

2. For which values of α (if any) does the function exhibit (weakly) decreasing returns to scale ($f(tL) \leq tf(L)$ for all $t > 1$ and all $L \geq 0$)? Does the function exhibit (weakly) increasing returns to scale ($f(tL) \geq tf(L)$ for all $t > 1$ and all $L \geq 0$) for $\alpha \geq 1$? Provide an analytical proof if you can. (7 points)

3. Consider now the first step of the cost minimization problem. The firm solves

$$\min wL$$
$$s.t. f(L) \geq y$$

   for $y > 0$. What is the solution for $L^*(w, y | \bar{L}, \alpha)$? (This notation stresses that the solution depends also on the parameters α and $\bar{L}$. Hint: You are better off not using Lagrangeans...) (5 points)

4. Write down the implied cost function $c(w, y | \bar{L}, \alpha)$. (2 points)

5. Derive an expression for the average cost $c(w, y | \bar{L}, \alpha)/y$ and the marginal cost $c'_y(w, y | \bar{L}, \alpha)$ for $y > 0$. Graph the average cost and marginal cost for $\alpha = .5$, $w = 1$ and $\bar{L} = 1$. Graph the supply function for the same values of the parameters. [remember, y is on the horizontal axis]. (7 points)

6. Now that we graphically solved for the supply function, we also derive it formally. Consider the second step of cost minimization

$$\max_y py - c(w, y | \bar{L}, \alpha) .$$

   Write down the first order condition and the second order conditions. Solve for $y^*(w, p | \bar{L}, \alpha)$. For what values of α is the second order condition satisfied? (5 points)

7. Assume now $\alpha < 1$ and write the condition under which firms are making positive profits, that is, under which $py^*(w, p | \bar{L}, \alpha) - c(w, y^* | \bar{L}, \alpha) > 0$. If you can, solve for the value of price $\bar{p}$ such that firms produce if the price p is larger than $\bar{p}$. (5 points)

8. Taking into account the answers to points 6 and 7, write down the supply function, that is, an expression for $y^*(w, p | \bar{L}, \alpha)$. Keep assuming $\alpha < 1$. (3 points)

9. Why do these conditions imply that the supply function is the portion of the marginal cost curve above average cost as long in its upward sloping portion, and zero otherwise (4 points)?

10. Keep assuming $\alpha < 1$. What happens to the supply function as $\bar{L}$ increases? What if the wage w increases? You can respond using the graphs or using the solution in point 8. Give an economic interpretation. (4 points)

Solution of Problem 1.

1. See Figure 1.

2. There is no $\alpha > 0$ such that the function exhibits decreasing returns to scale. Consider any L_0 such that $L_0 < \bar{L}$. If t is large enough (for example, $t = 2\bar{L}/L_0$), tL_0 will fall on the upward part of the production function, and therefore $f(tL_0) > 0 = tf(L_0)$. We now show that the function will exhibit increasing returns to scale as long as $\alpha \geq 1$. First, as we just proved, $f(tL) \geq tf(L)$ for any $L \leq \bar{L}$. As for $L > \bar{L}$, $f(tL) = (tL - \bar{L})^\alpha = \left(tL - t\frac{\bar{L}}{t}\right)^\alpha = t^\alpha \left(L - \frac{\bar{L}}{t}\right)^\alpha > t^\alpha \left(L - \bar{L}\right)^\alpha = t^\alpha f(L)$. Therefore, for $\alpha \geq 1$ the production function exhibits increasing returns to scale.

3. Given that the quantity to be produced y is positive, the quantity of labor to be employed is at least as large as $\bar{L}$. For all such values of L, the production function is increasing in L. Therefore the budget constraint will be satisfied with equality. (if it was not, the firm could cut costs by using less labor) Using $f(L) = (L - \bar{L})^\alpha = y$, we can invert the function and obtain $L - \bar{L} = y^{1/\alpha}$ or $L = \bar{L} + y^{1/\alpha}$. The solution to the problem therefore is $L^*\left(w, y | \bar{L}, \alpha\right) = \bar{L} + y^{1/\alpha}$.

4. The cost function is $c\left(w, y | \bar{L}, \alpha\right) = wL^*\left(w, y | \bar{L}, \alpha\right) = w\bar{L} + wy^{1/\alpha}$.

5. The average cost is $c\left(w, y | \bar{L}, \alpha\right)/y = \left(w\bar{L} + wy^{1/\alpha}\right)/y = w\bar{L}/y + wy^{(1-\alpha)/\alpha}$. The marginal cost is $\partial c\left(w, y | \bar{L}, \alpha\right)/\partial y = \partial\left(w\bar{L} + wy^{1/\alpha}\right)/\partial y = w\alpha y^{(1-\alpha)/\alpha}$. See Figure 2 for the second part of the exercise.

6. The first order condition is
$$p - c_y'\left(w, y | \bar{L}, \alpha\right) = 0$$
or
$$p = w\alpha y^{(1-\alpha)/\alpha}.$$
This implies
$$y^*\left(w, y | \bar{L}, \alpha\right) = p^{\frac{\alpha}{1-\alpha}} \left(\frac{1}{\alpha w}\right)^{\frac{\alpha}{1-\alpha}}$$
The second order condition is
$$-c_{y,y}''\left(w, y | \bar{L}, \alpha\right) < 0$$
or
$$-w\alpha(1-\alpha)y^{(1-2\alpha)/\alpha} < 0.$$
This condition is satisfied for $\alpha < 1$.

7. The firms make positive profits if
$$
\begin{aligned}
py^* - c\left(w, y^* | \bar{L}, \alpha\right) &= p * p^{\frac{\alpha}{1-\alpha}} \left(\frac{1}{\alpha w}\right)^{\frac{\alpha}{1-\alpha}} - w\bar{L} - wp^{\frac{1}{1-\alpha}} \left(\frac{1}{\alpha w}\right)^{\frac{1}{1-\alpha}} = \\
&= p^{\frac{1}{1-\alpha}} \left(\frac{1}{\alpha w}\right)^{\frac{\alpha}{1-\alpha}} - w\bar{L} - wp^{\frac{1}{1-\alpha}} \left(\frac{1}{\alpha w}\right)^{\frac{1}{1-\alpha}} = \\
&= p^{\frac{1}{1-\alpha}} \left[\left(\frac{1}{\alpha w}\right)^{\frac{\alpha}{1-\alpha}} - w\left(\frac{1}{\alpha w}\right)^{\frac{1}{1-\alpha}}\right] - w\bar{L} \geq 0
\end{aligned}
$$
or
$$p \geq \left(\frac{w\bar{L}}{\left(\frac{1}{\alpha w}\right)^{\frac{\alpha}{1-\alpha}} - w\left(\frac{1}{\alpha w}\right)^{\frac{1}{1-\alpha}}}\right)^{(1-\alpha)} \equiv \bar{p}.$$

8. Therefore the supply function is
$$y^* = \begin{cases} p^{\frac{\alpha}{1-\alpha}} \left(\frac{1}{\alpha w}\right)^{\frac{\alpha}{1-\alpha}} & \text{if } p \geq \bar{p} \\ 0 & \text{if } p < \bar{p} \end{cases}$$

9. In point 6, we saw that the first order condition is $p = c'_y$. Therefore, the optimal quantity y^* can be found graphically on the marginal cost curve. The second order condition warns us that this is only true if, at the candidate optimum, the second order condition is upward sloping. Finally, point 7 shows that firms will not find it profitable to produce is they make negative profits, that is, if at the price p is below average cost.

10. A change in the minimum quantity of labor $\bar{L}$ does not affect the marginal cost function, but it shifts upward the average cost function. Therefore it reduces the range of prices for which the firm is willing to produce. This makes sense since an increase in $\bar{L}$ is equivalent to an increase in fixed costs. An increase in the wage w shifts up both the marginal cost curve and the average cost curve. It both reduces the range or prices for which the firm produces and the quantity produced for each price. This is because it is an increase in the variable costs incurred by the firm.

1. **Problem 2. Uncertainty** (17 points). So far, in class we have assumed that the utility function depends only on consumption c. Now, we consider the case of Prospectus, whose utility function also on a reference point r (Kahneman and Tversky, 1979). Prospectus has utility function

$$U(c|r) = \begin{cases} (c-r)^{1/2} & \text{if } c \geq r \\ -(r-c)^{1/2} & \text{if } c < r \end{cases}$$

You can see this utility function plotted in the attached Figure.

Prospectus needs to choose between jobs G and I. Job G is in a government agency and guarantees consumption of $40,000 per year. Job I is in an investment bank and is very risky: with probability .5 the job will go well and Prospectus will consume $70,000 per year, but with probability .5 Prospects will be fired and have a yearly consumption of $10,000. Your task is to help Prospectus decide which job to take.

1. Assume that Prospectus just came out of school and has a low reference point, that is, r equals $10,000. Write down the expected utility $EU(c|r)$ from accepting job G and the expected utility $EU(c|r)$ from accepting job I. What would you recommend that Prospectus should do in order to maximize expected utility? (6 points)

2. Assume that Prospectus just quit a consulting job and has a high reference point, that is, r equals $70,000. Write down the expected utility $EU(c|r)$ from accepting job G and the expected utility $EU(c|r)$ from accepting job I. In this case, what would you recommend that Prospectus should do in order to maximize expected utility? (5 points)

3. Can you give an intuition for why the best job choice for Prospectus depends on the reference point? (Hint: use the Figure and possibly Jensen's inequality) (6 points)

Solution to Problem 2.

1. The expected utility from job G is

$$Eu_G = (40,000|r = 10,000) = (40,000 - 10,000)^{1/2} = (30,000)^{1/2}.$$

The expected utility from job I is

$$\begin{aligned} Eu_I &= .5u(70,000|r = 10,000) + .5u(10,000|r = 10,000) = \\ &= .5(70,000 - 10,000)^{1/2} + .5(10,000 - 10,000)^{1/2} = \\ &= .5(60,000)^{1/2} = \left(\frac{1}{4}\right)^{1/2}(60,000)^{1/2} = (15,000)^{1/2}. \end{aligned}$$

Prospects should choose job G in order to maximize expected utility.

2. The expected utility from job G is

$$Eu_G = (40{,}000|r = 70{,}000) = -(70{,}000 - 40{,}000)^{1/2} = -(30{,}000)^{1/2}.$$

The expected utility from job I is

$$
\begin{aligned}
Eu_I &= .5u(70{,}000|r = 70{,}000) + .5u(10{,}000|r = 70{,}000) = \\
&= .5 * \left(-(70{,}000 - 70{,}000)^{1/2}\right) + .5 * \left(-(70{,}000 - 10{,}000)^{1/2}\right) = \\
&= -.5(60{,}000)^{1/2} = -.5(2 * 30{,}000)^{1/2} = -\frac{\sqrt{2}}{2}(30{,}000)^{1/2}.
\end{aligned}
$$

Prospectus should choose job I in order to maximize expected utility.

3. Prospectus is risk averse for gambles involving only outcomes above the reference point, and risk seeking for gambles involving only oucomes below the reference point. When he just came out of school, therefore, he prefers not to take risk and chooses the government job. When, instead, he has a high reference point, he chooses the investment banking job because he is risk seeking. Formally, notice that, on outcomes above the reference point, the utility function is concave and therefore Jensen's inequeality implies $Eu(X) < u(EX)$. But job G guarantees exactly the expected value of job I, and therefore $Eu_G < u_I(X)$. The converse applies for the other case.

4. **Problem 3. Altruistic workers.** (20 points) Consider a worker in a firm that has to decide how hard to work. The worker's effort e, with $0 \le e \le 1$, has a disutility for the worker $-e^2/2$. The worker gets wage $0 < w < 1$. The worker's utility is therefore

$$U = w - e^2/2,$$

that is, the wage net of the effort cost. The production of the firm is e and the good is sold at price 1.

5. The worker chooses e to maximize own utility (subject to the constraint $0 \le e \le 1$). What is the optimal choice of effort e? (3 points)

6. The firm does not like the above outcome and decides to monitor the worker with probability p. With probability $p * (1 - e)$ the worker is caught shirking and is fired, and therefore gets 0 wage. With probability $p * e$ the worker is found at work and gets the wage w. With probability $(1 - p)$ there is no monitoring and the worker gets wage w. The worker maximizes the expected wage payment minus the effort cost. Write down the expected utility of the worker and solve for the utility maximizing effort e. What is the comparative statics with respect to p? Provide intuition. (6 points)

7. The firm now attempts to maximize profits given by revenue e minus the expected wage payment minus monitoring costs cp^2. (Hint: The firm pays the wage w with probability $1 - p + pe$) Solve for the profit-maximizing level of monitoring p^*. (5 points)

8. We now go back to point 1, that is, there is no monitoring. Assume now, however, that the worker is altruistic toward the firm. That is, the worker maximizes $w - e^2/2 + \alpha e$, where the last term is the product of the altruism coefficient α and the production of the firm. Solve for the utility maximizing effort in this case assuming no monitoring. What is the comparative statics with respect to the altruism coefficient α? (3 points)

9. A firm wants to improve productivity e. The firm has the choice between increasing the costly monitoring or selecting altruistic workers. What should the firm do? (3 points)

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Solution to Problem 3.

1. The worker maximizes $U = w - e^2/2$ by setting $e^* = 0$. The worker puts in no effort.

2. The expected utility of the worker is

$$p(1 - e) * 0 + pe * w + (1 - p) w - e^2/2 = pew + (1 - p) w - e^2/2.$$

   The first order condition with respect to e is

$$pw - e = 0.$$

   The worker therefore sets $e^* = pw$. Clearly, $\partial e^*/\partial p = w > 0$. The higher is the probbility of monitoring, the more effort the worker puts in.

3. The firm maximizes

$$\max_{p} e^* (p, w) - (1 - p) w - pe^* (p, w) w - cp^2 = pw - (1 - p) w - (pw)^2 - cp^2.$$

   The first order condition is $2w - 2pw^2 - 2cp = 0$, which implies

$$p^* = \frac{w}{(w^2 + c)}.$$

4. In case of an altruistic worker, the worker maximizes $w - e^2/2 + \alpha e$, which yields the first order condition $-e^* + \alpha = 0$, or $e^* = \alpha$. The effort of the worker is increasign in the altruism.

5. Either an increase in monitoring p or an increase in altruism lead to increase production. But one is costly, while the other is free. The firm therefore is better offer attracting altruistic workers.