Economics 101A
(Lecture 1)

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Outline

1. Prerequisites for the course
2. A test in maths
3. Optimization with 1 variable
4. Multivariate optimization

1 Prerequisites

- Mathematics
  - Good knowledge of multivariate calculus – Maths 1A or 1B
  - Basic knowledge of probability theory and matrix algebra
- Economics
  - Knowledge of fundamentals – Ec1 or 2 or 3
  - High interest!

2 A Test in Maths

1. Can you differentiate the following functions with respect to x?

   (a) $y = \mathsf{exp}(x)$

   (b) $y = a + bx + cx^2$

   (c) $y = \frac{\mathsf{exp}(x)}{b^x}$

2. Can you partially differentiate these functions with respect to x and w?

   (a) $y = axw + bx - c\frac{x}{w} + d\sqrt{xw}$

   (b) $y = \mathsf{exp}(x/w)$

   (c) $y = \int_0^1 (x + aw^2 + xs)ds$

3. Can you plot the following functions of one variable?

   (a) $y = \exp(x)$

   (b) $y = -x^2$

   (c) $y = \exp(-x^2)$

4. Are the following functions concave, convex or neither?

   (a) $y = x^3$

   (b) $y = -\exp(x)$

   (c) $y = x^{.5}y^{.5}$ for $x > 0$, $y > 0$

5. Consider an urn with 20 red and 40 black balls?

   (a) What is the probability of drawing a red ball?

   (b) What is the probability of drawing a black ball?

6. What is the determinant of the following matrices?

   (a) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix}$

3 Optimization with 1 variable

- Nicholson, Ch.2, pp. 22-26

- Example. Function $y = -x^2$

- What is the maximum?

- Maximum is at 0

- General method?

- Sure! Use derivatives

- Derivative is slope of the function at a point:

$$\frac{\partial f(x)}{\partial x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **Necessary condition for maximum** x^* is

$$\frac{\partial f(x^*)}{\partial x} = 0 \tag{1}$$

- Try with $y = -x^2$.

- $\frac{\partial f(x)}{\partial x} = \qquad\qquad\qquad = 0 \Longrightarrow x^* =$

- Does this guarantee a maximum? No!

- Consider the function $y = x^3$

- $\frac{\partial f(x)}{\partial x} = \qquad\qquad\qquad\qquad = 0 \Longrightarrow x^* =$

- Plot $y = x^3$.

- **Sufficient condition for a (local) maximum**:

$$\frac{\partial f(x^*)}{\partial x} = 0 \text{ and } \left.\frac{\partial^2 f(x)}{\partial^2 x}\right|_{x^*} < 0 \qquad (2)$$

- At a maximum, $f(x^* + h) - f(x^*) < 0$ for all h.

- Taylor Rule: $f(x^*+h)-f(x^*) = \frac{\partial f(x^*)}{\partial x}h+\frac{1}{2}\frac{\partial^2 f(x^*)}{\partial^2 x}h^2+$ higher order terms.

- Notice: $\frac{\partial f(x^*)}{\partial x} = 0$.

- $f(x^* + h) - f(x^*) < 0$ for all $h \Longrightarrow \frac{\partial^2 f(x^*)}{\partial^2 x}h^2 < 0 \Longrightarrow \frac{\partial^2 f(x^*)}{\partial^2 x} < 0$

- Careful: Maximum may not exist: $y = \exp(x)$

- Tricky examples:
  - *Minimum.* $y = x^2$
  - *No maximum.* $y = \exp(x)$ for $x \in (-\infty, +\infty)$
  - *Corner solution.* $y = x$ for $x \in [0, 1]$

4 Multivariate optimization

- Nicholson, Ch.2, pp. 26–32

- Function from R^n to R: $y = f(x_1, x_2, ..., x_n)$

- Partial derivative with respect to x_i:

$$\begin{aligned} & \frac{\partial f(x_1, ..., x_n)}{\partial x_i} \\ = & \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ...x_n) - f(x_1, ..., x_i, ...x_n)}{h} \end{aligned}$$

- Slope along dimension i

- Total differential:

$$df = \frac{\partial f(x)}{\partial x_1} dx_1 + \frac{\partial f(x)}{\partial x_2} dx_2 + ... + \frac{\partial f(x)}{\partial x_n} dx_n$$

- One important economic example

- Example 1: Partial derivatives of $y = f(L, K) = L^{.5}K^{.5}$

- $f'_L =$
  (marginal productivity of labor)

- $f'_K =$
  (marginal productivity of capital)

- $f''_{L,K} =$

Maximization over an open set (like R)

- **Necessary condition for maximum** x^* is

$$\frac{\partial f(x^*)}{\partial x_i} = 0 \ \forall i \qquad (3)$$

or in vectorial form

$$\nabla f(x) = 0$$

- These are commonly referred to as first order conditions (f.o.c.)

- Sufficient conditions? Define Hessian matrix H:

$$H = \begin{pmatrix} f''_{x_1,x_1} & f''_{x_1,x_2} & \cdots & f''_{x_1,x_n} \\ \ldots & \ldots & \ldots & \ldots \\ f''_{x_n,x_1} & f''_{x_n,x_2} & \cdots & f''_{x_n,x_n} \end{pmatrix}$$

- Subdeterminant $|H|_i$ of Matrix H is defined as the determinant of submatrix formed by first i rows and first i columns of matrix H.

- Examples.
  - $|H|_1$ is determinant of f''_{x_1,x_1}, that is, f''_{x_1,x_1}
  - $|H|_2$ is determinant of

$$H = \begin{pmatrix} f''_{x_1,x_1} & f''_{x_1,x_2} \\ f''_{x_2,x_1} & f''_{x_2,x_2} \end{pmatrix}$$

- **Sufficient condition for maximum** x^*.
  1. x^* must satisy first order conditions;
  2. Subdeterminants of matrix H must have alternating signs, with subdeterminant of H_1 negative.

- Case with $n = 2$

- Condition 2 reduces to $f''_{x_1,x_1} < 0$ and $f''_{x_1,x_1} f''_{x_2,x_2} - (f''_{x_1,x_2})^2 > 0$.

- Example 2: $h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2$

- First order condition w/ respect to x_1?

- First order condition w/ respect to x_2?

- $x_1^*, x_2^* =$

- For which p_1, p_2 is it a maximum?

- For which p_1, p_2 is it a minimum?

5 Next Class

- Comparative Statics
- Implicit Function Theorem
- Envelope Theorem
- An Example of Important Economics: The Economics of Discrimination

- Going toward:
  - Preferences
  - Utility Maximization (where we get to apply maximization techniques for the first time)