Economics 101A
(Lecture 16)

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Outline

1. Production Function II
2. Returns to Scale
3. Two-step Cost Minimization
4. Cost Minimization: Example
5. Geometry of Cost Curves

1 Production Function

- Mathematically, $d^2K/d^2L =$

2 Returns to Scale

- Nicholson, Ch. 7, pp. 190–193 [OLD: Ch. 11, pp. 275–278]

- Effect of increase in labor: f'_L

- Increase of all inputs: $f(t\mathbf{z})$ with t scalar, $t > 1$

- How much does output increase?

  – Decreasing returns to scale: for all $\mathbf{z}$ and $t > 1$,

$$f(t\mathbf{z}) < tf(\mathbf{z})$$

- Constant returns to scale: for all $\mathbf{z}$ and $t > 1$,

$$f(t\mathbf{z}) = tf(\mathbf{z})$$

- Increasing returns to scale: for all $\mathbf{z}$ and $t > 1$,

$$f(t\mathbf{z}) > tf(\mathbf{z})$$

- Example: $y = f(K, L) = AK^{\alpha}L^{\beta}$

- Marginal product of labor: $f'_L =$

- Decreasing marginal product of labor: $f''_L =$

- $MRTS =$

- Convex isoquant?

- Returns to scale: $f(tK, tL) = A(tK)^{\alpha}(tL)^{\beta} = t^{\alpha+\beta}AK^{\alpha}L^{\beta} = t^{\alpha+\beta}f(K, L)$

3 Two-step Cost minimization

- Nicholson, pp. 212–220 [OLD, Ch. 12 , pp. 298–307]

- Objective of firm: Produce output that generates maximal profit.

- Decompose problem in two:
  - Given production level y, choose cost-minimizing combinations of inputs
  - Choose optimal level of y.

- *First step.* Cost-Minimizing choice of inputs

- Two-input case: Labor, Capital

- Input prices:
  - Wage w is price of L
  - Interest rate r is rental price of capital K

- Expenditure on inputs: $wL + rK$

- Firm objective function:

$$\min_{L,K} wL + rK$$
$$s.t. f(L, K) \geq y$$

- Equality in constraint holds if:

  1. $w > 0$, $r > 0$;

  2. f strictly increasing in at least L or K.

- Counterexample if ass. 1 is not satisfied

- Counterexample if ass. 2 is not satisfied

- Compare with expenditure minimization for consumers

- First order conditions:

$$w - \lambda f'_L = 0$$

and

$$r - \lambda f'_K = 0$$

- Rewrite as

$$\frac{f'_L\,(L^*, K^*)}{f'_K\,(L^*, K^*)} = \frac{w}{r}$$

- MRTS (slope of isoquant) equals ratio of input prices

- Graphical interpretation

- Derived demand for inputs:
  - $L = L^* (w, r, y)$
  - $K = K^* (w, r, y)$
- Value function at optimum is **cost function:**

$$c(w, r, y) = wL^* (r, w, y) + rK^* (r, w, y)$$

- *Second step.* Given cost function, choose optimal quantity of y as well

- Price of output is p.

- Firm's objective:

$$\max py - c(w, r, y)$$

- First order condition:

$$p - c'_y(w, r, y) = 0$$

- Price equals marginal cost – very important!

- Second order condition:

$$-c''_{y,y}(w, r, y^*) < 0$$

- For maximum, need increasing marginal cost curve.

4 Cost Minimization: Example

- Continue example above: $y = f(L, K) = AK^{\alpha}L^{\beta}$

- Cost minimization:

$$\begin{aligned} &\min wL + rK \\ &s.t. AK^{\alpha}L^{\beta} = y \end{aligned}$$

- Solutions:

  – Optimal amount of labor:

$$L^*(r, w, y) = \left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}}$$

  – Optimal amount of capital:

$$\begin{aligned} K^*(r, w, y) &= \frac{w}{r}\frac{\alpha}{\beta}\left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} = \\ &= \left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}} \left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \end{aligned}$$

- Check various comparative statics:

  - $\partial L^*/\partial A < 0$ (technological progress and unemployment)

  - $\partial L^*/\partial y > 0$ (more workers needed to produce more output)

  - $\partial L^*/\partial w < 0$, $\partial L^*/\partial r > 0$ (substitute away from more expensive inputs)

- Parallel comparative statics for K^*

- Cost function

$$
\begin{aligned}
c(w,r,y) &= wL^*(r,w,y) + rK^*(r,w,y) = \\
&= \left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}} \left[\begin{array}{c} w\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} + \\ +r\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} \end{array} \right]
\end{aligned}
$$

- Define $B := w\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{-\frac{\alpha}{\alpha+\beta}} + r\left(\frac{w}{r}\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}}$

- Cost-minimizing output choice:

$$
\max py - B\left(\frac{y}{A}\right)^{\frac{1}{\alpha+\beta}}
$$

- First order condition:

$$p - \frac{1}{\alpha+\beta}\frac{B}{A}\left(\frac{y}{A}\right)^{\frac{1-(\alpha+\beta)}{\alpha+\beta}} = 0$$

- Second order condition:

$$-\frac{1}{\alpha+\beta}\frac{1-(\alpha+\beta)}{\alpha+\beta}\frac{B}{A^2}\left(\frac{y}{A}\right)^{\frac{1-2(\alpha+\beta)}{\alpha+\beta}}$$

- When is the second order condition satisfied?

- Solution:
  - $\alpha + \beta = 1$ (CRS):
    * S.o.c. equal to 0
    * Solution depends on p
    * For $p > \frac{1}{\alpha+\beta}\frac{B}{A}$, produce $y^* \to \infty$
    * For $p = \frac{1}{\alpha+\beta}\frac{B}{A}$, produce any $y^* \in [0, \infty)$
    * For $p < \frac{1}{\alpha+\beta}\frac{B}{A}$, produce $y^* = 0$

- $\alpha + \beta > 1$ (IRS):
  * S.o.c. positive
  * Solution of f.o.c. is a minimum!
  * Solution is $y^* \to \infty$.
  * Keep increasing production since higher production is associated iwth higher returns

- $\alpha + \beta < 1$ (DRS):
  * s.o.c. negative. OK!
  * Solution of f.o.c. is an interior optimum
  * This is the only "well-behaved" case under perfect competition
  * Here can define a supply function

5 Geometry of cost curves

- Nicholson, Ch. 8, pp. 220–228; Ch. 9, pp. 256–259 [OLD: Ch. 12, pp. 307–312 and Ch. 13, pp. 342–346.]

- Marginal costs $MC = \partial c/\partial y \longrightarrow$ Cost minimization

$$p = MC = \partial c\left(w, r, y\right)/\partial y$$

- Average costs $AC = c/y \longrightarrow$ Does firm break even?

$$\begin{aligned}
\pi &= py - c\left(w, r, y\right) > 0 \text{ iff} \\
\pi/y &= p - c\left(w, r, y\right)/y > 0 \text{ iff} \\
c\left(w, r, y\right)/y &= AC < p
\end{aligned}$$

- **Supply function** (quantity as function of price).

- Portion of marginal cost MC above average costs.(price equals marginal cost)

- Assume only 1 input (expenditure minimization is trivial)

- **Case 1.** Production function. $y = L^{\alpha}$

  – Cost function? (cost of input is w):

$$c(w, y) = wL^{*}(w, y) = wy^{1/\alpha}$$

  – Marginal cost?

$$\frac{\partial c(w, y)}{\partial y} = \frac{1}{\alpha} wy^{(1-\alpha)/\alpha}$$

  – Average cost $c(w, y)/y$?

$$\frac{c(w, y)}{y} = \frac{wy^{1/\alpha}}{y} = wy^{(1-\alpha)/\alpha}$$

- **Case 1a.** $\alpha > 1$. Plot production function, total cost, average and marginal. Supply function?

- **Case 1b.** $\alpha = 1$. Plot production function, total cost, average and marginal. Supply function?

- **Case 1c.** $\alpha < 1$. Plot production function, total cost, average and marginal. Supply function?

- **Case 2.** *Non-convex technology.* Plot production function, total cost, average and marginal. Supply function?

- **Case 3.** *Technology with setup cost.* Plot production function, total cost, average and marginal. Supply function?

6 Next Lecture

- Profit Maximization