Econ 101A – Midterm 1
Th 28 February 2008.

You have approximately 1 hour and 20 minutes to answer the questions in the midterm. Dan and Mariana will collect the exams at 11.00 sharp. Show your work, and good luck!

Problem 1. Consumption and Leisure Decision. (58 points) In class, we considered separately a consumption decision between goods x_1 and x_2 and a lesisure decision between consumption good x and leisure l. Now we consider those together. Yingyi likes three goods: consumption goods x_1, x_2, and leisure l. He maximizes the utility function

$$u(x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} l^{\gamma},$$

with $0 < \alpha_i < 1$ for $i = 1, 2$ and $0 < \gamma < 1$. The consumption good x_i has price p_i (for $i = 1, 2$), the hourly wage is w and the individual has total income M.

1. Write down the budget constraint. Consider that Yingyi has H hours to work and, if he does not work, he takes leisure. For example, H could be 24 hours if the time period is a day. Hence, the hours worked h equal $H - l$. There are no sources of income other than income from hours worked. Write down the budget constraint as a function of x_1, x_2, and l. [Hint: Money spent on goods has to be smaller than or equal to money earned] (5 points)
2. Write down the maximization problem of the worker with respect to x_1, x_2, and l with *all* the relevant constraints Assume that the budget constraint is satisfied with equality. Why can we assume that the budget constraint is satisfied with equality? Provide as complete an explanation as you can. (5 points)
3. Write down the Lagrangean and derive the first order conditions with respect to x_1, x_2, l, and λ. (4 points)
4. Solve for x_1^* as a function of the prices p_1, p_2, w, the total number of hours H, and the parameters α_1, α_2, and γ. [Hint: combine the first and second first-order condition, then combine the first and third first-order condition, and finally plug in budget constraint] Similarly solve for x_2^* and l^*. (6 points)
5. Plot the Engel function relating the demand for good 1 $x_1^*(H)$ to the number of hours available H. (Plot x_1 in the x axis and H in the y axis) In what sense H plays the role of income? Explain. (5 points)
6. Plot the demand function for good 1 $x_1^*(p_1)$ as a function of p_1. (Put x_1 in the x axis and price p_1 in the y axis) Is the demand function downward sloping? Interpret. (5 points)
7. Are goods x_1 and x_2 gross complements, gross substitutes, or neither? Define and answer. (5 points)
8. Plot the demand function for leisure $l^*(w)$ as a function of its (shadow) price w. (Put l in the x axis and price w in the y axis) Is the demand function downward sloping? Interpret. (6 points)
9. Relate the response to question 8 (leisure l and price w) to substitution and income effects. Be careful here, this is not exactly the case we saw in class. (6 points)
10. Using the envelope theorem, compute how the indirect utility $v(p_1, p_2, w, \alpha_1, \alpha_2, \gamma, H)$ changes as H changes: $\partial v/\partial H$. Remember that the indirect utility is the utility of Yingyi at the optimum level of the parameters: $v(p_1, p_2, w, \alpha_1, \alpha_2, \gamma, H) = u(x_1^*, x_2^*, l^*)$. What is the sign of $\partial v/\partial H$? Interpret (6 points)
11. (Extra credit) Solve for the Lagrangean multiplier $\lambda^*(p_1, p_2, w, \alpha_1, \alpha_2, \gamma)$ using the first order conditions above. Comment on what this implies for $\partial v/\partial H$. (6 points)

Solution to Problem 1.

1. The budget constraint is
$$p_1x_1 + p_2x_2 \leq w\left(H - l\right),$$
which can be rewritten as
$$p_1x_1 + p_2x_2 + wl \leq Hw$$

2. Yingyi maximizes
$$\begin{aligned} \max_{x_1,x_2,l} u\left(x_1, x_2, x_3\right) &= x_1^{\alpha_1}x_2^{\alpha_2}l^{\gamma} \\ s.t. p_1x_1 + p_2x_2 + wl &\leq Hw \\ s.t.\ x_1 &\geq 0 \\ s.t.\ x_2 &\geq 0 \\ s.t.\ l &\geq 0 \end{aligned}$$

3. The Lagrangean is
$$L\left(x_1, x_2, x_3, \lambda\right) = x_1^{\alpha_1}x_2^{\alpha_2}l^{\gamma} - \lambda\left(p_1x_1 + p_2x_2 + wl - Hw\right).$$
The first order conditions are
$$\begin{aligned} f.o.c.\text{ with respect to } x_1 &: \quad \alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2}l^{\gamma} - \lambda p_1 = 0 \\ f.o.c.\text{ with respect to } x_1 &: \quad \alpha_2 x_1^{\alpha_1}x_2^{\alpha_2-1}l^{\gamma} - \lambda p_2 = 0 \\ f.o.c.\text{ with respect to } x_1 &: \quad \gamma x_1^{\alpha_1}x_2^{\alpha_2}l^{\gamma-1} - \lambda w = 0 \\ f.o.c.\text{ with respect to } \lambda &: \quad -\left(p_1x_1 + p_2x_2 + wl - Hw\right) = 0 \end{aligned}$$

4. Following the hints provided:
   - From the first two f.o.c. we derive
   $$\frac{\alpha_1}{\alpha_2}\frac{x_2}{x_1} = \frac{p_1}{p_2} \text{ which implies } x_2 = \frac{p_1}{p_2}\frac{\alpha_2}{\alpha_1}x_1.$$
   - From the first and third f.o.c. we derive
   $$\frac{\alpha_1}{\gamma}\frac{l}{x_1} = \frac{p_1}{w} \text{ which implies } l = \frac{p_1}{w}\frac{\gamma}{\alpha_1}x_1.$$
   - Substituting the solutions for x_2 and x_3 into the budget constraint we obtain
   $$p_1x_1 + p_2\left(\frac{p_1}{p_2}\frac{\alpha_2}{\alpha_1}x_1\right) + w\left(\frac{p_1}{w}\frac{\gamma}{\alpha_1}x_1\right) = Hw$$
   which can be simplified to
   $$p_1x_1 + p_1\frac{\alpha_2}{\alpha_1}x_1 + p_1\frac{\gamma}{\alpha_1}x_1 = Hw$$
   which implies
   $$x_1^* = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \gamma}\frac{Hw}{p_1}.$$
   - Using the expressions above for x_2 and x_3, we obtain
   $$\begin{aligned} x_2^* &= \frac{\alpha_2}{\alpha_1 + \alpha_2 + \gamma}\frac{Hw}{p_2} \text{ and} \\ l^* &= \frac{\gamma}{\alpha_1 + \alpha_2 + \gamma}\frac{Hw}{w} = \frac{\gamma}{\alpha_1 + \alpha_2 + \gamma}H. \end{aligned}$$

5. As we can see,
$$\partial x_1^*/\partial H = \frac{\alpha_1}{\alpha_1+\alpha_2+\gamma}\frac{w}{p_1}$$
which is positive and independent of H. The Engel curve then is linear and increasing. In this case, the number of hours available for work has a relationship to demand akin to that of income because the individual has an implicit income of Hw.

6. As we can see,
$$\partial x_1^*/\partial p_1 = -\frac{\alpha_1 Hw}{(\alpha_1+\alpha_2+\gamma)\,p_1^2}$$
The demand function for x_1^* is downward sloping in price p_1. The higher the price, the less consumers want to purchase of the good that is now more expensive. This tells us that x_1 is not a Giffin good, but we knew this from 5, above, because there we proved that good one is a normal good, and a Giffin good must be inferior.

7. Good one is a gross complement (substitute) for good two if $\partial x_1^*/\partial p_2 > 0$ (< 0). In this case, $\partial x_1^*/\partial p_2 = 0$ (and also $\partial x_2^*/\partial p_1 = 0$), hence the two goods are neither gross substitutes nor gross complements. This is a feature of Cobb-Douglas preferences.

8. The demand function for l^* is independent of w, an unusual feature. This is because there are two opposing effects: (i) a substitution effect that leads to a reduction of leisure time when the shadow cost of leisure w goes up; (ii) an income effect that leads to an increase in leisure when an increase in the wage w occurs. For a Cobb-Douglas function these two effects cancel each other out. This is unusual because in this model a change in the shadow price w increases the income available (Hw), in addition to changing prices. To see this explicitly, consider the following:

   - Applying the envelope theorem to the expenditure minimization problem we get
   $$\frac{\partial e}{\partial w} = \frac{\partial}{\partial w}\left[p_1x_1 + p_2x_2 + w\,(l-H) - \lambda\,(x_1^{\alpha_1}x_2^{\alpha_2}l^{\gamma} - \overline{u})\right] = l^* - H$$
   - Plugging this into the Slutsky equation we get
   $$\frac{\partial l^*}{\partial w} = \frac{\partial h_l^*}{\partial w} - \frac{\partial l^*}{\partial M}\,(l^* - H)$$
   - Note that, unlike what we have seen before, the last term is negative, which means that the income effect is *positive* instead of negative. Increasing the "price" of leisure increases income.

9. See the answer to question 8, above.

10. The envelope theorem states that it is enough to compute the partial derivative of the Lagrangean function with respect to H. Hence,
$$\frac{\partial v^*}{\partial H} = \frac{\partial}{\partial H}\left[x_1^{\alpha_1}x_2^{\alpha_2}l^{\gamma} - \lambda\,(p_1x_1 + p_2x_2 + wl - Hw)\right] = \lambda^* w > 0.$$

    More total hours means that Yingyi can increase consumption or leisure or both, making him unambiguously better off, since his preferences are monotonic.

11. To solve for the Lagrangean multiplier λ^*, notice that from the first f.o.c.,
$$\lambda^* = \frac{\alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2}l^{\gamma}}{p_1}.$$

    Substituting the values for x_1^*, x_2^*, and l^*, we get
$$\begin{aligned}\lambda^* &= \frac{\alpha_1\left(\frac{\alpha_1}{\alpha_1+\alpha_2+\gamma}\frac{Hw}{p_1}\right)^{\alpha_1-1}\left(\frac{\alpha_2}{\alpha_1+\alpha_2+\gamma}\frac{Hw}{p_2}\right)^{\alpha_2}\left(\frac{\gamma}{\alpha_1+\alpha_2+\gamma}H\right)^{\gamma}}{p_1} = \\ &= \alpha_1^{\alpha_1}\alpha_2^{\alpha_2}\gamma^{\gamma}\left(\frac{1}{\alpha_1+\alpha_2+\gamma}\right)^{\alpha_1+\alpha_2+\gamma-1}(H)^{\alpha_1+\alpha_2+\gamma-1}(w)^{\alpha_1+\alpha_2-1}\left(\frac{1}{p_1}\right)^{\alpha_1}\left(\frac{1}{p_2}\right)^{\alpha_2}\end{aligned}$$

which in turn means that

$$\lambda^* w = \alpha_1^{\alpha_1} \alpha_2^{\alpha_2} \gamma^\gamma \left(\frac{1}{\alpha_1 + \alpha_2 + \gamma} \right)^{\alpha_1 + \alpha_2 + \gamma - 1} (H)^{\alpha_1 + \alpha_2 + \gamma - 1} (w)^{\alpha_1 + \alpha_2} \left(\frac{1}{p_1} \right)^{\alpha_1} \left(\frac{1}{p_2} \right)^{\alpha_2}.$$

Notice that the marginal utility of extra time is increasing in the wage, w, and decreasing in the prices of the two goods, p_1, and p_2. This makes sense. When Yingyi has a high wage, he can transform additional time more easiliy into utility from consumption. Conversely, when prices of goods are high, it is harder for Yingyi to transform extra time into utility from consumption.

Problem 2. (28 points) **Complements and Substitutes.**

1. (a) Define when two goods x_1 and x_2 are gross substitutes/complements and net substitutes/complements. (4 points)
   (b) Is it possible for two goods to be gross substitutes and net complements if both goods are normal goods? And if both goods are inferior goods? Use the general form of the Slutsky equation and provide an explanation for the result. (10 points)
2. Kim has utility function $u(x_1, x_2) = \exp(x_1 + x_2)$ for $x_i \geq 0$, $i = 1, 2$.
   (a) Plot the indifference curves of Kim. What kind of goods do they represent? (4 points)
   (b) Are the preferences represented by this utility function monotonic? Define. (4 points)
   (c) Are they rational? Define. (6 points)

Solution to Problem 2.

1. (a) Good one is a gross complement (substitute) for good two if $\partial x_1^*/\partial p_2 > 0$ (< 0) and vice versa. (Note that it is possible for good one to be a gross complement for good two while good two is a gross substitute for good one since one could be normal while the other is inferior.) Meanwhile, goods one and two are net complements (substitutes) if $\partial h_1^*/\partial p_2 > 0$ (< 0) where h_1^* is the Hicksian demand for good one. (Note that it is always true that $\partial h_1^*/\partial p_2 = \partial h_2^*/\partial p_1$ for any two goods, so net complementarity and/or substitutability are always symmetric.)
   (b) First consider the general form of the Slutsky equation for consumption goods:
   $$\frac{\partial x_i^*}{\partial p_j} = \frac{\partial h_i^*}{\partial p_j} - \frac{\partial x_i^*}{\partial M} x_j^*$$
   If the two goods are gross substitutes we have $\frac{\partial x_i^*}{\partial p_j} > 0$, and if they are net complements we have $\frac{\partial h_i^*}{\partial p_j} < 0$. By the non-negativity of demand we have $x_j^* \geq 0$. If both goods are normal we have $\frac{\partial x_i^*}{\partial M} \geq 0$ so there is no way to get $\frac{\partial h_i^*}{\partial p_j} - \frac{\partial x_i^*}{\partial M} x_j^* > 0$ as required, so the answer is no, it is not possible. However, if both goods are inferior we have $\frac{\partial x_i^*}{\partial M} \leq 0$ so that if $-\frac{\partial x_i^*}{\partial M} x_j^*$ is big enough we can get $\frac{\partial h_i^*}{\partial p_j} - \frac{\partial x_i^*}{\partial M} x_j^* > 0$ so the answer is yes, it is possible. (Note that two goods can be net complements in a model with more than two goods.)
2. Kim's preferences:
   (a) The indifference curves are straight lines with slope -1. The goods x_1 and x_2 are perfect substitutes, the individual only cares about the sum of the two goods. To see this, remember that we can apply any strictly increasing transformation to a utility function and we will get a new utility function that represents the same preferences. In particular, we take the natural log of Kim's utility function,
   $$\log(\exp(x_1 + x_2)) = x_1 + x_2$$
   This utility function should be familiar to you as representing perfect substitutes.
   (b) The preferences are monotonic if $x_i \geq y_i$ for all i implies $x \succcurlyeq y$. These preferences are indeed monotonic. If $x_i \geq y_i$ for all i, then $\exp(x_1 + x_2) \geq \exp(y_1 + y_2)$. To confirm this, check that $\frac{\partial u}{\partial x_i} = \exp(x_1 + x_2) > 0$ for $i = 1, 2$.
   (c) Preferences are rational if they are complete (for all x, y in the consumption set, X, either $x \succeq y$, or $y \succeq x$) and transitive (for all x, y, and z in X such that $x \succeq y$, and $y \succeq z$ we have $x \succeq z$). It is not necessary to prove these two properties for Kim's preferences because we already know that if preferences can be represented by a utility function, they must be rational.