Econ 101A – Midterm 2
Th 8 April 2009.

You have approximately 1 hour and 20 minutes to answer the questions in the midterm. I will collect the exams at 11.00 sharp. Show your work, and good luck!

Problem 1. Production (38 points)**.** Consider a farmer that produces corn using labor. The labor cost in dollar to produce y bushels of corn is $c(y) = Cy^2$, with $C > 0$. There are 100 identical farms which all behave competitively. Corn sells at a price p per bushel.

1. Derive the marginal cost $c'(y)$ and the average cost $c(y)/y$. Plot them. Derive graphically the supply curve. (Have the quantity y on the horizontal axis). (5 points)
2. Write the expression for the supply of corn $y(p)$ for each firm, and derive the expression for the aggregate supply function $Y^S(p)$ (5 points)
3. From now on, suppose that the demand curve of corn is $D(p) = 200 - 50p$ and assume $C = 1$. Derive the equilibrium price p^* and total quantity sold Y^{S*}. (5 points)
4. Derive the profits of each firm. (5 points)
5. Now assume that land is not free, and that the farmers are renting the land from a monopolist (that is, there is only one land owner, and it is impossible to rent land somewhere else). How much will the land-owner charge each of the farms? Explain. (8 points)
6. Taking the rent of the land into account, how much are the profits of each firm? (4 points)
7. In what sense there is a parallel between the presence of rents in the land and the entry of firms in the long-run equilibrium? (6 points)

Solution of Problem 1.

1. $c'(y) = 2Cy$ and $c(y)/y = Cy$. Since marginal cost is always larger than average cost, the supply curve is the marginal cost curve. See graph in appendix.
2. The condition $p = c'(y)$ implies

$$\begin{aligned} p &= 2Cy \text{ or} \\ y^*(p) &= \frac{p}{2C} \end{aligned}$$

The aggregate supply function, summing across all firms, is

$$y^*(p) = \sum_{j=1}^{100} \frac{p}{2C} = \frac{50p}{C}$$

3. The equilibrium is

$$\begin{aligned} y^*(p) &= 50p = D(p) = 200 - 50p \\ p(50 + 50) &= 200 \\ p^* &= 2 \end{aligned}$$

and

$$\begin{aligned} Y^{S*} &= 100 * y(p) \\ &= 200 - 50 * p^* \\ &= 100 \end{aligned}$$

4. Since $Y^{S*} = 100$, each firm produces 1 unit and earns profit $\pi = p * y^* - c(y^*) = 2 - 1^2 = 1$
5. Since land does not enter the corn production function, the land rental price (r) is a fixed cost the farmer must pay to operate. Fixed costs do not change the farmer's profit-maximizing choice of y, but if profit is negative at that y, the farmer will choose not to produce at all (exit the industry).

   In this case, given the profits calculated above, no farmer will pay more than \$1 for land. Every farmer will pay any price less than or equal to \$1, however, because operating will still be profitable. So the monopolist faces a land demand curve that is perfectly inelastic for $r \leq 1$:

$$D(r) = \begin{cases} 100 & \text{if } r \leq 1, \\ 0 & \text{if } r > 1. \end{cases}$$

   Given this demand, the monopolist maximizes his own profits by setting $r = 1$: the monopolist extracts all the farmers' profit as rent.
6. Each farmer's profit will be zero.
7. Both drive farmers' profits to zero, though in the case of rents the surplus is taken by the land-owner while in the case of perfect competition, it becomes consumer surplus.

Problem 2. Consumer Surplus (35 points) We evaluate here the change in consumer surplus associated with a change in the price of good 1 from p_1 to p_1' for a consumer with Cobb-Douglas utility $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, with $0 < \alpha < 1$.

1. Explain the intuition of why the change in consumer surplus is defined as $\Delta CS = e(p_1, p_2, u) - e(p_1', p_2, u)$, where e is the expenditure function. (6 points)

2. Define the expenditure function. (4 points)

3. Derive an expression for the expenditure function $e(p_1, p_2, u)$ for this Cobb-Douglas case given that the Hicksian demands in this case are

$$\begin{aligned} h_1^*(p_1, p_2, u) &= u \cdot (\frac{p_2}{p_1} \cdot \frac{\alpha}{1-\alpha})^{1-\alpha} \\ h_2^*(p_1, p_2, u) &= u \cdot (\frac{p_1}{p_2} \cdot \frac{1-\alpha}{\alpha})^{\alpha} \end{aligned}$$

   (5 points)

4. Solve for the change in consumer surplus $\Delta CS = e(p_1, p_2, u) - e(p_1', p_2, u)$. (5 points)

5. If you also substituted for u the expression for the indirect utility $v(p_1, p_2, M)$, you would get that ΔCS is proportional to

$$\left[1 - \left(\frac{p_1'}{p_1}\right)^{\alpha}\right] \cdot M \tag{1}$$

   (Do not attempt to do this, it involves a fair amount of algebra, take it as given) Using expression (1), show that the following statements are true, and provide intuitive explanations for each of them: (i) $\Delta CS > 0$ if and only if $p_1' < p_1$; (ii) holding constant p_1' and p_1 with $p_1' < p_1$, ΔCS is increasing in α; (iii) holding constant p_1' and p_1 with $p_1' < p_1$, ΔCS is increasing in M. (15 points)

Solution of Problem 2.

1. $e(p_1', p_2, u)$ is the minimum amount of money required to attain the initial utility level u at the new price of good 1 (p_1'). Subtracting this from the expenditure that is required before the price change ($e(p_1, p_2, u)$) gives the maximum amount the consumer would be willing to pay to enact the price change. If the consumer were to pay this amount (payment may be positive or negative), then she would reach an identical utility level after the price change. In the absence of this compensating payment, she will achieve either greater utility (if $p_1' < p_1$) or less utility (if $p_1' > p_i$), but we can monetize the value of this gained/lost utility as the amount she would pay to get it.

2. The expenditure function is defined as:

$$e(p_1, p_2, u) = \min_{h_1, h_2} p_1 h_1 + p_2 h_2 \text{ s.t. } u(x_1, x_2) \geq u$$

   OR:

$$e(p_1, p_2, u) = p_1 h_1^* + p_2 h_2^*$$

   where h_1^* and h_2^* are the quantities of goods 1 and 2 that minimize spending while attaining utility level u.

3.

$$\begin{aligned} e(p_1,p_2,u) &= p_1 h_1^* + p_2 h_2^* \\ &= p_1 * u \cdot \left(\frac{p_2}{p_1}\cdot\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + p_2 * u \cdot \left(\frac{p_1}{p_2}\cdot\frac{1-\alpha}{\alpha}\right)^{\alpha} \\ &= u p_1^{\alpha} p_2^{1-\alpha}\left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{1-\alpha}{\alpha}\right)^{\alpha}\right] \\ &= u p_1^{\alpha} p_2^{1-\alpha}\left[\frac{\alpha^{-\alpha}}{(1-\alpha)^{1-\alpha}}\right] \end{aligned}$$

4. Using the expression found above,

$$\begin{aligned} \Delta CS &= e(p_1,p_2,u) - e\left(p_1',p_2,u\right) \\ &= u p_2^{1-\alpha}\left[\frac{\alpha^{-\alpha}}{(1-\alpha)^{1-\alpha}}\right]\left(p_1^{\alpha} - p_1'^{\alpha}\right) \end{aligned}$$

By the fundamental theorem of calculus, the same expression can be found by integrating $\frac{\partial e(p_1,p_2,u)}{\partial p_1} = h_1^*$ over the range $p_1'^{\alpha}$ to p_1.

5. If we take the previous expression and substitute

$$u = v\left(p_1,p_2,M\right) = \left(\frac{\alpha}{p_1}\right)^{\alpha}\cdot\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha}\cdot M,$$

we obtain

$$\begin{aligned} \Delta CS &= \left(\frac{\alpha}{p_1}\right)^{\alpha}\left(\frac{1-\alpha}{p_2}\right)^{1-\alpha} M p_2^{1-\alpha}\left[\frac{\alpha^{-\alpha}}{(1-\alpha)^{1-\alpha}}\right]\left(p_1^{\alpha} - p_1'^{\alpha}\right) = \\ &= \frac{1}{p_1^{\alpha}} M\left(p_1^{\alpha} - p_1'^{\alpha}\right) = M\left[1-\left(\frac{p_1'}{p_1}\right)^{\alpha}\right] \end{aligned}$$

(You were not required to show this) (i) Proof of $\Delta CS > 0 \Leftrightarrow p_1' < p_1$:

Given that ΔCS is proportional to the expression in (1), its sign is the sign of $\left[1-\left(\frac{p_1'}{p_1}\right)^{\alpha}\right]$. This quantity is positive if and only if $\left(\frac{p_1'}{p_1}\right)^{\alpha} < 1. \Leftrightarrow \frac{p_1'}{p_1} < 1^{1/\alpha} \Leftrightarrow p_1' < p_1$.

Intuition: Given Cobb-Douglas utility, the consumer is buying a positive quantity of x_1 at the original price. After a price decrease, s/he will be able to purchase the same bundle for less money and use the leftover money to buy more goods: the increase in utility that follows corresponds to an increase in CS.

(ii) Proof that ΔCS is increasing in α:

$$\frac{d\Delta CS}{d\alpha} \propto \frac{d\left[1-\left(\frac{p_1'}{p_1}\right)^{\alpha}\right]\cdot M}{d\alpha} = -M\frac{d\left(\frac{p_1'}{p_1}\right)^{\alpha}}{d\alpha} = -M\left(\frac{p_1'}{p_1}\right)^{\alpha}\ln\left(\frac{p_1'}{p_1}\right)$$

Given that $p_1' < p_1$, $\ln\left(\frac{p_1'}{p_1}\right) < 0$. Therefore, the quantity above is positive and $\frac{d\Delta CS}{d\alpha} > 0$.

Intuition: α is the share of income spent on x_1. When more x_1 is purchased, the consumer has more to gain from a decrease in p_1.

(iii) Proof that ΔCS is increasing in M:

$$\frac{d\Delta CS}{dM} \propto \frac{d\left[1-\left(\frac{p_1'}{p_1}\right)^{\alpha}\right]\cdot M}{dM} = \left[1-\left(\frac{p_1'}{p_1}\right)^{\alpha}\right]$$

From (i), this quantity is positive when $p_1 > p_1\prime$.

Intuition: A consumer with higher M buys a larger quantity of x_1 at p_1, and thus gets more “extra" income when p_1 falls to p_1'.

Problem 3. Uncertainty (45 points)

1. A first consumer has an expected utility function of the form $u(w) = \sqrt{w}$. She initially has a wealth of \$4. She has a lottery ticket that will be worth \$14 with probability 1/2 and will be worth \$0 with probability 1/2. What is her expected utility? What is the lowest price p she is willing to accept to sell her ticket? (10 points)

2. A second consumer has an expected utility function of the form $u(w) = \ln(w)$. A friend that is fond of gambling offers him the opportunity to bet on the flip of a coin that has probability π of coming up heads. If he bets $\$x$, he will have $w + x$ if head comes up and $w - x$ is tails comes up. Notice that x has to be non-negative ($x \geq 0$).

   - Provide a definition of risk-aversion and show that the agent is risk-averse (or not). (6 points)
   - What is his expected utility? (4 points)
   - Solve for the optimal x^* as a function of π. Discuss the qualitative features of the solution (8 points)
   - In particular, discuss the optimal x^* for $\pi < 1/2$ and for $\pi > 1/2$. Draw a parallel with the case of investment in risky asset that we saw in class. (8 points)
   - True or not true. Explain as precisely as you can. 'Given that the agent is risk-averse, he will not bet in the lottery unless the lottery has a substantially positive expected value' (9 points)

Solution of Problem 3.

1. Expected utility with lottery ticket:

$$\begin{aligned}
E[U] &= \frac{1}{2}u(4+14) + \frac{1}{2}u(4) \\
&= \frac{1}{2}\sqrt{18} + \frac{1}{2}\sqrt{4} \\
&= \frac{3\sqrt{2}}{2} + 1 \\
&= \frac{3\sqrt{2}+2}{2}
\end{aligned}$$

   Utility of selling at price p is: $u(4+p) = \sqrt{4+p}$. She will sell the ticket if this utility is greater than the expected utility of keeping the ticket.

$$\begin{aligned}
\sqrt{4+p} &\geq \frac{3\sqrt{2}+2}{2} \text{or (squaring both sides)} \\
4+p &\geq \frac{18+4+12\sqrt{2}}{4} = \frac{11}{2} + 3\sqrt{2} \\
p &\geq \frac{3}{2} + 3\sqrt{2}
\end{aligned}$$

2.
   - Risk aversion means $E[U(x)] < U(E(x))$ for any probabilistic outcome x, meaning a risk-averse agent will always reject a fair bet. Risk-aversion results from diminishing marginal utility: the utility that is gained from winning $\$y$ is less than the utility that is lost when losing $\$y$.

     We can show an agent is risk averse by showing that her utility function is concave. Check for $u(x) = \ln(x)$:

     $u'(x) = \frac{1}{x} > 0$ for all x. $u''(x) = -\frac{1}{x^2} < 0$ for all x.

     $u(x)$ is concave, so the agent is risk-averse.
   - $E[U] = \pi \ln(w+x) + (1-\pi)\ln(w-x)$

- Solving for $x^*(\pi)$:

$$\max_x \pi \quad \ln(w+x) + (1-\pi)\ln(w-x) \text{ s.t.} x \geq 0$$

$$\text{f.o.c.} \quad \frac{\pi}{w+x^*} - \frac{1-\pi}{w-x^*} = 0$$

$$x^* = \max\{w(2\pi - 1), 0\}$$

$$\text{s.o.c.} \quad -\frac{\pi}{(w+x^*)^2} - \frac{1-\pi}{(w-x^*)^2} < 0 \text{ for all } x\text{, so } x^* \text{ is a maximum when } x^* > 0$$

(when $w(2\pi - 1) < 0$, a negative bet would be utility-maximizing, but this is not allowed).

  x^* is proportional to w, so the amount a consumer wants to put at risk is a constant fraction of her wealth. It is also increasing in π conditional on $\pi > 1/2$, which means she will take larger bets as the probability of winning (and thus, the expected value of the bet) increases.

- x^* is strictly positive whenever $\pi > 1/2$, otherwise it is 0.

  Parallel to the example we saw in class on risky investments: In allocating savings between a risky and safe investment, any risk-averse person will put a strictly positive (albeit small) amount in the risky investment as long as its expected value is strictly greater than the certain value of the safe investment. Here, a risk averse agent will always participate in the bet if $\pi > 1/2$, which is equivalent to the bet having strictly positive expected value.

- The statement is not true. It is shown above that even if π is only very slightly greater than 1/2, bringing the expected value of the lottery arbitrarily close to zero, the agent will still bet a positive amount of money. The amount will certainly get smaller as the expected value of the lottery approaches zero, but it will remain positive. This result does not rely on the particular utility function, because any continuous function is locally linear; thus, for small enough changes in wealth, a risk-averse agent will behave as if he is risk-neutral.