Economics 101A
(Lecture 2)

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Outline

1. Optimization with 1 variable

2. Multivariate optimization

3. Comparative Statics

4. Implicit function theorem

1 Optimization with 1 variable

- Nicholson, Ch.2, pp. 20-23 (20-24, 9th Ed)

- Example. Function $y = -x^2$

- What is the maximum?

- Maximum is at 0

- General method?

- Sure! Use derivatives

- Derivative is slope of the function at a point:

$$\frac{\partial f(x)}{\partial x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- **Necessary condition for maximum** x^* is

$$\frac{\partial f(x^*)}{\partial x} = 0 \qquad (1)$$

- Try with $y = -x^2$.

- $\frac{\partial f(x)}{\partial x} =$ $\qquad\qquad = 0 \Longrightarrow x^* =$

- Does this guarantee a maximum? No!

- Consider the function $y = x^3$

- $\frac{\partial f(x)}{\partial x} = \qquad\qquad\qquad = 0 \Longrightarrow x^* =$

- Plot $y = x^3$.

- **Sufficient condition for a (local) maximum**:

$$\frac{\partial f(x^*)}{\partial x} = 0 \text{ and } \left.\frac{\partial^2 f(x)}{\partial^2 x}\right|_{x^*} < 0 \qquad (2)$$

- Proof: At a maximum, $f(x^* + h) - f(x^*) < 0$ for all h.

- Taylor Rule: $f(x^*+h)-f(x^*) = \frac{\partial f(x^*)}{\partial x}h+\frac{1}{2}\frac{\partial^2 f(x^*)}{\partial^2 x}h^2+$ higher order terms.

- Notice: $\frac{\partial f(x^*)}{\partial x} = 0$.

- $f(x^* + h) - f(x^*) < 0$ for all $h \Longrightarrow \frac{\partial^2 f(x^*)}{\partial^2 x}h^2 < 0 \Longrightarrow \frac{\partial^2 f(x^*)}{\partial^2 x} < 0$

- Careful: Maximum may not exist: $y = \exp(x)$

- Tricky examples:
  - *Minimum.* $y = x^2$
  - *No maximum.* $y = \exp(x)$ for $x \in (-\infty, +\infty)$
  - *Corner solution.* $y = x$ for $x \in [0, 1]$

2 Multivariate optimization

- Nicholson, Ch.2, pp. 23-30 (24-32, 9th Ed)

- Function from R^n to R: $y = f(x_1, x_2, ..., x_n)$

- Partial derivative with respect to x_i:

$$\begin{aligned}&\frac{\partial f(x_1, ..., x_n)}{\partial x_i} \\ = \;& \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, ...x_n) - f(x_1, ..., x_i, ...x_n)}{h}\end{aligned}$$

- Slope along dimension i

- Total differential:

$$df = \frac{\partial f(x)}{\partial x_1} dx_1 + \frac{\partial f(x)}{\partial x_2} dx_2 + ... + \frac{\partial f(x)}{\partial x_n} dx_n$$

- One important economic example

- Example 1: Partial derivatives of $y = f(L, K) = L^{.5}K^{.5}$

- $f'_L =$
  (marginal productivity of labor)

- $f'_K =$
  (marginal productivity of capital)

- $f''_{L,K} =$

Maximization over an open set (like R)

- **Necessary condition for maximum** x^* is

$$\frac{\partial f(x^*)}{\partial x_i} = 0 \ \forall i \tag{3}$$

or in vectorial form

$$\nabla f(x) = 0$$

- These are commonly referred to as first order conditions (f.o.c.)

- Sufficient conditions? Define Hessian matrix H:

$$H = \begin{pmatrix} f''_{x_1,x_1} & f''_{x_1,x_2} & \cdots & f''_{x_1,x_n} \\ \ldots & \ldots & \ldots & \ldots \\ f''_{x_n,x_1} & f''_{x_n,x_2} & \cdots & f''_{x_n,x_n} \end{pmatrix}$$

- Subdeterminant $|H|_i$ of Matrix H is defined as the determinant of submatrix formed by first i rows and first i columns of matrix H.

- Examples.

  - $|H|_1$ is determinant of f''_{x_1,x_1}, that is, f''_{x_1,x_1}

  - $|H|_2$ is determinant of

$$H = \begin{pmatrix} f''_{x_1,x_1} & f''_{x_1,x_2} \\ f''_{x_2,x_1} & f''_{x_2,x_2} \end{pmatrix}$$

- **Sufficient condition for maximum** x^*.

  1. x^* must satisy first order conditions;

  2. Subdeterminants of matrix H must have alternating signs, with subdeterminant of H_1 negative.

- Case with $n = 2$

- Condition 2 reduces to $f''_{x_1,x_1} < 0$ and $f''_{x_1,x_1} f''_{x_2,x_2} - (f''_{x_1,x_2})^2 > 0$.

- Example 2: $h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2$

- First order condition w/ respect to x_1?

- First order condition w/ respect to x_2?

- $x_1^*, x_2^* =$

- For which p_1, p_2 is it a maximum?

- For which p_1, p_2 is it a minimum?

3 Comparative statics

- Economics is all about ‘comparative statics’

- What happens to optimal economic choices if we change one parameter?

- Example: Car production. Consumer:

  1. Car purchase and increase in oil price

  2. Car purchase and increase in income

- Producer:

  1. Car production and minimum wage increase

  2. Car production and decrease in tariff on Japanese cars

- Next two sections

4 Implicit function theorem

- Implicit function: Ch. 2, pp. 31-32 (32–33, 9th Ed)

- Consider function $x_2 = g(x_1, p)$

- Can rewrite as $x_2 - g(x_1, p) = 0$

- **Implicit function** has form: $h(x_2, x_1, p) = 0$

- Often we need to go from implicit to explicit function

- Example 3: $1 - x_1 * x_2 - e^{x_2} = 0$.

- Write x_1 as function of x_2:

- Write x_2 as function of x_1:

- **Univariate implicit function theorem (Dini)**: Consider an equation $f(p, x) = 0$, and a point (p_0, x_0) solution of the equation. Assume:

  1. f continuous and differentiable in a neighbourhood of (p_0, x_0);

  2. $f'_x(p_0, x_0) \neq 0$.

- Then:

  1. There is one and only function $x = g(p)$ defined in a neighbourhood of p_0 that satisfies $f(p, g(p)) = 0$ and $g(p_0) = x_0$;

  2. The derivative of $g(p)$ is

$$g'(p) = -\frac{f'_p(p, g(p))}{f'_x(p, g(p))}$$

- Example 3 (continued): $1 - x_1 * x_2 - e^{x_2} = 0$

- Find derivative of $x_2 = g(x_1)$ implicitely defined for $(x_1, x_2) = (1, 0)$

- Assumptions:

  1. Satisfied?

  2. Satisfied?

- Compute derivative

- **Multivariate implicit function theorem (Dini)**: Consider a set of equations ($f_1(p_1, ..., p_n; x_1, ..., x_s) = 0; ...; f_s(p_1, ..., p_n; x_1, ..., x_s) = 0$), and a point ($p_0$,$x_0$) solution of the equation. Assume:

  1. $f_1, ..., f_s$ continuous and differentiable in a neighbourhood of (p_0,x_0);

  2. The following Jakobian matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ evaluated at (p_0,x_0) has determinant different from 0:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & & \frac{\partial f_1}{\partial x_s} \\ ... & ... & ... \\ \frac{\partial f_s}{\partial x_1} & ... & \frac{\partial f_s}{\partial xs} \end{pmatrix}$$

- Then:

  1. There is one and only set of functions $x = \mathbf{g}(p)$ defined in a neighbourhood of p_0 that satisfy $\mathbf{f}(p, \mathbf{g}(p)) = \mathbf{0}$ and $\mathbf{g}(p_0) = x_0$;

  2. The partial derivative of x_i with respect to p_k is

$$\frac{\partial g_i}{\partial p_k} = -\frac{\det\left(\frac{\partial(f_1,\ldots,f_s)}{\partial(x_1,\ldots x_{i-1},p_k,x_{i+1}\ldots,x_s)}\right)}{\det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)}$$

- Example 2 (continued): Max $h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2$

- f.o.c. x_1 : $2p_1 * x_1 - 2 = 0 = f_1(p,x)$

- f.o.c. x_2 : $2p_2 * x_2 - 5 = 0 = f_2(p,x)$

- Comparative statics of x_1^* with respect to p_1?

- First compute det $\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

- Then compute $\mathbf{det}\left(\frac{\partial(f_1,\ldots,f_s)}{\partial(x_1,\ldots x_{i-1},p_k,x_{i+1}\ldots,x_s)}\right)$

$$\begin{pmatrix} \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} & \end{pmatrix}$$

- Finally, $\frac{\partial x_1}{\partial p_1} =$

- Why did you compute $\mathbf{det}\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)$ already?

5 Next Class

- Next class:
  - Envelope Theorem
  - Convexity and Concavity
  - Constrained Maximization
  - Envelope Theorem II

- Going toward:
  - Preferences
  - Utility Maximization (where we get to apply maximization techniques the first time)