Econ 101A – Midterm 1
Th 29 September 2004.

You have approximately 1 hour and 20 minutes to answer the questions in the midterm. I will collect the exams at 12.30 sharp. Show your work, and good luck!

Problem 1. Utility maximization. (57 points) In class, we have considered the case of maximization of utility with Cobb-Douglas utility function with goods x (baseball) and y (cupcakes) as arguments. We have also considered the case of intertemporal maximization of utility with consumption of goods today, c_0, and in the future, c_1. This problem asks you to combine the two elements and reconsider some of the properties we had found in the two separate cases. Consider a consumer with utility function

$$u(x_0, y_0, x_1, y_1) = (x_0)^{\alpha} (y_0)^{\beta} + \frac{1}{1+\delta} (x_1)^{\alpha} (y_1)^{\beta} \tag{1}$$

where x_t is the quantity of baseball x consumed at time $t = 0, 1$. Similarly, y_t is the quantity of cupcakes y consumed at time $t = 0, 1$. For reasons that we will see below, we will assume $\alpha + \beta < 1$, so do not substitute $\beta = 1 - \alpha$. As usual, $\alpha > 0$ and $\beta > 0$ also hold.

1. What is the interpretation of δ in the utility function? Interpret in particular the special cases $\delta = 0$ and $\delta \to \infty$. (6 points)

2. Explain the procedure to determine if $u(x_0, y_0, x_1, y_1)$ is concave in (x_0, y_0, x_1, y_1). You should not do the computations. (4 points)

3. We now consider the budget constraint. Both in period 0 and in period 1 the price of music x is p_x and the price of cupcakes y is p_y. We denote income at time 0 by M_0 and income at time 1 as M_1. Consumers can borrow across periods 0 and 1 at interest rate r. Show that this implies that the (intertemporal) budget constraint is:

$$p_x x_0 + p_y y_0 + \frac{1}{1+r} (p_x x_1 + p_y y_1) \leq M_0 + \frac{1}{1+r} M_1. \tag{2}$$

   (6 points)

4. The consumer maximizes utility (1) with respect to the four consumption variables (x_0, y_0, x_1, y_1), subject to budget constraint (2). Assume that the budget constraint is satisfied with equality (it is) and write the Lagrangean function. Derive the set of first order conditions. (5 points).

5. Show that the first order conditions with respect to (x_0, y_0, x_1, y_1) lead to the familiar conditions

$$\frac{\alpha}{\beta} \frac{y_0}{x_0} = \frac{p_x}{p_y} \tag{3}$$

   and

$$\frac{\alpha}{\beta} \frac{y_1}{x_1} = \frac{p_x}{p_y}. \tag{4}$$

   Give an economic interpretation of these conditions. Are you surprised that the same conditions that hold in the one-period Cobb-Douglas utility maximization also hold in the 2-period one? (5 points)

6. [Harder] If you can, show that the first order conditions imply

$$\left(\frac{y_1}{y_0}\right)^{1-(\alpha+\beta)} = \left(\frac{x_1}{x_0}\right)^{1-(\alpha+\beta)} = \frac{1+r}{1+\delta} \tag{5}$$

   (You may want to leave this for last – you can continue answering the rest of Problem 1 without solving this point) (10 points)

7. Keep assuming $\alpha + \beta < 1$ and consider equation (5). How does relative consumption in period 1 compared to period 0 ($y_1/y_0 = x_1/x_0$) vary as r increases? Explain intuitively why an increase in the interest rate increase consumption in the future. Similarly, how does relative consumption in period 1 compared to period 0 ($y_1/y_0 = x_1/x_0$) vary as δ increases? Provide intuition. (6 points)

8. What does equation (5) imply about y_1/y_0 for $r = \delta$? Provide intuition on why at the optimum the agent smooths consumption of goods x and y across periods. (6 points) Extra credit: Why is it crucial for this answer that $\alpha + \beta < 1$?

9. For $r = \delta$, solve for the optimal value of x_0^*. Hint: use the budget constraint and equation (3). (5 points)

10. Use the expression for x_0^* that you obtained in point 9. Differentiate x_0^* with respect to M_0, that is, compute $\partial x_0^*/\partial M_0$. Is music x a normal good (for all levels of price and income)? (4 points)

Solution to Problem 1.

1. Parameter δ has the interpretation of psychological discount rate. It indicates how much the individual discounts future utility as of the present. The higher δ, the more impatient the individual is, the less she value the future. Special case $\delta = 0$: in this case the individual gives equal weight to consumption of baseball (or cupcakes) in the present or in the future. Special case $\delta \to \infty$: the individual is myopic, does not give any weight to future utility.

2. To determine whether the utility function u is concave in (x_0, y_0, x_1, y_1) we should set up the matrix:

$$H = \begin{vmatrix} \frac{\partial^2 u}{\partial x_0, \partial x_0} & \frac{\partial^2 u}{\partial x_0, \partial x_1} & \frac{\partial^2 u}{\partial x_0, \partial y_0} & \frac{\partial^2 u}{\partial x_0, \partial y_1} \\ \frac{\partial^2 u}{\partial x_1, \partial x_0} & \frac{\partial^2 u}{\partial x_1, \partial x_1} & \frac{\partial^2 u}{\partial x_1, \partial y_0} & \frac{\partial^2 u}{\partial x_1, \partial y_1} \\ \frac{\partial^2 u}{\partial y_0, \partial x_0} & \frac{\partial^2 u}{\partial y_0, \partial x_1} & \frac{\partial^2 u}{\partial y_0, \partial y_0} & \frac{\partial^2 u}{\partial y_0, \partial y_1} \\ \frac{\partial^2 u}{\partial y_1, \partial x_0} & \frac{\partial^2 u}{\partial y_1, \partial x_1} & \frac{\partial^2 u}{\partial y_1, \partial y_0} & \frac{\partial^2 u}{\partial y_1, \partial y_1} \end{vmatrix}$$

The condition for concavity is that the first subdeterminant be negative or equal to zero, the second be positive (or equal to zero), the third be negative (or equal to zero), the fourth (the whole matrix) be positive (or equal to zero). Importantly, these conditions have to hold for *all* (x_0, y_0, x_1, y_1) in R_4^+.

3. In period 0 the consumer can spend income M_0 on good x_0 or on good y_0. The savings in period 0 $s_0 = M_0 - (p_x x_0 + p_y y_0)$ can be saved for period 1 and earn interest rate r. Notice that s_0 can be negative, that is, the individual may wish to borrow from period 1. In period 2 the individual can spend $M_1 + (1+r)\, s_0 = M_1 + (1+r)\left[M_0 - (p_x x_0 + p_y y_0)\right]$. The budget constraint in period 1 therefore can be written as $p_x x_1 + p_y y_1 \leq M_1 + (1+r)\left[M_0 - (p_x x_0 + p_y y_0)\right]$. By rearranging terms, we obtain

$$(1+r)\,(p_x x_0 + p_y y_0) + (p_x x_1 + p_y y_1) \leq M_1 + (1+r)\, M_0,$$

which is equivalent to (2) (just divide by $(1+r)$).

4. The consumer maximizes

$$\max_{(x_0, y_0, x_1, y_1)} (x_0)^\alpha (y_0)^\beta + \frac{1}{1+\delta} (x_1)^\alpha (y_1)^\beta$$
$$s.t.\ p_x x_0 + p_y y_0 + \frac{1}{1+r} (p_x x_1 + p_y y_1) \leq M_0 + \frac{1}{1+r} M_1$$

This is equivalent to maximizing the Lagrangean

$$\begin{aligned} L(x_0, y_0, x_1, y_1, \lambda) \quad = \quad & (x_0)^\alpha (y_0)^\beta + \frac{1}{1+\delta} (x_1)^\alpha (y_1)^\beta - \\ & \lambda \left[p_x x_0 + p_y y_0 + \frac{1}{1+r} (p_x x_1 + p_y y_1) - \left(M_0 + \frac{1}{1+r} M_1 \right) \right]. \end{aligned}$$

This yields the following five first order conditions:

$$\begin{aligned}
f.o.c.\ x_0 &: \quad \alpha (x_0)^{\alpha-1} (y_0)^{\beta} - \lambda p_x = 0 \\
f.o.c.\ y_0 &: \quad \beta (x_0)^{\alpha} (y_0)^{\beta-1} - \lambda p_y = 0 \\
f.o.c.\ x_1 &: \quad \frac{1}{1+\delta}\left(\alpha (x_1)^{\alpha-1} (y_1)^{\beta}\right) - \lambda \frac{1}{1+r} p_x = 0 \\
f.o.c.\ y_1 &: \quad \frac{1}{1+\delta}\left(\beta (x_1)^{\alpha} (y_1)^{\beta-1}\right) - \lambda \frac{1}{1+r} p_y = 0 \\
f.o.c.\ \lambda &: \quad -\left[p_x x_0 + p_y y_0 + \frac{1}{1+r}(p_x x_1 + p_y y_1) - \left(M_0 + \frac{1}{1+r} M_1\right)\right] = 0
\end{aligned}$$

5. First order conditions 1 and 2 lead to

$$\begin{aligned}
\alpha (x_0)^{\alpha-1} (y_0)^{\beta} &= \lambda p_x \\
&\text{and} \\
\beta (x_0)^{\alpha} (y_0)^{\beta-1} &= \lambda p_y
\end{aligned}$$

Taking the ratio of these two equalities, we obtain

$$\frac{\alpha}{\beta}\frac{y_0}{x_0} = \frac{p_x}{p_y}. \tag{6}$$

Similarly, the ratio of first order conditions 3 and 4 lead to

$$\frac{\alpha}{\beta}\frac{y_1}{x_1} = \frac{p_x}{p_y}. \tag{7}$$

These conditions indicates the Marginal Rate of Substitution between goods x and y in period 0 equate the ratio of the prices, and similarly so for goods x and y in period 1. The condition MSR=ratio of prices that denotes the tangency of indifference curve and budget set, therefore, extends to the 2-period case.

6. First order conditions 1 and 3 lead to

$$\begin{aligned}
\alpha (x_0)^{\alpha-1} (y_0)^{\beta} &= \lambda p_x \\
&\text{and} \\
\frac{1}{1+\delta}\left(\alpha (x_1)^{\alpha-1} (y_1)^{\beta}\right) &= \lambda \frac{1}{1+r} p_x.
\end{aligned}$$

Taking the ratio of these two equalities, we obtain

$$\left(\frac{x_0}{x_1}\right)^{\alpha-1}\left(\frac{y_0}{y_1}\right)^{\beta} = \frac{1+r}{1+\delta}. \tag{8}$$

Note now that (6) and (7) can be combined to yield

$$\frac{y_0}{y_1} = \frac{x_0}{x_1}.$$

We label $R = \frac{y_0}{y_1} = \frac{x_0}{x_1}$ the ratio of consumption in period 0 and 1. Then (8) implies

$$R^{\alpha+\beta-1} = \left(\frac{y_1}{y_0}\right)^{1-(\alpha+\beta)} = \left(\frac{x_1}{x_0}\right)^{1-(\alpha+\beta)} = \frac{1+r}{1+\delta}$$

7. As r increases, the right-hand side of the equality increases. This implies that the left-hand side must increase as well. Given $\alpha + \beta < 1$, this implies that y_1/y_0 and x_1/x_0 must increase. Intuitively, as the interest rate increases, it becomes more expensive to consume in period 0 and more advantageous to consume in the later period, period 1. The opposite effect applies when impatience increases. The right-hand side decreases, and y_1/y_0 and x_1/x_0 must decrease as well. Intuitively, if individuals become more impatient they consume more in the present.

8. For $r = \delta$, we obtain $y_1^* = y_0^* = y^*$ and $x_1^* = x_0^* = x^*$, that is, the consumer consumes the same quantity of baseball (cupcakes) in each of the periods. The reason for this is that the utility function is concave in the amount of consumption in each period. This implies that marginal utility is decreasing in the amount of consumption in each period. The consumer prefers to consume equal amounts rather than consuming a lot one period and very little the next.

9. Under the assumption $r = \delta$, we know that $y_1^* = y_0^* = y^*$ and $x_1^* = x_0^* = x^*$. Further, equation (3) implies $p_y y^* = (\beta/\alpha)\, p_x x^*$. We substitute into the budget constraint to get

$$p_x x^* + (\beta/\alpha)\, p_x x^* + \frac{1}{1+r}\left(p_x x^* + (\beta/\alpha)\, p_x x^*\right) = M_0 + \frac{1}{1+r} M_1$$

or

$$p_x x^* \frac{\alpha+\beta}{\alpha}\left(1 + \frac{1}{1+r}\right) = M_0 + \frac{1}{1+r} M_1$$

or

$$x^* = \frac{1+r}{2+r}\frac{\alpha}{\alpha+\beta}\frac{M_0 + \frac{1}{1+r} M_1}{p_x}. \tag{9}$$

10. The derivative of (9) with respect to M_0 is

$$\frac{\partial x_0^*}{\partial M_0} = \frac{1+r}{2+r}\frac{\alpha}{\alpha+\beta}\frac{1}{p_x}.$$

The good is a normal good since increases in income M_0 induce an increase in the quantity consumed x_0^* for all levels of p_x and r.

Problem 2. (18 points)

1. Consider the maximization problem
$$\max_x f(x,p)$$
where both x and p are real numbers (that is, not vectors). This leads to the first order condition
$$f'_x(x^*,p)=0.$$
Use the implicit function theorem to give an expression for dx^*/dp. Argue that, as long as x^* is a maximum, the sign of dx^*/dp is given by $f''_{x,p}(x^*,p)$. (8 points)

2. Consider a consumer with preferences over two goods, music x and cupcakes y, with $x \geq 0$ and $y \geq 0$. The consumer is addicted to cupcakes y and is 'loss averse' to consuming less than $Y > 0$ units of cupcakes. Here are the preferences of the consumer. A bundle (x,y) is weakly preferred ($\succsim$) to a bundle (x',y') if and only if the following is true:
$$\begin{aligned} (y-Y)+x &\geq (y'-Y)+x' \text{ for } y \geq Y \text{ and } y' \geq Y \\ 2(y-Y)+x &\geq (y'-Y)+x' \text{ for } y < Y \text{ and } y' \geq Y \\ (y-Y)+x &\geq 2(y'-Y)+x' \text{ for } y \geq Y \text{ and } y' < Y \\ 2(y-Y)+x &\geq 2(y'-Y)+x' \text{ for } y < Y \text{ and } y' < Y \end{aligned}$$
Can you provide a utility function that represents these preferences? Plot some of the indifference curves. Why could these indifference curves capture an aspect of addiction? (10 points)

Solution to Problem 2.

1. Using the implicit function theorem we have
$$\frac{dx^*}{dp} = -\frac{\frac{\partial\left(f'_x(x^*,p)\right)}{\partial p}}{\frac{\partial\left(f'_x(x^*,p)\right)}{\partial x}} = -\frac{f''_{x,p}(x^*,p)}{f''_{x,x}(x^*,p)}.$$
Since in a point of maximum the second order condition guarantees $f''_{x,p}(x^*,p) < 0$, we can conclude that the sign of dx^*/dp is equal to the sign of the numerator, that is, to the sign of $f''_{x,p}(x^*,p)$.

2. A utility function that represents the preferences in the problem is
$$u(x,y) = \begin{matrix} x+(y-Y) & \text{if} & y \geq Y \\ x+2(y-Y) & \text{if} & y < Y \end{matrix}$$
See the Figure for indifference curves. The consumer has the same preferences for x and y as long as y is at least as large as Y. That is, she is willing to trade one unit of music for one unit of cupcake. If y falls below the minimum level Y, though, the preferences of the consumer change. For y below Y, the consumer wants two units of cupcake in exchange for one unit of music. Intuitively, this captures the existence of a reference point with respect to which the consumer is loss averse: the relative preference for good y increases when y falls below the reference point. This may capture a feature of addiction insofar as falling below the addictive level Y makes the consumer desire y more.