Economics 101A
(Lecture 3)

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Outline

1. Implicit Function Theorem II
2. Envelope Theorem
3. Convexity and concavity
4. Constrained Maximization
5. Envelope Theorem II

1 Implicit function theorem II

- **Univariate implicit function theorem (Dini)**: Consider an equation $f(p, x) = 0$, and a point (p_0, x_0) solution of the equation. Assume:

  1. f continuous and differentiable in a neighbourhood of (p_0, x_0);

  2. $f'_x(p_0, x_0) \neq 0$.

- Then:

  1. There is one and only function $x = g(p)$ defined in a neighbourhood of p_0 that satisfies $f(p, g(p)) = 0$ and $g(p_0) = x_0$;

  2. The derivative of $g(p)$ is

$$g'(p) = -\frac{f'_p(p, g(p))}{f'_x(p, g(p))}$$

- Example 3 (continued): $1 - x_1 * x_2 - e^{x_2} = 0$

- Find derivative of $x_2 = g(x_1)$ implicitely defined for $(x_1, x_2) = (1, 0)$

- Assumptions:

  1. Satisfied?

  2. Satisfied?

- Compute derivative

- **Multivariate implicit function theorem (Dini)**: Consider a set of equations ($f_1(p_1, ..., p_n; x_1, ..., x_s) = 0; ...; f_s(p_1, ..., p_n; x_1, ..., x_s) = 0$), and a point ($p_0$,$x_0$) solution of the equation. Assume:

  1. $f_1, ..., f_s$ continuous and differentiable in a neighbourhood of (p_0,x_0);

  2. The following Jakobian matrix $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ evaluated at (p_0,x_0) has determinant different from 0:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & & \frac{\partial f_1}{\partial x_s} \\ ... & ... & ... \\ \frac{\partial f_s}{\partial x_1} & ... & \frac{\partial f_s}{\partial xs} \end{pmatrix}$$

- Then:

  1. There is one and only set of functions $x = \mathbf{g}(p)$ defined in a neighbourhood of p_0 that satisfy $\mathbf{f}(p, \mathbf{g}(p)) = \mathbf{0}$ and $\mathbf{g}(p_0) = x_0$;

  2. The partial derivative of x_i with respect to p_k is

$$\frac{\partial g_i}{\partial p_k} = -\frac{\det\left(\frac{\partial(f_1,\ldots,f_s)}{\partial(x_1,\ldots x_{i-1},p_k,x_{i+1}\ldots,x_s)}\right)}{\det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)}$$

- Example 2 (continued): Max $h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2$

- f.o.c. x_1 : $2p_1 * x_1 - 2 = 0 = f_1(p,x)$

- f.o.c. x_2 : $2p_2 * x_2 - 5 = 0 = f_2(p,x)$

- Comparative statics of x_1^* with respect to p_1?

- First compute $\det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

- Then compute $\det\left(\frac{\partial(f_1,\ldots,f_s)}{\partial(x_1,\ldots x_{i-1},p_k,x_{i+1}\ldots,x_s)}\right)$

$$\begin{pmatrix} \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} & & & \\ & & & \end{pmatrix}$$

- Finally, $\frac{\partial x_1}{\partial p_1} =$

- Why did you compute $\det\left(\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right)$ already?

2 Envelope Theorem

- Ch. 2, pp. 32-36 (33–37, 9th Ed)

- You now know how x_1^* varies if p_1 varies.

- How does $h\left(\mathbf{x}^*\left(\mathbf{p}\right)\right)$ vary as p_1 varies?

- Differentiate $h(x_1^*(p_1,p_2),x_2^*(p_1,p_2),p_1,p_2)$ with respect to p_1:

$$\begin{aligned}
& \frac{dh(x_1^*(p_1,p_2),x_2^*(p_1,p_2),p_1,p_2)}{dp_1} \\
= \quad & \frac{\partial h(\mathbf{x}^*,\mathbf{p})}{\partial x_1} * \frac{\partial x_1^*(\mathbf{x}^*,\mathbf{p})}{\partial p_1} \\
& + \frac{\partial h(\mathbf{x}^*,\mathbf{p})}{\partial x_2} * \frac{\partial x_2^*(\mathbf{x}^*,\mathbf{p})}{\partial p_1} \\
& + \frac{\partial h(\mathbf{x}^*,\mathbf{p})}{\partial p_1}
\end{aligned}$$

- The first two terms are zero.

- **Envelope Theorem** for unconstrained maximization. Assume that you maximize function $f(\mathbf{x};\mathbf{p})$ with respect to x. Consider then the function f at the optimum, that is, $f(\mathbf{x}^*(\mathbf{p}),\mathbf{p})$. The total differential of this function with respect to p_i equals the partial derivative with respect to p_i:

$$\frac{df(\mathbf{x}^*(\mathbf{p}),\mathbf{p})}{dp_i} = \frac{\partial f(\mathbf{x}^*(\mathbf{p}),\mathbf{p})}{\partial p_i}.$$

- You can disregard the indirect effects. Graphical intuition.

3 Convexity and concavity

- Function f from $C \subset R^n$ to R is concave if

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$$

for all $x, y \in C$ and for all $t \in [0, 1]$

- Notice: C must be convex set, i.e., if $x \in C$ and $y \in C$, then $tx + (1-t)y \in C$, for $t \in [0, 1]$

- Function f from $C \subset R^n$ to R is strictly concave if

$$f(tx + (1-t)y) > tf(x) + (1-t)f(y)$$

for all $x, y \in C$ and for all $t \in (0, 1)$

- Function f from R^n to R is convex if $-f$ is concave.

- Alternative characterization of convexity.

- A function f, twice differentiable, is concave if and only if **for all** x the subdeterminants $|H_i|$ of the Hessian matrix have the property $|H_1| \leq 0, |H_2| \geq 0, |H_3| \leq 0$, and so on.

- For the univariate case, this reduces to $f'' \leq 0$

- For the bivariate case, this reduces to $f''_{x,x} \leq 0$ and $f''_{x,x} * f''_{y,y} - \left(f''_{x,y}\right)^2 \geq 0$

- A twice-differentiable function is strictly concave if the same property holds with strict inequalities.

- Examples.

    1. For which values of a, b, and c is $f(x) = ax^3 + bx^2 + cx + d$ is the function concave over R? Strictly concave? Convex?

    2. Is $f(x, y) = -x^2 - y^2$ concave?

- For Example 2, compute the Hessian matrix

    - $f'_x =$ $\quad\quad\quad\quad\quad\quad\quad\quad$ $, f'_y =$

    - $f''_{x,x} =$ $\quad\quad\quad\quad\quad\quad\quad\quad$ $, f''_{x,y} =$

    - $f''_{y,x} =$ $\quad\quad\quad\quad\quad\quad\quad\quad$ $, f''_{y,y} =$

    - Hessian matrix H :

    $$H = \begin{pmatrix} f''_{x,x} = & \quad\quad\quad\quad f''_{x,y} = & \quad\quad\quad\quad \\ f''_{y,x} = & \quad\quad\quad\quad f''_{y,y} = & \quad\quad\quad\quad \end{pmatrix}$$

- Compute $|H_1| = f''_{x,x}$ and $|H_2| = f''_{x,x} * f''_{y,y} - \left(f''_{x,y}\right)^2$

- Why are convexity and concavity important?

- Theorem. Consider a twice-differentiable concave (convex) function over $C \subset R^n$. If the point $\mathbf{x}_0$ satisfies the fist order conditions, it is a global maximum (minimum).

- For the proof, we need to check that the second-order conditions are satisfied.

- These conditions are satisfied by definition of concavity!

- (We have only proved that it is a local maximum)

4 Constrained maximization

- Ch. 2, pp. 36-42 (38–44, 9th Ed)

- So far unconstrained maximization on R (or open subsets)

- What if there are constraints to be satisfied?

- Example 1: $\max_{x,y} x * y$ subject to $3x + y = 5$

- Substitute it in: $\max_{x,y} x * (5 - 3x)$

- Solution: $x^* =$

- Example 2: $\max_{x,y} xy$ subject to $x \exp(y) + y \exp(x) = 5$

- Solution: ?

- Graphical intuition on general solution.

- Example 3: $\max_{x,y} f(x,y) = x * y$ s.t. $h(x,y) = x^2 + y^2 - 1 = 0$

- Draw $0 = h(x,y) = x^2 + y^2 - 1$.

- Draw $x * y = K$ with $K > 0$. Vary K

- Where is optimum?

- Where dy/dx along curve $xy = K$ equals dy/dx along curve $x^2 + y^2 - 1 = 0$

- Write down these slopes.

- Idea: Use implicit function theorem.

- Heuristic solution of system

$$\max_{x,y} f(x,y)$$
$$\text{s.t. } h(x,y) = 0$$

- Assume:
  - continuity and differentiability of h
  - $h'_y \neq 0$ (or $h'_x \neq 0$)

- Implicit function Theorem: Express y as a function of x (or x as function of y)!

- Write system as $\max_x f(x, g(x))$

- f.o.c.: $f'_x(x, g(x)) + f'_y(x, g(x)) * \frac{\partial g(x)}{\partial x} = 0$

- What is $\frac{\partial g(x)}{\partial x}$?

- Substitute in and get: $f'_x(x, g(x)) + f'_y(x, g(x)) * (-h'_x/h'_y) = 0$ or

$$\frac{f'_x(x, g(x))}{f'_y(x, g(x))} = \frac{h'_x(x, g(x))}{h'_y(x, g(x))}$$

- **Lagrange Multiplier Theorem, necessary condition.** Consider a problem of the type

$$\max_{x_1,\ldots,x_n} f\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right)$$

$$\text{s.t.} \qquad \begin{cases} h_1\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right) = 0 \\ h_2\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right) = 0 \\ \ldots \\ h_m\left(x_1, x_2, \ldots, x_n; \mathbf{p}\right) = 0 \end{cases}$$

with $n > m$. Let $\mathbf{x}^* = \mathbf{x}^*(\mathbf{p})$ be a local solution to this problem.

- Assume:

  – f and h differentiable at x^*

  – the following Jacobian matrix at $\mathbf{x}^*$ has maximal rank

$$J = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\mathbf{x}^*) & \ldots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}^*) \\ \ldots & \ldots & \ldots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}^*) & \ldots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

- Then, there exists a vector $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_m)$ such that $(\mathbf{x}^*, \boldsymbol{\lambda})$ maximize the Lagrangean function

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}; \mathbf{p}) - \sum_{j=0}^{m} \lambda_j h_j(\mathbf{x}; \mathbf{p})$$

- Case $n = 2, m = 1$.

- First order conditions are

$$\frac{\partial f(\mathbf{x}; \mathbf{p})}{\partial x_i} - \lambda \frac{\partial h(\mathbf{x}; \mathbf{p})}{\partial x_i} = 0$$

for $i = 1, 2$

- Rewrite as

$$\frac{f'_{x_1}}{f'_{x_2}} = \frac{h'_{x_1}}{h'_{x_2}}$$

Constrained Maximization, Sufficient condition for the case $n = 2, m = 1$.

- If $\mathbf{x}^*$ satisfies the Lagrangean condition, and the determinant of the bordered Hessian

$$H = \begin{pmatrix} 0 & -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial^2 x_1}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2 \partial x_1}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2 \partial x_2}(\mathbf{x}^*) \end{pmatrix}$$

  is positive, then $\mathbf{x}^*$ is a constrained maximum.

- If it is negative, then $\mathbf{x}^*$ is a constrained minimum.

- Why? This is just the Hessian of the Lagrangean L with respect to λ, x_1, and x_2

- Example 4: $\max_{x,y} x^2 - xy + y^2$ s.t. $x^2 + y^2 - p = 0$

- $\max_{x,y,\lambda} x^2 - xy + y^2 - \lambda(x^2 + y^2 - p)$

- F.o.c. with respect to x:

- F.o.c. with respect to y:

- F.o.c. with respect to λ:

- Candidates to solution?

- Maxima and minima?

5 Envelope Theorem II

- Envelope Theorem II: Ch. 2, pp. 42-43 (44, 9th Ed)

- **Envelope Theorem for Constrained Maximization.** In problem above consider $F(p) \equiv f(\mathbf{x}^*(\mathbf{p}); \mathbf{p})$. We are interested in $dF(p)/dp$. We can neglect indirect effects:
$$\frac{dF}{dp_i} = \frac{\partial f(\mathbf{x}^*(\mathbf{p}); \mathbf{p})}{\partial p_i} - \sum_{j=0}^{m} \lambda_j \frac{\partial h_j(\mathbf{x}^*(\mathbf{p}); \mathbf{p})}{\partial p_i}$$

- Example 4 (continued). $\max_{x,y} x^2 - xy + y^2$ s.t. $x^2 + y^2 - p = 0$

- $df(x^*(p), y^*(p))/dp$?

- Envelope Theorem.

6 Next Class

- Next class:
  - Preferences
  - Utility Maximization (where we get to apply maximization techniques the first time)