Econ 101A – Solutions to Midterm 2
Th 10 November 2005.

You have approximately 1 hour and 20 minutes to answer the questions in the midterm. We will collect the exams at 12.30 sharp. Show your work, and good luck!

Problem 1. (Self-control problems) (33 points) Consider an individual with self-control problems, Arnold. Arnold is deciding how much to exercise. The quantity of exercise is e, with $e > 0$. The benefit of exercise is e, which is received one period after the exercise. The effort cost of exercising is $c(e)$, with $c'() > 0$ and $c''() > 0$, a cost felt immediately.

1. At the moment of exercising, therefore, Arnold maximizes the discounted utility
$$\max_e -c(e) + \frac{\beta}{1+\delta}e.$$
with $\beta < 1$. Compute the first order condition that defines the solution e^*. (3 points)

2. Show that the function $-c(e) + \beta e/(1+\delta)$ is concave. What does this imply about the solution e^*? (5 points)

3. Now consider Arnold one period before the actual exercise decision. In this period Arnold receives no additional payoff. Arnold has a commitment device that allows him to choose the attendance for next period. Write down the discounted utility function that Arnold maximizes and solve for the first-order condition defining e^*_C, the exercise level chosen with commitment. (8 points)

4. Compare e^* and e^*_C. Discuss with reference to self-control problems. (6 points)

5. How do e^* and e^*_C compare when β equals 1? Provide intuition (3 points)

6. (Harder) Suppose now that at each attendance Arnold pays a price p per unit of exercise, that is, he pays $p * e$ overall. (the price could be negative, allowing for a subsidy for attendance) With this additional price, now Arnold chooses the new attendance decision $e^{*\prime}$ to maximize
$$\max_e -c(e) - pe + \frac{\beta}{1+\delta}e.$$
What is the level of price p^* such that the attendance $e^{*\prime}$ with price equals the attendance e^*_C with commitment device? That is, what does the price on exercise need to be to attain the attendance chosen with commitment? Is this price p^* positive or negative? Provide intuition on this result. (8 points)

Solution to Problem 1.

1. The first-order condition is
$$-c'(e^*) + \frac{\beta}{1+\delta} = 0 \quad (1)$$
or
$$c'(e^*) = \frac{\beta}{1+\delta}. \quad (2)$$

2. In order for the function $-c(e) + \beta e/(1+\delta)$ to be concave, the second derivative should be negative for all values of e. The first derivative is $-c'(e) + \beta/(1+\delta)$ and the second derivative is $-c''(e)$. Given that, by assumption, $c''(e) > 0$ for all e, the concavity condition is satisfied. This implies that the solution e^* to the first-order-condition (1) will be the global maximum.

3. One period before, the discounted utility is

$$0-\beta\frac{1}{1+\delta}c\left(e\right)+\beta\frac{1}{\left(1+\delta\right)^{2}}e$$

After dividing by $\beta/\left(1+\delta\right)$,(this is a monotonic transformation that represents the same preferences) we obtain the maximization problem

$$\max_{e}-c\left(e\right)+\frac{1}{1+\delta}e$$

The first order condition is

$$-c'\left(e_{C}^{*}\right)+\frac{1}{1+\delta}=0$$

or

$$c'\left(e_{C}^{*}\right)=\frac{1}{\left(1+\delta\right)}. \tag{3}$$

4. We compare (2) and (3). The right-hand-side of (2) is lower since $\beta<1$. Therefore, $c'\left(e^{*}\right)<c'\left(e_{C}^{*}\right)$ follows. Since $c'\left(\right)$ is an increasing function, this implies $e^{*}<e_{C}^{*}$. Without a commitment device, Arnold exercises too little. This is because the short-run impatience β (capturing the self-control problems) kicks in.

5. For $\beta=1$, the agent has no self-control problems, and $e^{*}=e_{C}^{*}$, as can be easily checked comparing (2) and (3).

6. In order to equate the level of attendance $e^{*\prime}$ to the level of attendance e_{C}^{*} with commitment device, we want to equate the maximization problem

$$\max_{e}-c\left(e\right)-pe+\frac{\beta}{1+\delta}e$$

to the problem

$$\max_{e}-c\left(e\right)+\frac{1}{1+\delta}e.$$

We therefore need

$$-c\left(e\right)-pe+\frac{\beta}{1+\delta}e=-c\left(e\right)+\frac{1}{1+\delta}e$$

or

$$p=-\frac{1-\beta}{1+\delta}.$$

This price has to be negative, and more so the more the self-control parameter β deviates from 1. Arnold needs to be incentivised to attend the gym. If he gets paid for each attendance (that is, p is negative), Arnold will attend the optimal amounts of times. The payment per attendance works as a commitment device.

Problem 2. Production in two locations. (57 points) In this exercise, we consider a farm harvesting papayas y in two locations. Notoriously, papaya harvesting requires no capital, so the production function involves only labor L. Papayas sell at a price $p > 0$.

1. Consider now just the first location. In this location there is ample availability of unskilled workers. The production function is therefore linear in the number of workers: $y = AL$, where L is the number of workers and A is the productivity of each worker. Assume that the wage of a worker is w. Assume also $L \geq 0$, and $A > 0$. Solve the cost minimization problem of a farm that wants to produce y papayas in the first location, that is, determine $L_1^* (w, y|A)$ and the cost function $c_1 (w, y|A)$. (6 points)

2. Solve for marginal cost $c'_{y1} (w, y|A)$ and average cost $c_1 (w, y|A) / y$. Still assuming that only the first location operates, graph and write out the supply function $y_1^S (p, w|A)$. (6 points)

3. Consider now the second location in isolation. In this second location the very first workers are very capable, but the productivity of the workers declines steeply. The production function is $y = L^{1/3}$, where L is the number of workers. Assume that the wage of a worker is w (same as above). Assume also $L \geq 0$. Solve the cost minimization problem of a farm that wants to produce y papayas in this second location, that is, determine $L_2^* (w, y)$ and the cost function $c_2 (w, y)$. (5 points)

4. Solve for marginal cost $c'_{y2} (w, y)$ and average cost $c_2 (w, y) / y$ in this second location. Assuming that only this second location operates, graph and write out the supply function $y_2^S (p, w)$. (5 points)

5. Now the company decides that it is more efficient to operate the two locations together. In particular, the farm minimizes the total cost from operating the two locations $c_1 (w, y_1|A) + c_2 (w, y_2)$, subject to producing a total production y of papayas, where $y = y_1 + y_2$. Set up the problem and solve for the cost-minimizing $y_1^* (p, w, y)$ and $y_2^* (p, w, y)$. That is, find how much a given y will be produced in one location and how much in another location. Assume $y > (1/3A)^{1/2}$ (10 points)

6. Compute $\partial y_1^* (p, w, y) / \partial y$ and $\partial y_2^* (p, w, y) / \partial y$. Use these derivatives to provide intuition on how the overall production of y is divided into the two locations. (Keep assuming $y > (1/3A)^{1/2}$) (6 points)

7. Characterize the solution for $y_1^* (p, w, y)$ and $y_2^* (p, w, y)$ in the case $y < (1/3A)^{1/2}$. (Hint: It is a corner solution) (5 points)

8. (Harder) Use what you did in the previous points to derive the overall cost function for the firm, that is, $c^* (p, w, y)$, where the firm optimally allocates the quantity produced between the two locations. If you cannot do it analytically, try graphically. Provide intuition. (10 points)

9. Even if you were not able to solve point 8 analytically, comment on how using the two locations allows the firm to reduce costs relative to using esclusively one or the other. (4 points)

Solution of Problem 2.

1. The firm uses only labor and the cost minimization becomes

$$\min_L wL$$
$$s.t.\ AL \geq y.$$

   We already solved this in class. We know that the constraint will be binding and the firm will use just about enough labor to produce y. Therefore, $L_1^* (w, y|A)$ is the solution of $AL_1^* (w, y|A) = y$, or $L_1^* (w, y|A) = y/A$. It follows that the cost function $c_1(w, y)$ is $wL_1^* (w, y|A) = wy/A$.

2. The marginal cost function is $c'_{y1} (w, y|A) = w/A$, which also equals the average cost function $c_1 (w, y|A) / y$. The supply function will equal the marginal cost function as long as this function is above the average

cost. Since the marginal cost equals the average cost here, we do not need to worry about the latter condition. The supply function $y^S(p, w|A)$ is

$$y^S(p, w|A) = \begin{cases} y \to +\infty & \text{if} \quad p > w/A \\ \text{any } y \in [0, +\infty) & \text{if} \quad p = w/A \\ 0 & \text{if} \quad p < w/A \end{cases}$$

3. In the second location the firm solves

$$\min_L wL$$
$$s.t.\ L^{1/3} \geq y.$$

As in point 1, the firm will use just about enough labor to produce y. Therefore, $L_2^*(w, y)$ is the solution of $L_2^*(w, y)^{1/3} = y$, or $L_2^*(w, y) = y^3$. It follows that the cost function $c_2(w, y)$ is $wL_2^*(w, y) = wy^3$.

4. The marginal cost function is $c'_{y2}(w, y) = 3wy^2$, and the average cost function is $c_2(w, y)/y = wy^2$. The supply function will equal the marginal cost function as long as this function is above the average cost. Since the marginal cost is always higher than the average cost here, we do not need to worry about the latter condition. The supply function $y_2^S(p, w)$ is obtained by setting $p = c'_{y2}(w, y) = 3wy^2$ and therefore

$$y_2^S(p, w) = (p/3w)^{1/2}.$$

5. The problem is

$$\min_{y_1, y_2} c_1(w, y_1|A) + c_2(w, y_2)$$
$$s.t.\ y_1 + y_2 = y.$$

We can substitute in $y_2 = y - y_1$ from the constraint and get

$$\min_{y_1} c_1(w, y_1|A) + c_2(w, y - y_1) = \frac{wy_1}{A} + w(y - y_1)^3$$

which leads to the first order condition:

$$\frac{w}{A} - 3w(y - y_1)^2 = 0$$

or

$$y_1^*(p, w, y) = y - \left(\frac{1}{3A}\right)^{1/2} \tag{4}$$

Using $y_2 = y - y_1^*$, we get

$$y_2^*(p, w, y) = y - y_1^*(p, w, y) = \left(\frac{1}{3A}\right)^{1/2}. \tag{5}$$

Assuming $y > (1/3A)^{1/2}$, expressions (4) and (5) are well-defined.

6. Using (4) and (5), we can compute

$$\frac{\partial y_1^*(p, w, y)}{\partial y} = 1$$

and

$$\frac{\partial y_2^*(p, w, y)}{\partial y} = 0.$$

The firm produces the first units always in location 2 because the first units of labor in location 2 are very productive. However, the marginal productivity in location 2 decreases quickly in the number of units produced there, so the firm stops producing in location 2 once it hits $y = (1/3A)^{1/2}$. At this point it becomes more profitable to produce in location 1, where all the rest of the output is produced. So, every additional unit of output being produced is produced in the first location, the one with constant marginal cost.

7. In the case $y < (1/3A)^{1/2}$, the solution for (4) involves producing nothing in the first location, that is, $y_1^*(p,w,y) = 0$. It follows from point 4 that $y_2^*(p,w,y) = y$.

8. The overall cost function is

$$c(w,y|\alpha) = \begin{cases} \frac{w}{A}\left(y - \left(\frac{1}{3A}\right)^{1/2}\right) + w\left(\frac{1}{3A}\right)^{3/2} = w\left[\frac{1}{A}y - 2\left(\frac{1}{3A}\right)^{3/2}\right] & \text{if} \quad y > (1/3A)^{1/2} \\ wy^3 & \text{if} \quad y < (1/3A)^{1/2} \end{cases}$$

Notice that for $y < (1/3A)^{1/2}$ the company is producing only in the second location, while for $y > (1/3A)^{1/2}$ the company is producing in both locations.

9. By using the two locations, the firm is able to produce more efficiently. For $y < (1/3A)^{1/2}$ the farm uses the second location, which has the lower costs initially. As soon as y gets larger than $(1/3A)^{1/2}$, however, the costs of firm 2 have grown larger than the cost of growing the additional papaya in firm 1. The farm therefore grows all the additional papayas in location 1 which has a constant marginal cost of growing.