Econ 101A – Problem Set 6 Solutions
Due on Monday Dec. 9. No late Problem Sets accepted, sorry!

This Problem set tests the knowledge that you accumulated mainly in lectures 24 to 26. The problem set is focused on dynamic games and general equilibrium. General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

Problem 1. Dynamic Games (48 points) Two companies produce the same good. In the first period, firm1 seels its product as a monopolist on the West Coast. In the second period, firm 1 competes with firm 2 on the East Coast as a Cournot duopolist. There is no discounting between the two periods. Firm 1 produces quantity $x_W \leq c/\alpha$ at the West at cost cx_W. On the East Coast, and that's what makes this problem interesting, firm 1 produces quantity x_E at cost $(c - \alpha x_W)\, x_E$, where $0 < \alpha < c < 1/2$. The parameter α captures a form of learning by doing. The more firm 1 produces on the West Coast, the lower the marginal costs are going to be on the East Coast. As for firm 2, it produces in the East market with cost cx_2. The inverse demand functions are $p_W(x_W) = 1 - x_W$ and $p_E(x_E, x_2) = 1 - x_E - x_2$. Each firm maximizes profit. In particular, firm 1 maximizes the total profits from its West and East coast operations.

1. Consider first the case of **simultaneous choice**. Assume that firm 2 does not observe x_W before making its production decision. This means that, although formally firm 1 chooses output x_W first, that you should analyze the game as a symultaneous game between firm 1 and firm 2. Use Nash Equilibrium. Write down the profit function that firm 1 maximizes (careful here) and the profit function that firm 2 maximizes (5 points)
2. Write down the first order conditions of firm 1 with respect to x_W and x_E, and the first order condition of firm 2 with respect to x_2. Solve for x_W^*, x_E^*, and x_2^*. (4 points)
3. Check the second order conditions for firm 1 and for firm 2. (3 points)
4. What is the comparative statics of x_W^* and x_E^* with respect to α? Does it make sense? How about the comparative statics of x_2^* with respect to α? (4 points)
5. Compute the profits of firm 2 in equilibrium. How do they vary as α varies? (compute the comparative statics) Why are firm 2's profits affected by α even though the parameter α does not directly affect the costs of firm 2? (5 points)
6. Now consider the case of **sequential choice**. Assume that firm 2 observes x_W before making its production decision x_2. This means that you should analyze the game as a dynamic game between firm 1 and firm 2, and use the concept of subgame-perfect equilibrium. Remember, we start from the last period. Write down the profit functions that firm 1 and firm 2 maximize on the East Coast (4 points)
7. Write down the first order conditions of firm 1 with respect to x_E, and the first order condition of firm 2 with respect to x_2. Solve for x_E^* and x_2^* as a function of x_W^* (4 points)
8. Compute the comparative statics of x_E^* and x_2^* with respect to x_W^*. Do these results make sense? (3 points)
9. Compute the profits of firm 1 on the East Coast as a function of x_W^*. (2 points)
10. Using the answer to point 9, write down the maximization problem of firm 1 in the first period, that it, when it decides the production on the West Coast. (3 points)
11. Write down the first order conditions of firm 1 with respect to x_W. Solve for x_W^* and then, using the solution for x_W^*, find the solution for x_E^* and x_2^*. (5 points)

12. Compare the solutions for x_W^* under simultaneous and under sequential choice. What can you conclude? Under which conditions the firm does more preemption, that is, produces more on the West Coast in order to reduce the production in equilibrium of firm 2? (6 points)

Solution to Problem 1.

1. Firm 1 maximizes the sum of the profits on the East and West coast, that is, the maximization program is

$$\max_{x_W, x_E} x_W (1 - x_W) - c x_W + x_E (1 - x_E - x_2) - (c - \alpha x_W) x_E.$$

Firm 2 simply maximizes the profits on the East Coast:

$$\max x_2 (1 - x_E - x_2) - c x_2.$$

2. The first order conditions of firm 1 with respect to x_W and x_E are

$$1 - 2x_W - c + \alpha x_E = 0 \tag{1}$$

and

$$1 - 2x_E - x_2 - (c - \alpha x_W) = 0. \tag{2}$$

The first order condition of firm 2 with respect to x_2 is

$$1 - x_E - 2x_2 - c = 0 \tag{3}$$

From equation (1) we obtain $x_W^* = (1 - c + \alpha x_E^*)/2$ which we can substitute into (2) to obtain $1 - 2x_E^* - x_2^* - c + \alpha (1 - c + \alpha x_E^*)/2 = 0$ or

$$\left(2 - \alpha^2/2\right) x_E^* = 1 - x_2^* - c + \alpha (1 - c)/2. \tag{4}$$

From (3) we obtain

$$x_E^* = 1 - 2x_2^* - c$$

which we can substitue into (4) to get

$$\left(2 - \alpha^2/2\right)(1 - 2x_2^* - c) = 1 - x_2^* - c + \alpha (1 - c)/2$$

or

$$x_2^* \left(3 - \alpha^2\right) = -\left(-1 + \alpha^2/2 + \alpha/2\right)(1 - c)$$

or

$$x_2^* = \frac{\left(2 - \alpha^2 - \alpha\right)(1 - c)}{6 - 2\alpha^2}. \tag{5}$$

We can use (3) to obtain

$$x_E^* = (1 - c) - 2x_2^* = \frac{\left[\left(6 - 2\alpha^2\right) - 2\left(2 - \alpha^2 - \alpha\right)\right](1 - c)}{6 - 2\alpha^2} = \frac{(2 + 2\alpha)(1 - c)}{6 - 2\alpha^2}. \tag{6}$$

Finally, from (1) we get

$$x_W^* = (1 - c)/2 + \alpha x_E/2 = \frac{\left[\left(3 - \alpha^2\right) + \alpha (1 + \alpha)\right](1 - c)}{6 - 2\alpha^2} = \frac{(3 + \alpha)(1 - c)}{6 - 2\alpha^2} \tag{7}$$

3. The Hessian matrix for firm 1 is

$$H = \begin{bmatrix} -2 & \alpha \\ \alpha & -2 \end{bmatrix}$$

where the first minor is -1 and the determinant is $4 - \alpha^2$ which is positive since $\alpha < 2$. As for firm 2, the second derivative of the profit function with respect to x_2 is $-2 < 0$. The second order conditions are satisfied.

4. From expressions (6) and (7) it is clear that both x_E^* and x_W^* are increasing in α: the higher the degree of learning by doing, the higher the production by firm 1 on the Eas and West market. The learning by doing induces the firm to produce more on the West coast, since this will reduce the costs of producion on the East Coast. Once this high production has taken place, the firm has effecively reduced the costs of producing on the East coast and therefore produces more. As for firm 2, we can differentiate expression (5) to get

$$\begin{aligned}\frac{\partial x_2^*}{\partial \alpha} &= (1-c)\frac{(-2\alpha-1)\left(6-2\alpha^2\right)-\left(2-\alpha^2-\alpha\right)(-4\alpha)}{\left(6-2\alpha^2\right)^2} = \frac{1-c}{\left(6-2\alpha^2\right)^2}\left[-12\alpha-6+4\alpha^3+2\alpha^2+8\alpha-4\alpha^3-4\alpha^2\right] \\ &= \frac{1-c}{\left(6-2\alpha^2\right)^2}\left[-4\alpha-6-2\alpha^2\right]<0.\end{aligned}$$

Therefore, as the degree of learning-by-doing of firm 1 increases, the production of firm 2 decreases. Essentially, by learning by doing, firm 1 decreases the own production costs and pushes firm 2 to lower and lower levels of production.

5. The profits of firm 2 are

$$\begin{aligned}\pi_2 &= \frac{\left(2-\alpha^2-\alpha\right)(1-c)}{6-2\alpha^2}\left(1-\frac{(2+2\alpha)(1-c)}{6-2\alpha^2}-\frac{\left(2-\alpha^2-\alpha\right)(1-c)}{6-2\alpha^2}\right)-c\frac{\left(2-\alpha^2-\alpha\right)(1-c)}{6-2\alpha^2}= \\ &= \frac{\left(2-\alpha^2-\alpha\right)(1-c)}{6-2\alpha^2}\left((1-c)-\frac{(1-c)\left(4+\alpha-\alpha^2\right)}{6-2\alpha^2}\right)=\left[\frac{\left(2-\alpha^2-\alpha\right)(1-c)}{6-2\alpha^2}\right]^2\end{aligned}$$

Since the profits of firm 2 π_2 equal $\left(x_2^*\right)^2$, it is clear that $\partial\pi_2/\partial\alpha$ has the same sign as $\partial x_2^*/\partial\alpha$, which is negative. As the learning by doing of firm 1 increases, the profits of firm 2 decrease. This is since the increased production of firm 1 induces firm 2 to produce less and reduces its profits. Interestingly, this occurs despite firm 2 not observing the decisions of firm 1 before it takes its own decisions. It all happens through equilibrium arguments.

6. In the sequential version, the firms produce on the East Coast after observing production x_W^* on the West Coast. The profit funcion of firm 1 in period 2 is

$$x_E\left(1-x_E-x_2\right)-\left(c-\alpha x_W^*\right)x_E$$

and the profit function of firm 2 is

$$x_2\left(1-x_E-x_2\right)-cx_2.$$

7. The first order conditions are

$$1-2x_E^*-x_2^*-c+\alpha x_W^*=0$$

and

$$1-x_E^*-2x_2^*-c=0.$$

Solving for x_2^* in the first equation we get $x_2^*=1-2x_E^*-c+\alpha x_W^*$ which we can substitute in the second expression to get

$$1-x_E^*-2\left(1-2x_E^*-c+\alpha x_W^*\right)-c=0$$

or

$$x_E^*\left(x_W^*\right)=\frac{1-c+2\alpha x_W^*}{3}.$$

We can then obtain

$$x_2^*\left(x_W^*\right)=\frac{1-c}{2}-\frac{1-c+2\alpha x_W^*}{6}=\frac{(1-c)-\alpha x_W^*}{3}$$

8. Notice that x_E^* is increasing in x_W^* since higher past production decreases currect marginal costs for firm 1. As for firm 2, x_2^* is decreasing in x_W^* since higher α implies that firm 1 will produce more on the East coast.

9. The profits for firm 1 are

$$\begin{aligned}\pi_1\left(x_W^*\right) &= x_E^*\left(1-x_E^*-x_2^*-\left(c-\alpha x_W^*\right)\right)=\frac{1-c+2\alpha x_W^*}{3}\left(1-c+\alpha x_W^*-\frac{1-c+2\alpha x_W^*}{3}-\frac{(1-c)-\alpha x_W^*}{3}\right)= \\ &= \frac{1-c+2\alpha x_W^*}{3}\left(1-c+\alpha x_W^*-\frac{2(1-c)+\alpha x_W^*}{3}\right)=\frac{\left(1-c+2\alpha x_W^*\right)^2}{9}\end{aligned}$$

10. The profit function in period 1 for firm 1 is

$$x_W\left(1-x_w\right)-cx_W+\frac{\left(1-c+2\alpha x_W\right)^2}{9}$$

11. The first order condition with respect to x_W is

$$1-2x_W^*-c+\frac{2}{9}\left(1-c+2\alpha x_W^*\right)2\alpha=0$$

or

$$x_W^*\left(2-\frac{8}{9}\alpha^2\right)=(1-c)\left(1+\frac{4}{9}\alpha\right)$$

or

$$x_W^*=(1-c)\left(9+4\alpha\right)/\left(18-8\alpha^2\right)$$

We can then plug x_W^* into $x_E^*\left(x_W^*\right)$ to obtain

$$x_E^*\left(x_W^*\right)=\frac{1-c+2\alpha\left(1-c\right)\left(9+4\alpha\right)/\left(18-8\alpha^2\right)}{3}.$$

Similarly, we get

$$x_2^*\left(x_W^*\right)=\frac{(1-c)-\alpha\left(1-c\right)\left(9+4\alpha\right)/\left(18-8\alpha^2\right)}{3}$$

12. The production on the West coast under simultaneous production is

$$x_W^*=\frac{(3+\alpha)(1-c)}{6-2\alpha^2}$$

whereas in the sequential version we obtain

$$x_W^*=(1-c)\left(9+4\alpha\right)/\left(18-8\alpha^2\right).$$

The production in the sequential game is higher than in the simultaneous game if

$$(1-c)\left(9+4\alpha\right)/\left(18-8\alpha^2\right)\geq\frac{(3+\alpha)(1-c)}{6-2\alpha^2}$$

or

$$(9+4\alpha)\left(6-2\alpha^2\right)\geq(3+\alpha)\left(18-8\alpha^2\right)$$

or

$$54-18\alpha^2+24\alpha-8\alpha^3\geq54-24\alpha^2+18\alpha-8\alpha^3$$

or

$$6\alpha^2+6\alpha=6\alpha\left(1+\alpha\right)\geq0$$

or $\alpha\geq0$. That is, as long as there is learning by doing ($\alpha>0$), firm 1 produces more on the West coast if the game is sequential than if it is simultaneous. In other words, if firm 2 can observe x_W^* then, as in a standard Stackelberg duopoly, the leader gets to produce more and earn more profits. I realize that computing the profits here would be very cumbersone (sorry), but i am ready to bet 10:1 that firm 1 overall makes more profits in the sequential than in the simultaneous case. :)

Problem 2. General Equilibrium (32 points) Consider the case of pure exchange with two consumers. Both consumers have Cobb-Douglas preferences, but with different parameters. Consumer 1 has utility function $u(x_1^1, x_2^1) = (x_1^1)^{\alpha}(x_2^1)^{1-\alpha}$. Consumer 2 has utility function $u(x_1^2, x_2^2) = (x_1^2)^{\beta}(x_2^2)^{1-\beta}$. The endowment of good j owned by consumer i is ω_j^i. The price of good 1 is p_1, while the price of good 2 is normalized to 1 without loss of generality.

1. Only for point 1, assume $\omega_1^1 = 1, \omega_1^2 = 3, \omega_2^1 = 3, \omega_2^2 = 1$. (that is, total endowment of each good is 4). Assume further $\alpha = 1/2$, $\beta = 1/2$. Draw the Pareto set and the contract curve for this economy in an Edgeworth box. (you do not need to give the exact solutions, only a graphical representation) What is the set of points that could be the outcome under barter in this economy? (5 points)

2. For each consumer, compute the utility maximization problem. Solve for x_j^{i*} for $j = 1, 2$ and $i = 1, 2$ as a function of the price p_1 and of the endowments. [This problem to be solved with closed eyes!] (5 points)

3. Assume again $\omega_1^1 = 1, \omega_1^2 = 3, \omega_2^1 = 3, \omega_2^2 = 1$ and $\alpha = 1/2$, $\beta = 1/2$. Do a qualitative plot of the offer curve for consumer 1. [Trick to do this is to compute the values of x_1^{1*} and x_2^{1*} as you increase p_1 from 0] What happens to the consumption of good 1 and good 2 as the price p_1 increases? Plot also the offer curve of consumer 2. Graphically, find the intersection, the general equilibrium point. (7 points)

4. We now solve analytically for the general equilibrium. Require that the total sum of the demands for good 1 equals the total sum of the endowments, that is, that $x_1^{1*} + x_1^{2*} = \omega_1^1 + \omega_1^2$. Solve for the general equilibrium price p_1^*. (6 points)

5. What is the comparative statics of p_1^* with respect to the endowment of good 1, that is, with respect to ω_1^i for $i = 1, 2$? What about with respect to the endowment of the other good? Does this make sense? What is the comparative statics of p_1^* with respect to the taste for good 1, that is, with respect to α and β? Does this make sense? (4 points)

6. Now require the same general equilibrium condition in market 2. Solve for p_1^* again, and check that this solution is the same as the one you found in the point above. In other words, you found a property that is called Walras'Law.In an economy with n markets, if $n - 1$ markets are in equilibrium, the nth market will be in equilibrium as well. (5 points)

Solution to Problem 2.

1. See Figure.

2. Instead of solving the Lagrangean problem (make sure you know how to do this), i will use a general feature of Cobb-Douglas utility functions. The consumer consumes share α of income in the first commodity. This implies
$$x_1^{1*} = \alpha \frac{\left(p_1\omega_1^1 + \omega_2^1\right)}{p_1}$$
and
$$x_2^{1*} = (1-\alpha)\left(p_1\omega_1^1 + \omega_2^1\right).$$
For consumer 2
$$x_1^{2*} = \beta \frac{\left(p_1\omega_1^2 + \omega_2^2\right)}{p_1}$$
and
$$x_2^{1*} = (1-\beta)\left(p_1\omega_1^2 + \omega_2^2\right).$$

3. See Figure.

4. We have already imposed the first part of the definition of Walrasian Equilibrium, that is, that each consumer chooses the optimal allocation given the price p_1. We now impose the second part of the definition, that is, that each market be in equilibrium. We first impose the condition for the first market, that is, $x_1^{1*} + x_1^{2*} = \omega_1^1 + \omega_1^2$. This implies

$$\alpha \frac{\left(p_1^* \omega_1^1 + \omega_2^1\right)}{p_1^*} + \beta \frac{\left(p_1^* \omega_1^2 + \omega_2^2\right)}{p_1^*} = \omega_1^1 + \omega_1^2$$

or

$$\alpha \omega_1^1 + \beta \omega_1^2 + \frac{\left(\alpha \omega_2^1 + \beta \omega_2^2\right)}{p_1^*} = \omega_1^1 + \omega_1^2$$

or

$$\left(\alpha \omega_2^1 + \beta \omega_2^2\right) = p_1^* \left[(1-\alpha)\, \omega_1^1 + (1-\beta)\, \omega_1^2\right]$$

or

$$p_1^* = \left(\alpha \omega_2^1 + \beta \omega_2^2\right) / \left[(1-\alpha)\, \omega_1^1 + (1-\beta)\, \omega_1^2\right].$$

5. Notice that, as the endowment of good 1 ω_1^1 or ω_1^2 increases, the price of good 1 decreases. this makes sense. If the quantity available of good 1 increases, while holding constant the quantity of good 2, good 2 becomes relatively scarcer and this induces the relative price of good 1 to decrease. This is like an increase in supply that decreases the price. Notice that p_1 is not the absolute price of good 1, but rather the relative price of good 1 relative to the price of good 2, which we normalized to 1. Simmetrically, if the endowment of good 2 increases, good 1 becomes scarcer and the price p_1^* increases. As for the other comparative statics, as the taste for good 1 increases (increase in α or β) the equilibrium price of good 1 goes up. Since increase in taste for good 1 means a positive shift in demand, we are not surprised that this induces an increase in the equilibrium price of good 1. Notice a difference, though, between the general equilibrium increase in the demand and the partial equilibrium. If both α and β increase, that is, if both consumers like good 1 more, the price of good 1 will go up, but the quantity consumed of good 1 will go down (or al least not go up) for one of the consumers. In this seting, the ttal quantity available of each good is constant.

6. We now impose the condition for the second market, that is, $x_2^{1*} + x_2^{2*} = \omega_2^1 + \omega_2^2$. This implies

$$(1-\alpha)\left(p_1^* \omega_1^1 + \omega_2^1\right) + (1-\beta)\left(p_1^* \omega_1^2 + \omega_2^2\right) = \omega_2^1 + \omega_2^2$$

or

$$\left[(1-\alpha)\, \omega_1^1 + (1-\beta)\, \omega_1^2\right] p_1^* = \left(\alpha \omega_2^1 + \beta \omega_2^2\right)$$

or

$$p_1^* = \left(\alpha \omega_2^1 + \beta \omega_2^2\right) / \left[(1-\alpha)\, \omega_1^1 + (1-\beta)\, \omega_1^2\right]$$

which is the same solution that we found before. If there are n markets, it's enough to impose equilibrium conditions in $n-1$ of them, and the n-th market will automatically be in equilibrium.

Problem 3. Moral Hazard (46 points, due to Botond Koszegi). We analyze here a principal-agent problem with hidden action (moral hazard). The principal is hiring an agent. The agent can put high effort e_H or low effort e_L. If the agent puts high effort e_H, the output is $y_H = 18$ with probability 3/4 and $y_L = 1$ with probability 1/4. If the agent puts low effort e_L, the output is $y_H = 18$ with probability 1/4 and $y_L = 1$ with probability 3/4. The principal decides the pay of the agent w. The utility of the agent is $\sqrt{w} - c(e)$, where $c(e_H) = .1$ and $c(e_L) = 0$. The reservation utility of the agent is .1. The principal maximizes expected profits.

1. Assume first that the principal can observe the effort of the agent (that is, there is no hidden action). We want to determine the optimal contract. In this case, the principal pays a wage $w(e)$ that can depend on the effort. The way you solve a problem like this with a discrete number of effort levels is to study first the case in which the principal wants to implement high effort e_H. In this case, the wage will be w if the agent chooses e_H and 0 otherwise. Solve the problem

$$\max_w \frac{3}{4}18 + \frac{1}{4}1 - w$$
$$s.t. \sqrt{w} - c(e_H) \geq .1$$

   Argue that the constraint is satisfied with equality and solve for w^*. Compute the expected profit in this case $E\pi_H$. (5 points)

2. We are still in the case of no hidden action. Assume that the principal wants to implement low effort e_L. Similarly to above, set up the maximization problem and solve for w^*. Compute the expected profit in this case $E\pi_L$ and compare with $E\pi_H$. Which profit level is higher? The higher one is the contract chosen by the principal and hence the action implemented. (5 points)

3. Now consider the case with hidden action. The wages can only be a function of the outcomes: w_H when $y = y_H$ and w_L when $y = y_L$. We study this case in two steps. Assume first that the principle wants to implement e_H. We study the optimal behavior of the agent after signing the contract. Write the inequality that indicates under what condition the agent prefers action e_H to action e_L. (This is a function of w_H and w_L and goes under the name of incentive compatibility constraint). (5 points)

4. Now consider the condition under which the agent prefers the contract offered to the reservation utility .1. (This is the individual rationality constraint) (4 points)

5. Argue that the two inequalities you just derived will be satisfied with equality. Solve the two equations to derive w_L^* and w_H^*. Compute the profits for the principal from implementing the high action under hidden action: $E\pi_H^{HA}$. (5 points)

6. Now, assume that the principal wants the agent to take the low action e_L. In this case, you do not need to worry that the agent will deviate to the high action, since that will take more effort. The principal will pay a flat wage w. Write the individual rationality constraint for the agent if he takes the action e_L and the pay is w. Set this constraint to equality (Why?) and derive w^*. Compute the implied profits for the principal from implementing the low action under hidden action: $E\pi_L^{HA}$.(5 points)

7. Compare the profits in point 6 (profits from low effort) and in point 7 (profits from low effort). Under hidden action, what contract does the principal choose to implement, the one that guarantees high effort or the one that guarantees low effort? (4 points)

8. Compare the profits for the principal under perfect observability and under hidden action. Why are they different despite the fact that the action chosen by the agent in equilibrium will be the same? (6 points)

9. There is a monitoring system that allows the principal to perfectly observe the actions (and hence to implement the perfect observability contract). How much would the principal be willing to pay for it? (3 points)

10. Compare the utility of the agent under perfect observability and under hidden action. (4 points)

Solution to Problem 3.

1. In the problem

$$\begin{aligned} & \max_w \frac{3}{4}18+\frac{1}{4}1-w \\ s.t. \sqrt{w}-c\left(e_H\right) &\geq .1 \end{aligned}$$

   the principal would be foolish to pay a salary w higher than the one that satisfies the constraint with equality, since that would lover expected profits. The solution for w hence is $\sqrt{w}-.1=.1$, which implies $w=4/100$. This implies an expected profit of $E\pi=55/4-4/100$.

2. In this case, we follow the same logic, but the individual rationality constraint is now $\sqrt{w}-c\left(e_L\right)=.1$, or $\sqrt{w}=.1$, which implies $w=1/100$. This implies an expected profit of $E\pi=\frac{1}{4}18+\frac{3}{4}1-w=21/4-1/100$.

3. In the case of hidden action, the principal cannot enforce a particular action. S/he can only pay as a function of the observe output y_L or y_H. The agent prefers the action e_H to the action e_L if

$$\frac{3}{4}\sqrt{w_H}+\frac{1}{4}\sqrt{w_L}-c\left(e_H\right)\geq\frac{1}{4}\sqrt{w_H}+\frac{3}{4}\sqrt{w_L}-c\left(e_L\right).$$

   If this condition holds, the expected utility of putting high effort e_H is higher than the expected utility of putting low effort e_L. Hence, the agent will choose to put high effort.

4. The agent prefers to put high effort to the reservation utility if

$$\frac{3}{4}\sqrt{w_H}+\frac{1}{4}\sqrt{w_L}-c\left(e_H\right)\geq .1.$$

5. The individual rationality inequality will be satisfied with equlity because otherwise the principal could lower the pay under w_L and increase profits. (Notice that this would make the first constraint more likely to be satisfied, so the argument is kosher) As for the first constraint, the incentive compatibility one, if the constraint were not satisfied with equality, the principal could increase the profits by lowering w_H and increasing w_L so as to keep the individual rationality constraint satisfied. This would increase profits because of the risk-aversion. Given that the agents are risk-averse, they value more the increase in w_L than the decrease in w_H (since in the contract it will be the case that $w_H>w_L$). This is not a full proof, but it tries to convey the intuition. If this is hard to comprehend, do not worry too much. The proof of this part is beyond the level required in this class. Moving on, now that we know that the constraints are satisfied with equality, we can solve for w_L^* and w_H^*. The system is

$$\begin{aligned} \frac{3}{4}\sqrt{w_H}+\frac{1}{4}\sqrt{w_L}-c\left(e_H\right) &= \frac{1}{4}\sqrt{w_H}+\frac{3}{4}\sqrt{w_L}-c\left(e_L\right) \\ \frac{3}{4}\sqrt{w_H}+\frac{1}{4}\sqrt{w_L}-c\left(e_H\right) &= .1 \end{aligned}$$

   The equations reduce to

$$\begin{aligned} \frac{1}{2}\sqrt{w_H} &= \frac{1}{2}\sqrt{w_L}+.1 \\ \frac{3}{4}\sqrt{w_H}+\frac{1}{4}\sqrt{w_L} &= .2 \end{aligned}$$

   Substituting $\sqrt{w_H}$ into the second equation gives

$$\frac{3}{4}\left(\sqrt{w_L}+.2\right)+\frac{1}{4}\sqrt{w_L}=.2$$

   or $\sqrt{w_L}=1/20$, or $w_L^*=1/400$. It follows that $w_H^*=25/400$. Hence, the expected profits are

$$E\pi=\frac{55}{4}-\frac{3}{4}\frac{25}{400}-\frac{1}{4}\frac{1}{400}=\frac{55}{4}-\frac{19}{400}.$$

6. If the principal wants to implement the low action, she will pay a constant wage for the high and low output given the risk-aversion of the agents. The individual rationality constraint is

$$\frac{1}{4}\sqrt{w}+\frac{3}{4}\sqrt{w}-c\left(e_L\right)\geq .1$$

The inequality is satisfied with equality because otherwise the principal could lower the wage w and increase the profits. It follows that $\sqrt{w^*}=.1$, or $w^*=1/100$. The implied profits are

$$E\pi=\frac{1}{4}18+\frac{3}{4}1-\frac{1}{100}=\frac{21}{4}-\frac{1}{100}.$$

7. The profits from implementing the high effort $(\frac{55}{4}-\frac{19}{400})$ are substantially higher than the profits from implementing the low effort $(\frac{21}{4}-\frac{1}{100})$. Hence, the principal chooses to implement the contract with high effort.

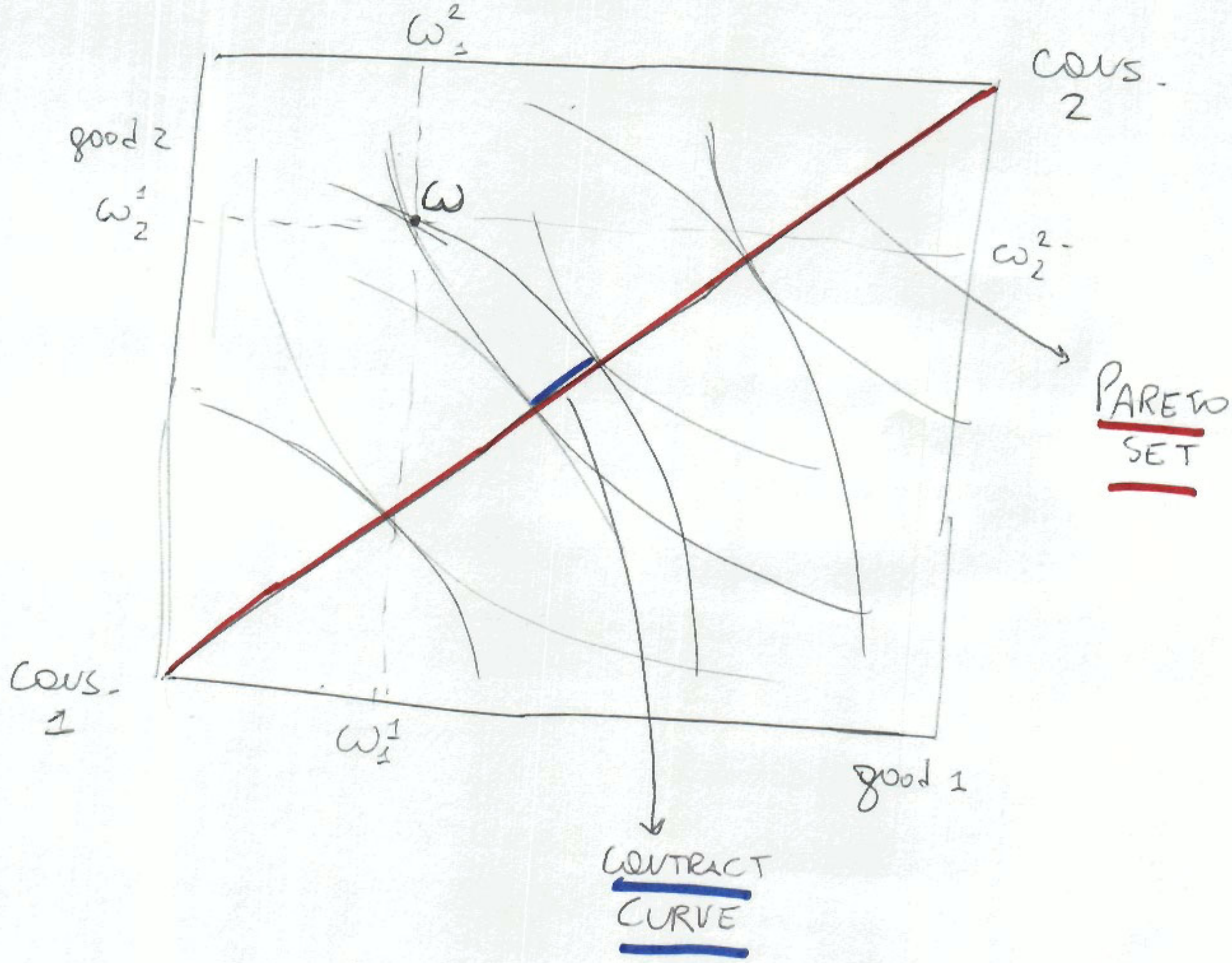
8. It is interesting that both under perfect observability and under hidden action the principal implements the high action e_H. That is, in equilibrium, the agent will choose e_H in both cases. However, the profits are higher in the case of perfect observability. Compare $\frac{55}{4}-\frac{4}{100}$and $\frac{55}{4}-\frac{19}{400}$. The difference is due to the fact that under hidden action the principal has to introduce different wages in case of high outcome and low outcome, paying the high outcome more, in order to make sure that the agent puts high effort. Since the agent is risk-averse, she does not like this, and is less willing to take the contract, which in turn reduces the profits of the firm.

9. The principal would be willing to pay

$$\frac{55}{4}-\frac{4}{100}-\left(\frac{55}{4}-\frac{19}{400}\right)=\frac{3}{400}.$$

10. Notice that in all cases the agent is offered a contract that makes her indifferent relative to the outside option. Therefore, her utility is always .1. This is because the principal here is a monopolist which can extract all the surplus.

Problem 2.1



ω_1^2
Cons. 2
good 2
ω
ω_2^1
ω_2^2
Pareto set
Cons. 1
ω_1^1
good 1
Contract curve