

### Econ 101A – Problem Set 4

Due in class on Th 3 April. No late Problem Sets accepted, sorry!

This Problem set tests the knowledge that you accumulated mainly in lectures 14 to 17. The problem set is focused on production. General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

**Problem 1. Production: cost minimization and profit maximization** (11 points) In class we introduced two different characterizations of firm decisions: cost minimization and profit maximization. In this exercise you are asked to show that the two optimization problems are equivalent ways of solving the same problem. Consider a mango farm in the Philippines. The farm has a production function  $y = f(L, K, F)$ , where  $L$  is labor,  $K$  is capital, and  $F$  is farm land. The price of labor is  $w$ , the price of capital is  $r$ , the price of land is  $s$ , with all of the prices positive. Assume also  $f'_i > 0$  and  $f''_{i,i} < 0$ , with  $i = L, K, F$ .

1. Write down the first step of the cost minimization problem and the first order conditions with respect to  $L$ ,  $K$ , and  $F$ . (2 points)
2. The first order conditions in step 1, together with the constraint  $y = f(L, K, F)$ , allow us to solve for  $L^*(w, r, s, y)$ ,  $K^*(w, r, s, y)$  and  $F^*(w, r, s, y)$ . We can therefore write the cost function  $c(w, r, s, y) = wL^*(w, r, s, y) + rK^*(w, r, s, y) + sF^*(w, r, s, y)$ . In the second step of the cost minimization, the firm maximizes  $py - c(w, r, s, y)$ . Write down the first order conditions with respect to  $y$ . (1 point)
3. Now, instead of going down the cost-minimization path, we go down the one-stage profit-maximization path. Write down the profit-maximization problem and the first order conditions with respect to  $L$ ,  $K$ , and  $F$ . (2 points)
4. Your task now is to show that the four first order conditions in the cost minimization (from points 1 and 2) are equivalent to the three first order condition from the profit maximization (from point 3). In other words, profit maximization and cost minimization are two different ways to do the same thing. To do this, you will need to use the envelope theorem to find an alternative expression for  $\partial c(w, r, s, y) / \partial y$ . [Hint:  $c(w, r, s, y)$  is the objective function of the minimization program  $\min wL + rK + sF$  s.t.  $f(L, K, F) - y = 0$ .] (6 points)

**Problem 2. Production: perfect competition** (42 points) In this exercise, we characterize the short-run solution to the Philippine farmers problem for the case of perfect competition. We assume that the firm has the production function  $y = AL^\alpha K^\beta F^\gamma$ . In the short-run, however, the quantity of land farmed is fixed to  $\bar{F}$ , so there effectively are only two factors of production with respect to which the firm maximizes.

1. Write down the cost minimization problem with respect to  $L$  and  $K$  and the first order conditions with respect to  $L$  and  $K$  (2 points)
2. Solve for  $L^*(w, r, s, y, \bar{F})$  and  $K^*(w, r, s, y, \bar{F})$ . (3 points)
3. How does  $L^*$  vary as  $w$  increases? Compute  $\partial L^* / \partial w$  using the solution in point 2. Does it make sense? How about if there is technological progress and  $A$  increases? What happens to  $L^*$ ? (3 points)
4. Write down the cost function  $c(w, r, s, y, \bar{F})$  and derive the expressions for the marginal cost  $c'_y(w, r, s, y, \bar{F})$  and the average cost  $c(w, r, s, y, \bar{F}) / y$ . (4 points)
5. Assume  $\alpha + \beta < 1$  and plot the marginal cost  $c'_y(w, r, s, y, \bar{F})$  and the average cost  $c(w, r, s, y, \bar{F}) / y$  as a function of  $y$ . [I am interested in the shape of the curves, not in exact drawings]. Draw the supply curve for the firm, that is,  $y = y^*(w, r, s, \bar{F})$ . Is the supply curve (weakly) increasing in  $p$ ? (5 points)

6. Assume  $\alpha + \beta = 1$  and plot the marginal cost  $c'_y(w, r, s, y, \bar{F})$  and the average cost  $c(w, r, s, y, \bar{F})/y$  as a function of  $y$ . How does the supply curve for the firm look like? Write it down analytically. Why does the firm never want to produce a positive and finite quantity of output despite constant returns to scale in labor and capital? (5 points)
7. Assume  $\alpha + \beta > 1$  and plot the marginal cost  $c'_y(w, r, s, y, \bar{F})$  and the average cost  $c(w, r, s, y, \bar{F})/y$  as a function of  $y$ . How does the supply curve for the firm look like? (3 points)
8. We now consider market supply and demand for the only well-behaved case, that is, for  $\alpha + \beta < 1$ . First, let's look at aggregation of the supply side. There are  $J$  producers with production function as above. Each one of them, therefore, has a supply function as at point 5. Assuming that we are on the increasing portion of the supply curve, we can write the equation for the supply function as

$$\frac{(y^j)^{\frac{1-(\alpha+\beta)}{\alpha+\beta}}}{\alpha+\beta} D = p \quad (1)$$

where

$$D = \left( \frac{1}{A(\bar{F})^\gamma} \right)^{\frac{1}{\alpha+\beta}} \left[ w \left( \frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} + r \left( \frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} \right].$$

We can invert expression (1) to get

$$y^j = \left[ \frac{(\alpha + \beta)p}{D} \right]^{\frac{\alpha+\beta}{1-(\alpha+\beta)}}$$

This is the firm supply function provided that the price  $p$  is high enough (we are not checking this, enough algebra!). Now, your turn. Aggregate over the  $J$  firms to generate the industry supply function  $Y^S$ . (2 points)

9. Now, the consumer side. Assume that consumers have Cobb-Douglas preferences for mangoes (good  $y$  with price  $p$ ) and all the other goods (good  $m$  with price 1). Their utility function is

$$u(y, m) = y^\alpha m^{1-\alpha}.$$

There are  $J$  consumers, each with income  $M_j$ ,  $j = 1, \dots, J$ . From the many exercises we did on Cobb-Douglas utility it follows that in equilibrium

$$y_j^* = \alpha \frac{M_j}{p}.$$

This is the individual demand function for good  $y$ . Compute now the aggregate demand function  $Y^D$  for the whole economy. Notice that in this special case aggregate demand does not depend on the distribution of income. (3 points)

10. Finally, the market equilibrium! Equate the demand  $Y^D$  and supply  $Y^S$  to solve for the equilibrium price  $p_M$ . What is  $p_M$ ? Plug back the value of  $p_M$  into your expression for  $Y^D$  to get the market level of production  $Y$  (4 points). You should get the following solutions

$$p_M = \left[ \alpha \frac{\sum_{j=1}^J M_j}{J} \right]^{1-(\alpha+\beta)} \frac{(\alpha + \beta)^{-(\alpha+\beta)}}{A(\bar{F})^\gamma} \left[ w \left( \frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} + r \left( \frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} \right]^{\alpha+\beta}$$

and

$$Y = \left[ \alpha \sum_{j=1}^J M_j \right]^{\alpha+\beta} J^{1-(\alpha+\beta)} A(\bar{F})^\gamma (\alpha + \beta)^{\alpha+\beta} \left[ w \left( \frac{\beta w}{\alpha r} \right)^{-\frac{\beta}{\alpha+\beta}} + r \left( \frac{\beta w}{\alpha r} \right)^{\frac{\alpha}{\alpha+\beta}} \right]^{-(\alpha+\beta)}$$

11. Determine now the sign of the following changes on  $p_M$  and  $Y$ . Provide intuition. Also, make sure that the results square with your graphical intuition about moving supply and demand function. (8 points)
- an increase in total income  $\sum_{j=1}^J M_j$ ;
  - an increase in productivity  $A$ .