Econ 101A – Problem Set 3
Due in class on Tu 21 October. No late Problem Sets accepted, sorry!

This Problem set tests the knowledge that you accumulated mainly in lectures 12 to 14, but it builds on the work of the previous weeks. It is focused on choice under uncertainy and time-inconsistency. General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

Problem 1. Relative and Absolute Risk aversion (6 points) In class we introduced the concepts of relative and absolute risk aversion, but we have not used them. This exercise introduces you to two useful classes of utility functions.

1. Consider the exponential utiliy function $-\exp(-\rho c)$. Show that it is increasing ($u' > 0$) and concave ($u'' < 0$) for all c as long as $\rho > 0$, that is, as long as the agent is risk-averse. Show that this function has constant absolute risk aversion coefficient r_A given by ρ. (2 points)
2. Consider the power utiliy function $\frac{c^{1-\rho}}{1-\rho}$ for $\rho \neq 1$. Show that it is increasing ($u' > 0$) and concave ($u'' < 0$) for all $c > 0$. Show that this function has constant relative risk aversion coefficient r_R given by ρ. (2 points)
3. Consider the log utility function $\ln(c)$. Show that it is increasing ($u' > 0$) and concave ($u'' < 0$) for all $c > 0$. Show that this function has constant relative risk aversion coefficient r_R equal to 1. (in fact, it is possibile to show $\lim_{\rho\to1} \frac{c^{1-\rho}-1}{1-\rho} = \ln(c)$ – you are not required to prove this) (2 points).

Solution to Problem 1.

1. $u'(c) = \rho\exp(-\rho c) > 0$ and $u''(c) = -\rho^2\exp(-\rho c) < 0$. The coefficient of absolute risk aversion is
$$r_A = -u''(c)/u'(c) = -\left[-\rho^2\exp(-\rho c)/\rho\exp(-\rho c)\right] = \rho$$
2. $u'(c) = c^{-\rho} > 0$ and $u''(c) = -\rho c^{-\rho-1} < 0$. The coefficient of relative risk aversion is
$$r_R = -u''(c)\,c/u'(c) = -\left[-\rho c^{-\rho-1} * c/c^{-\rho}\right] = \rho$$
3. $u'(c) = 1/c > 0$ and $u''(c) = -1/c^2 < 0$. The coefficient of relative risk aversion is
$$r_R = -u''(c)/u'(c) = -\left[\left(-c/c^2\right)/(1/c)\right] = 1.$$

   For those interested in the proof that $\lim_{\rho\to1} \frac{c^{1-\rho}-1}{1-\rho} = \ln(c)$. When you try to compute $\lim_{\rho\to1} \frac{c^{1-\rho}-1}{1-\rho}$, you realize that you get a 0/0 kind of indeterminacy. We can apply De L'Hopital Theorem. We differentiate both numerator and denominator with respect to ρ. We get
$$\lim_{\rho\to1}\frac{c^{1-\rho}-1}{1-\rho} = \lim_{\rho\to1}\frac{\partial\left(c^{1-\rho}-1\right)/\partial c}{\partial(1-\rho)/\partial c} = \lim_{\rho\to1}\frac{-c^{1-\rho}\ln c}{-1} = \ln c\lim_{\rho\to1}c^{1-\rho} = \ln c.$$

Problem 2. Investment in Risky Asset (26 points) We consider here a standard problem of investment in risky assets, similar to the one that we covered in class. The agent can invest in bonds or stocks. Bonds have a return $r > 0$. (in class we asumed $r = 0$) Stocks have a stochastic return, $r_+ > r$ with probability p, and $r_- < r$ with probability $1 - p$. In expectations, the stocks outperform bonds, that is, $pr_+ + (1-p) r_- > r$. The agent has income w and utility function u, with $u'(x) > 0$ and $u''(x) < 0$ for all x. The agents wants to decide the optimal share α of his wealth to invest in stocks. The agent maximizes

$$\begin{aligned}
&\max_{\alpha} (1-p)\, u\left(w\left[(1-\alpha)(1+r) + \alpha(1+r_-)\right]\right) + \\
&+ pu\left(w\left[(1-\alpha)(1+r) + \alpha(1+r_+)\right]\right) \\
&s.t. 0 \leq \alpha \leq 1
\end{aligned}$$

or, after some semplification,

$$\begin{aligned}
&\max_{\alpha} (1-p)\, u\left(w\left[1+r+\alpha(r_- - r)\right]\right) + pu\left(w\left[1+r+\alpha(r_+ - r)\right]\right) \\
&s.t. 0 \leq \alpha \leq 1
\end{aligned}$$

1. Assume that the solution is interior and write down the first order conditions for this problem with respect to α. (1 point)

2. Write down the second order condition. Is it satisfied? (3 points)

3. Use the first order conditions to derive the comparative statics of α^* with respect to w. Use the implicit function theorem to write down $\partial\alpha^*/\partial w$. (this is a long expression – sorry!) (4 points)

4. What is the sign of the denominator? You have checked this already. Where? (3 points)

5. Argue that, given your answer to point 4, the sign of $\partial\alpha^*/\partial w$ is given by the sign of the numerator. Simplify the numerator using the first order conditions. Once you do this simplification, you should get the following expression for the numerator:

$$\begin{aligned}
&(1-p)\, w (r_- - r) \left[1+r+\alpha(r_- - r)\right] u''\left(w\left[1+r+\alpha(r_- - r)\right]\right) + \\
&+ pw (r_+ - r) \left[1+r+\alpha(r_+ - r)\right] u''\left(w\left[1+r+\alpha(r_+ - r)\right]\right).
\end{aligned}$$

   (4 points) Now, let me do one piece of the argument for you. We are interested in the sign of this expression, since it coincides with the sign of $\partial\alpha^*/\partial w$. We can rewrite it as

$$\begin{aligned}
&(1-p)(r_- - r)\, u'\left(w\left[1+r+\alpha(r_- - r)\right]\right) \left\{ \frac{u''\left(w\left[1+r+\alpha(r_- - r)\right]\right)}{u'\left(w\left[1+r+\alpha(r_- - r)\right]\right)} w\left[1+r+\alpha(r_- - r)\right] \right\} \quad (1) \\
&+ p(r_+ - r)\, u'\left(w\left[1+r+\alpha(r_- - r)\right]\right) \left\{ \frac{u''\left(w\left[1+r+\alpha(r_+ - r)\right]\right)}{u'\left(w\left[1+r+\alpha(r_+ - r)\right]\right)} w\left[1+r+\alpha(r_+ - r)\right] \right\}.
\end{aligned}$$

   All we did was to multiply and divide by $u'\left(w\left[1+r+\alpha(r_- - r)\right]\right)$ in the first half of the expression and by $u'\left(w\left[1+r+\alpha(r_+ - r)\right]\right)$ in the second half.

6. Your turn again. What are the expressions in curly brackets? They should be familiar to you. Show that for a power utility function $\frac{c^{1-\rho}}{1-\rho}$ the two expressions in curly brackets are both equal to $-\rho$ (you can use point 2 in the previous problem). Using this nice result, rewrite expression (1) substituting the two expressions in curly brackets with $-\rho$ (4 points)

7. Consider the simplified expression (1) where you substituted $-\rho$ for the curly brackets. Argue, using the first order conditions, that the resulting expression is in fact equal to zero! Now, if you go back and look at the steps of this exercise, you will realize that you have proven the following important result: With power utility function, the ratio of wealth invested in stocks (α) is independent of wealth w, i.e., $\partial\alpha/\partial w = 0$. Therefore, the model predicts that individuals earning \$20,000 should invest the same *fraction* of their earnings in stocks as individuals earning \$100,000. (3 points)

8. How would you test the above prediction? What would you expect to find? (4 points)

Solution to Problem 2.

1. The first order condition with respect to α is

$$0 = (1-p)\, w\, (r_- - r)\, u'\left(w\left[1 + r + \alpha\,(r_- - r)\right]\right) + pw\,(r_+ - r)\, u'\left(w\left[1 + r + \alpha\,(r_+ - r)\right]\right).$$

2. The second order condition with respect to α is

$$(1-p)\, w^2\, (r_- - r)^2\, u''\left(w\left[1 + r + \alpha\,(r_- - r)\right]\right) + pw^2\,(r_+ - r)^2\, u''\left(w\left[1 + r + \alpha\,(r_+ - r)\right]\right)$$

which is negative given the assumption $u''(x) < 0$ for all x.

3. Using the implicit function theorem on the first-order conditions, we get

$$\begin{aligned}
\partial\alpha^*/\partial w &= -\frac{\partial\,(\text{first order condition})\,/\partial w}{\partial\,(\text{first order condition})\,/\partial\alpha} = \qquad (2)\\
&= -\frac{(1-p)\,(r_- - r)\, u'\left(w\left[1 + r + \alpha\,(r_- - r)\right]\right) + p\,(r_+ - r)\, u'\left(w\left[1 + r + \alpha\,(r_+ - r)\right]\right)}{(1-p)\, w^2\, (r_- - r)^2\, u''\left(1 + w\left[r + \alpha\,(r_- - r)\right]\right) + pw^2\,(r_+ - r)^2\, u''\left(w\left[1 + r + \alpha\,(r_+ - r)\right]\right)}\\
&\quad -\frac{(1-p)\, w\, (r_- - r)\left[1 + r + \alpha\,(r_- - r)\right] u''\left(w\left[1 + r + \alpha\,(r_- - r)\right]\right)}{(1-p)\, w^2\, (r_- - r)^2\, u''\left(w\left[1 + r + \alpha\,(r_- - r)\right]\right) + pw^2\,(r_+ - r)^2\, u''\left(w\left[1 + r + \alpha\,(r_+ - r)\right]\right)}\\
&\quad +\frac{pw\,(r_+ - r)\left[1 + r + \alpha\,(r_+ - r)\right] u''\left(w\left[1 + r + \alpha\,(r_+ - r)\right]\right)}{(1-p)\, w^2\, (r_- - r)^2\, u''\left(w\left[1 + r + \alpha\,(r_- - r)\right]\right) + pw^2\,(r_+ - r)^2\, u''\left(w\left[1 + r + \alpha\,(r_+ - r)\right]\right)}
\end{aligned}$$

4. The denominator is the derivative of the first order condition with respect to α, and therefore equals the second order conditions that we derived at point 2. We saw already that the second order condition is satisfied since the expression is negative.

5. Since the denominator is negative, and there is a minus of front of the fraction in expression (2), the sign of $\partial\alpha^*/\partial w$ is given by the sign of the numerator. We can simplify the first part of the numerator once we realize that that part is identical to the first order conditions divided by w. But since the first order condition is equal to 0, this part is zero as well. We get the following expression for the numerator

$$\begin{aligned}
&(1-p)\, w\, (r_- - r)\left[1 + r + \alpha\,(r_- - r)\right] u''\left(w\left[1 + r + \alpha\,(r_- - r)\right]\right) +\\
&+pw\,(r_+ - r)\left[1 + r + \alpha\,(r_+ - r)\right] u''\left(w\left[1 + r + \alpha\,(r_+ - r)\right]\right).
\end{aligned}$$

6. The expressions in curly brackets are the negative of the coefficient of relative risk aversion computed, respectively, for $c = w\left[1 + r + \alpha\,(r_+ - r)\right]$ and for $c = w\left[1 + r + \alpha\,(r_- - r)\right]$! Therefore, for a power utility function they both equal $-\rho$. This is great because we can simplify the ugly expression (1) by plugging in $-\rho$ and get

$$(1-p)\,(r_- - r)\, u'\left(w\left[1 + r + \alpha\,(r_- - r)\right]\right)(-\rho) + p\,(r_+ - r)\, u'\left(w\left[1 + r + \alpha\,(r_- - r)\right]\right)(-\rho). \qquad (3)$$

7. Expression (3) equals the first order conditions multiplied by ρ/w. Therefore, since the expression in the first order conditions is zero, this expression must be zero as well. This proves that $\partial\alpha^*/\partial w = 0$ with power utility function.

8. The prediction that the fraction of wealth invested in stocks should be constant at differente levels of wealth is testable. If we can observe income and financial decisions for different groups of individuals, we can compute the ratio α and see if it is independent of wealth. The stylized fact from this type of studies is that the fraction of wealth invested in stocks increases as wealth increases.

Problem 3. Time inconsistent preferences. (35 points) In this exercise, we reconsider the topic of choice over time, with the twist that consumers have time-inconsistent preferences, as introduced in lecture 14. We assume three periods, $t = 0$, $t = 1$, and $t = 2$. We will call this time-inconsistent agent Tim. To make things simpler, assume that Tim only receives income in period 0, that is, $M_0 > 0$, $M_1 = M_2 = 0$. He earns per-period interest r on each dollar saved. We denote M_1' the income saved from period 1, i.e, $M_1' = (1+r)(M_0 - c_0)$. Similarly, $M_2' = (1+r)(M_1' - c_1)$. We assume that in period t Tim has utility function

$$u(c_t, c_{t+1}, c_{t+2}) = \ln(c_t) + \frac{\beta}{1+\delta}\ln(c_{t+1}) + \beta\left(\frac{1}{1+\delta}\right)^2 \ln(c_{t+2}).$$

To make things clearer, imagine that c is ice cream, and that Tim has an immediate gratification problems with ice cream. If he can consume ice cream, he will eat too much, and leave too little income saved for the future. This is what the case $\beta < 1$ captures.

1. In this sort of intertemporal problems, you need to start from the last period and work backward. In period 2 Tim receives M_2' in income. How much ice cream will Tim consume in period 2? [Remember, period 2 is the last period, any ice cream that the agent does not consume in the last period is wasted. Therefore, the agent maximizes $\ln(c_2)$ s.t. $c_2 \leq M_2'$.] (1 point)

2. Let us now go back to period 1. In period 1 Tim has income M_1' and has to decide how much ice cream to consume, and how much money to save for period 2. Argue that this leads to the budget constraint

$$c_1 + \frac{c_2}{1+r} \leq M_1'.$$

   (3 points)

3. Now that we have derived the budget constraint, consider the maximization problem of Tim in period 1:

$$\begin{aligned} &\max_{c_1, c_2} \ln(c_1) + \frac{\beta}{1+\delta}\ln(c_2) \\ &s.t.\ c_1 + \frac{c_2}{1+r} \leq M_1'. \end{aligned} \tag{4}$$

   In this case, the easiest way to solve the problem is to solve for c_2 in the budget constraint (which is satisfied with equality), plug it into the objective function, and then maximize the objective function with respect to c_2. Once you find the solution for c_2^*, use the budget constraint to obtain c_1^*. If you prefer, you can alternatively use the Lagrangean system. You will get the same result, if you do the calculations right! What are the solutions for c_1^* and c_2^* as a function of M_1', r, δ, and β? (5 points)

4. We now consider several features of this solution. Are you surprised that c_1^* is independent of r? What does this tell you about the strength of the income and substitution effect? Explain in words the income and substitution effects of a change in r on c_1^*. (no math here) (4 points)

5. What is the effect on c_1^* and c_2^* of an increase in impatience δ? Is it reasonable? (3 points)

6. What is the effect on c_1^* and c_2^* of an increase in β? Remember that higher β is associated with less time-inconsistency, i.e., less taste for immediate gratification? Does it make sense that qualitatively an increase in δ has the same effects as a decrease in β? (4 points)

7. Now we go back to period 0. Suppose that Tim, at time 0, could decide already the ice cream consumption of the future selves. In other words, he has a commitment device: for example, he may ask his friends at time 0 to perpetually make fun of him if he consumes more than a predetermined level of ice cream. What quantity of consumption would Tim decide for periods 1 and 2 as a function of M_1'? Here is how we solve this problem. Consider the utility function at time 0:

$$\ln(c_0) + \frac{\beta}{1+\delta}\ln(c_1) + \beta\left(\frac{1}{1+\delta}\right)^2 \ln(c_2).$$

Tim maximizes this utility function subject to the budget constraint $c_1 + \frac{c_2}{1+r} \leq M_1'$. In addition, Tim is taking the choice of c_0 for given, at least for now. The terms with c_0 drop out. The maximization problem therefore is:

$$\max_{c_1,c_2} \frac{\beta}{1+\delta}\ln(c_1) + \beta\left(\frac{1}{1+\delta}\right)^2 \ln(c_2)$$
$$s.t.\ c_1 + \frac{c_2}{1+r} \leq M_1'.$$

Notice the similarity to the maximization problem in (4). As in point 3, solve for c_2 in the budget constraint (which is satisfied with equality), and plug it into the objective function, and then maximize the objective function with respect to c_2. We label the solution for c_2 $c_2^{*,c}$, that is the level of c_2 chosen with commitment. Once you find the solution for $c_2^{*,c}$, use the budget constraint to obtain $c_1^{*,c}$. What are the solutions for $c_1^{*,c}$ and $c_2^{*,c}$ as a function of M_1', r, δ, and β? (5 points)

8. This is the key part of the exercise. You should now compare the solutions to point 7 and the solutions to point 3. Are they equal? No! They are different precisely because of the time inconsistency. Show that, however, they coincide ($c_1^* = c_1^{*,c}$) for $\beta = 1$. That is, when there is no time inconsistency ($\beta = 1$), the solutions with and without commitment are the same. (3 points)
9. Show that $c_1^{*,c} < c_1^*$. Why is this the case? (3 points)
10. Argue, formally or informally, that Tim at time 0 is happier with commitment (that is, with $c_1^{*,c}$ and $c_2^{*,c}$) than without commitment (with c_1^* and c_2^*). (4 points)

Solution to Problem 3.

1. In the last period Tim consumes all the income, that is, she consumes $c_2^* = M_2'$. This follows from the fact that the utility function $\ln(c)$ is increasing in c.
2. The savings of Tim in period 1 ($M_1' - c_1$) earn interest r. Therefore, in period 2 Tim will be able to consumer $c_2 \leq (1+r)(M_1' - c_1)$. If we divide this by $(1+r)$, we obtain $c_2/(1+r) \leq M_1' - c_1$, which we can easlity transform into $c_1 + \frac{c_2}{1+r} \leq M_1'$ by adding c_1 on both sides.
3. From the budget constraint, which is statisfied with equality since the individual will consume all the income available, we get $c_2 = (1+r)(M_1' - c_1)$. If we substitute $c_2 = (1+r)(M_1' - c_1)$ into the objective function, we get $\ln(c_1) + \frac{\beta}{1+\delta}\ln((1+r)(M_1' - c_1))$. The first order condition with respect to c_1 is

$$\frac{1}{c_1^*} - \frac{\beta}{1+\delta}\frac{(1+r)}{(1+r)(M_1' - c_1^*)} = 0$$

or

$$\frac{1}{c_1^*} = \frac{\beta}{1+\delta}\frac{1}{(M_1' - c_1^*)}$$

or

$$M_1' - c_1^* = \frac{\beta}{1+\delta}c_1^*$$

which gives

$$c_1^* = \frac{1+\delta}{1+\delta+\beta}M_1'. \tag{5}$$

We can then obtain c_2^* using $c_2^* = (1+r)(M_1' - c_1^*)$ from the budget constraint:

$$c_2^* = (1+r)\left(\frac{\beta}{1+\delta+\beta}M_1'\right). \tag{6}$$

4. The solution for c_1^* is independent or r since the substitution effect and the income effect exactly equal and offset each other. (This is a feature typical of the log function, but you could not have known this!) The substitution effect of an increase in r is negative. As r increases, Tim substitutes away from consuming today into saving since the returns to saving went up. The income effect says that the increase in the interest rate makes the individual richer, since Tim is saving ($c_1^* < M_1'$). For a normal good (like in this case), higher income means higher consumption, hence higher c_1^*.

5. We can easily compute $\partial c_1^*/\partial\delta$ and $\partial c_2^*/\partial\delta$ using expressions (5) and (6):
$$\frac{\partial c_1^*}{\partial \delta} = \frac{(1+\delta+\beta)-(1+\delta)}{(1+\delta+\beta)^2} M_1' = \frac{\beta}{(1+\delta+\beta)^2} M_1' > 0$$
and
$$\frac{\partial c_2^*}{\partial \delta} = -\frac{\beta}{(1+\delta+\beta)^2} (1+r)\, M_1' < 0$$
The more impatient Tim gets (the higher δ), the higher the consumption in the first period and the lower the consumption in the second period.

6. We can easily compute $\partial c_1^*/\partial\beta$ and $\partial c_2^*/\partial\beta$ using expressions (5) and (6):
$$\frac{\partial c_1^*}{\partial \beta} = -\frac{1+\delta}{(1+\delta+\beta)^2} M_1' < 0$$
and
$$\frac{\partial c_2^*}{\partial \beta} = \frac{(1+\delta+\beta)-\beta}{(1+\delta+\beta)^2} (1+r)\, M_1' = \frac{1+\delta}{(1+\delta+\beta)^2} (1+r)\, M_1' > 0$$
The more time-inconsistent Tim gets (the lower β), the higher the consumption in the first period and the lower the consumption in the second period. In this sense, lower β and higher δ are about the same. As you will see, the difference comes later!

7. From the budget constraint, which is statisfied with equality since the individual will consume all the income available, we get $c_2 = (1+r)\,(M_1' - c_1)$. If we substitute $c_2 = (1+r)\,(M_1' - c_1)$ into the objective function, we get $\frac{\beta}{1+\delta}\ln(c_1) + \frac{\beta}{(1+\delta)^2}\ln((1+r)\,(M_1'-c_1))$. The first order condition with respect to c_1 is
$$\frac{\beta}{1+\delta}\frac{1}{c_1^{*,c}} - \frac{\beta}{(1+\delta)^2}\frac{(1+r)}{(1+r)\left(M_1' - c_1^{*,c}\right)} = 0$$
or
$$\frac{1}{c_1^{*,c}} = \frac{1}{1+\delta}\frac{1}{\left(M_1' - c_1^{*,c}\right)}$$
or
$$M_1' - c_1^{*,c} = \frac{1}{1+\delta} c_1^{*,c}$$
which gives
$$c_1^{*,c} = \frac{1+\delta}{1+\delta+1} M_1'.$$
We can then obtain $c_2^{*,c}$ using $c_2^{*,c} = (1+r)\left(M_1' - c_1^{*,c}\right)$ from the budget constraint:
$$c_2^{*,c} = (1+r)\left(\frac{1}{1+\delta+1} M_1'\right).$$

8. Notice that
$$c_1^{*,c} = \frac{1+\delta}{1+\delta+1} M_1' \neq c_1^* = \frac{1+\delta}{1+\delta+\beta} M_1'.$$
As a consequence,
$$c_2^{*,c} = (1+r)\left(\frac{1}{1+\delta+1} M_1'\right) \neq c_2^* = (1+r)\left(\frac{\beta}{1+\delta+\beta} M_1'\right).$$

This is exactly the time inconsistency: Tim has different preferences at time 0 and at time 1 and therefore makes different choices at time 0 and at time 1. However, the choices with and without commitment are equal if $\beta = 1$. The case $\beta = 1$ is the case with no time inconsistency, since there no drive for immediate gratification.

9. To prove

$$c_1^{*,c} = \frac{1+\delta}{1+\delta+1} M_1' < c_1^* = \frac{1+\delta}{1+\delta+\beta} M_1'$$

just notice that c_1^* is decreasing in β, and that $c_1^* = c_1^{*,c}$ for $\beta = 1$. The intuition is the following. Time has an urge for immediate gratification in period 1 and, if unconstrained, he will consume too much. When he can resort to a commitment device, he manages to control his future self, so that in period 1 he will consume less ice cream.

10. Tom at time 0 is happier with commitment (that is, with $c_1^{*,c}$ and $c_2^{*,c}$) than without commitment (with c_1^* and c_2^*). Remember, with commitment Tom can control his future consumption of ice cream, and he avoids overeating. Formally, observe that the levels $c_1^{*,c}$ and $c_2^{*,c}$ are obtained maximizing the utility function of Tim at time 0, and therefore must yield the highest possible utility.