Econ 101A – Problem Set 1 Solution

Problem 1. Univariate unconstrained maximization. (10 points) Consider the following maximization problem:

$$\max_x f(x; x_0) = \exp(-(x - x_0)^2)$$

- 1. Write down the first order conditions for this problem with respect to x (notice that x_0 is a parameter, you should not maximize with respect to it). (1 point)
  2. Solve explicitely for x^* that satisfies the first order conditions. (1 point)
  3. Compute the second order conditions. Is the stationary point that you found in point 2 a maximum? Why (or why not)? (2 points)
  4. As a comparative statics exercise, compute the change in x^* as x_0 varies. In other words, compute dx^*/dx_0. (2 points)
  5. We are interested in how the value function $f(x^*(x_0); x_0)$ varies as x_0 varies. We do it two ways. First, plug in $x^*(x_0)$ from point 2 and then take the derivative with respect to x_0. Second, use the envelope theorem. You should get the same result! (2 points)
  6. Is the function f concave in x? (2 points)

Solution to Problem 1:

1. The first order condition with respect to x is $df(x; x_0)/dx = 0$. Plugging in the function in question we obtain:

$$-2(x - x_0) \cdot exp(-(x - x_0)^2) = 0$$

2. Solving the first order condition yields: $x^* = x_0$.

3. Computing the second order conditions, we find that:

$$d^2 f(x; x_0)/dx^2 = (-2 + 4(x - x_0)^2) exp(-(x - x_0)^2)$$

   Plugging in $x^* = x_0$ we find that the second derivative is negative, a sufficient condition for x^* to be a local maximum. Furthermore, since $x^* = x_0$ is the only point that satisfies the first order conditions, it follows that it is the *only* local maximum. Since the function $exp(-(x - x_0)^2)$ tends to zero as x tends to either plus or minus infinity, it follows that the point $x^* = x_0$ is a global maximum.

4. Since $x^* = x_0,\ dx^*/dx_0 = 1$.

5. Method 1: Plug the formula for x^* into the original function to obtain $f(x^*; x_0) = exp(-(x_0 - x_0)^2) = exp(0) = 1$. Hence, $df(x^*, x_0)/dx_0 = 0$.

   Method 2: By the envelope theorem, we know that:

$$df(x^*; x_0)/dx_0 = \partial f(x; x_0)/\partial x_0|_{x=x^*}$$

   Computing the latter partial we find that it equals (at $x^* = x_0$): $2(x^* - x_0) exp(-(x^* - x_0)^2) = 0$.

6. No, it is not concave. Recall that the second derivative of $f(x; x_0)$ with respect to x must be non-positive for concavity. However, we found that this derivative was equal to: $[-2 + 4(x - x_0)^2] exp(-(x - x_0)^2)$. Note that whenever $x >> x_0$ the bracketed term is positive.

Problem 2. Multivariate unconstrained maximization. (13 points) Consider the following maximization problem:

$$\max_{x,y} f(x,y;a,b) = ax^2 - x + by^2 - y$$

- 1. Write down the first order conditions for this problem with respect to x and y (notice that a and b are parameters, you do not need to maximize with respect to them). (1 point)
  2. Solve explicitely for x^* and y^* that satisfy the first order conditions. (1 point)
  3. Compute the second order conditions. Under what conditions for a and b is the stationary point that you found in point 2 a maximum? (2 points)
  4. Asume that the conditions for a and b that you found in point 3 are met. As a comparative statics exercise, compute the change in y^* as a varies. In other words, compute dy^*/da. Compute it both directly using the solution that you obtained in point 2 and using the general method presented in class that makes use of the determinant of the Hessian. The two results should coincide! (3 points)
  5. We are interested in how the value function $f(x^*(a,b);y^*(a,b))$ varies as a varies. We do it two ways. First, plug in $x^*(a,b)$ and $y^*(a,b)$ from point 2 into f and then take the derivative of $f(x^*(a,b);y^*(a,b))$ with respect to a. Second, use the envelope theorem. You should get the same result! Which method is faster? (3 points)
  6. Under what conditions on a and b is the function f concave in x and y? When is it convex in x and y? (3 points)

Solution to Problem 2:

1. The first order conditions with respect to a and b are, respectively:
   (i) $2ax - 1 = 0$
   (ii) $2by - 1 = 0$

2. $x^* = 1/2a$ and $y^* = 1/2b$.

3. We first compute the Hessian matrix:
$$\begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix}$$
   The second order conditions for a local maximum are that $2a < 0$ and $4ab > 0$, where $4ab$ is the determinant of the Hessian. These inequalities imply that the function assumes a local maximum at $(x^*,\ y^*)$ whenever $a < 0$ and $b < 0$. Furthermore, since this is the only point that satisfies the first order condition, it follows that it is the only local maximum. Lastly, the fact that $a, b < 0$ implies that as either x or y tend to infinity, the function becomes increasingly negative. It follows that whenever $a, b < 0$, the function obtains a global maximum at (x^*, y^*).

4. Method 1: We found that $y^* = 1/2b$ (a, b are assumed to be less than zero). Therefore, $dy^*/da = 0$.

   Method 2: Assume $a, b < 0$. By the implicit function theorem, we find that:
$$dy^*/da = -\frac{det\begin{bmatrix} 2a & 2x^* \\ 0 & 0 \end{bmatrix}}{det\begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix}} = 0$$

5. Method 1: Plugging the values of x^* and y^* back into the original function we find: $f(x^*, y^*; a, b) = -1/4a - 1/4b$. Therefore, $\partial f(x^*, y^*; a, b)/\partial a = +1/4a^2 > 0$.

   Method 2: Use the envelope theorem, which tells us that
$$df(x^*, y^*; a, b)/da = \partial(x, y : a, b)/\partial a|_{x=x^*, y=y^*}$$
   Computing the partial and evaluating at $x = x^*, y = y^*$ we get: $(x^*)^2 = (1/2a)^2 = 1/4a^2$.

6. Recall the Hessian matrix for $f(x, y; a, b)$

$$\begin{bmatrix} 2a & 0 \\ 0 & 2b \end{bmatrix}$$

The conditions for concavity are $a \leq 0$ and $4ab \geq 0$. The conditions for convexity are $a \geq 0$ and $4ab \geq 0$.

Problem 3. Multivariate constrained maximization. (19 points) Consider the following maximization problem:

$$\begin{aligned} \max_{x,y} u(x,y) &= x^\alpha y^\beta \\ \text{s.t. } p_x x + p_y y &= M, \end{aligned}$$

with $0 < \alpha < 1$, $0 < \beta < 1$. The problem above is a classical maximization of utility subject to a budget constraint. The utility function $x^\alpha y^\beta$ is also called a Cobb-Douglas utility function. You can interpret p_x as the price of good x and p_y as the price of good y. Finally, M is the total income. I provide these details to motivate this problem. In order to solve the problem, you only need to apply the theory of constrained maximization that we covered in class. But beware, you are going to see a lot more Cobb-Douglas functions in the next few months!

- 1. Write down the Lagrangean function (1 point)
  2. Write down the first order conditions for this problem with respect to x, y, and λ. (1 point)
  3. Solve explicitely for x^* and y^* as a function of p_x, p_y, M, α, and β. (3 points)
  4. Notice that the utility function $x^\alpha y^\beta$ is defined only for $x > 0, y > 0$. Does your solution for x^* and y^* satisfies these constraints? What assumptions you need to make about p_x, p_y and M so that $x^* > 0$ and $y^* > 0$? (1 point)
  5. Write down the bordered Hessian. Compute the determinant of this 3x3 matrix and check that it is positive (this is the condition that you need to check for a constrained maximum) (3 points)
  6. As a comparative statics exercise, compute the change in x^* as p_x varies. In order to do so, use directly the expressions that you obtained in point 3, and differentiate x^* with respect to p_x. Does your result make sense? That is, what happens to the quantity of good x^* consumed as the price of good x increases? (2 points)
  7. Similarly, compute the change in x^* as p_y varies. Does this result make sense? What happens to the quantity of good x^* consumed as the price of good y increases? (2 points)
  8. Finally, compute the change in x^* as M varies. Does this result make sense? What happens to the quantity of good x^* consumed as the total income M increases? (2 points)
  9. We have so far looked at the effect of changes in p_x, p_y, and M on the quantities of goods consumed. We now want to look at the effects on the utility of the consumer at the optimum. Use the envelope theorem to calculate $\partial u(x^*(p_x, p_y, M), y^*(p_x, p_y, M))/\partial p_x$. What happens to utility at the optimum as the price of good x increases? Is this result surprising? (2 points)
  10. Use the envelope theorem to calculate $\partial u(x^*(p_x, p_y, M), y^*(p_x, p_y, M))/\partial M$. What happens to utility at the optimum as total income M increases? Is this result surprising? (2 points)

Solution of Problem 3. In steps:

1. The Lagrangean function is $x^\alpha y^\beta - \lambda(p_x x + p_y y - M)$
2. The first order conditions are

$$\begin{aligned} \frac{\partial L}{\partial x} &= \alpha x^{\alpha-1} y^\beta - \lambda p_x = 0 \\ \frac{\partial L}{\partial y} &= \beta x^\alpha y^{\beta-1} - \lambda p_y = 0 \\ \frac{\partial L}{\partial \lambda} &= p_x x + p_y y - M = 0 \end{aligned}$$

3. We can solve for x^* and y^* as follows. Rewrite the first two equations as

$$\begin{aligned} \alpha x^{\alpha-1} y^\beta / p_x &= \lambda, \\ \beta x^\alpha y^{\beta-1} / p_y &= \lambda. \end{aligned} \tag{1}$$

This implies $\alpha x^{\alpha-1}y^{\beta}/p_x = \beta x^{\alpha}y^{\beta-1}/p_y$. Assuming $x > 0$ and $y > 0$ (see point 4 below), we divide both terms by $x^{\alpha-1}y^{\beta-1}$ and simplify this to $\alpha y/p_x = \beta x/p_y$ or $p_y y = \frac{\beta}{\alpha}p_x x$. Substitute $p_y y$ back into the third first order condition (the constraint) to obtain $p_x x + \frac{\beta}{\alpha}p_x x - M = 0$. This implies $p_x x = \frac{\alpha}{\alpha+\beta}M$ or

$$x^* = \frac{\alpha}{\alpha+\beta}\frac{M}{p_x}. \tag{2}$$

Using the relationship $p_y y = \frac{\beta}{\alpha}p_x x$, it follows that

$$y^* = \frac{\beta}{\alpha+\beta}\frac{M}{p_y}. \tag{3}$$

4. The unique solution for x^* and y^* satisfies $x^* > 0$ and $y^* > 0$ as long as $\alpha > 0$, $\beta > 0$. $M > 0$, $p_x > 0$, and $p_y > 0$. The first two conditions are assumed since the outstart. $M > 0$ implies positive income to be spent, a natural assumption. Positive prices is also a natural assumption (not so many goods are sold for free:)

5. The bordered Hessian is

$$H = \begin{pmatrix} 0 & -p_x & -p_y \\ -p_x & \alpha(\alpha-1)(x^*)^{\alpha-2}y^{*\beta} & \alpha\beta\,(x^*)^{\alpha-1}\,(y^*)^{\beta-1} \\ -p_y & \alpha\beta(x^*)^{\alpha-1}\,(y^*)^{\beta-1} & \beta(\beta-1)(x^*)^{\alpha}\,(y^*)^{\beta-2} \end{pmatrix}$$

and the determinant is

$$\begin{aligned} & +p_x\left(-p_x\beta(\beta-1)(x^*)^{\alpha}\,(y^*)^{\beta-2} + p_y\alpha\beta(x^*)^{\alpha-1}\,(y^*)^{\beta-1}\right) \\ & -p_y\left(-p_x\alpha\beta\,(x^*)^{\alpha-1}\,(y^*)^{\beta-1} + p_y\alpha(\alpha-1)(x^*)^{\alpha-2}y^{*\beta}\right) \\ = \; & x^{\alpha-2}y^{\beta-2}\left(-\beta\,(\beta-1)\,p_x^2\,(x^*)^2 + 2\alpha\beta p_x p_y xy - \alpha\,(\alpha-1)\,p_y^2\,(y^*)^2\right). \end{aligned}$$

Notice that the bordered Hessian does not depend on λ (unlike in the example that we examined in class), and therefore we did not need to solve explicetely for the value of λ^* to answer this question. As for the sign of the determinant, given the assumptions $0 < \alpha < 1$ and $0 < \beta < 1$ it is clear that the determinant is positive. Therefore the stationary point (x^*, y^*) defined by (2) and (3) is a maximum. Notice that in this case we do not even need to substitute the value of x^* and y^* into the determinant. It is enough to know $x^* > 0$, $y^* > 0$.

6. We do the comparative statics using the expressions (2) and (3) for x^* and y^*. $\partial x^*/\partial p_x = -\frac{\alpha}{\alpha+\beta}M/p_x^2 < 0$. This makes sense. As the price of a good increases, the quantity of that good consumed decreases.

7. Once again, we do the comparative statics using the expressions (2) and (3) for x^* and y^* and get $\partial x^*/\partial p_y = 0$. This is a very specific case. It means that good x and y are neither complements nor substitutes. The consumption of good x is not affected by changes in the consumption of good y. We will go back in class to this case. We will say then that in the Cobb-Douglas case the substitution effect perfectly counterbalances the income effect. (Do not worry if you do not understand this now)

8. Finally, using the expressions (2) and (3) for x^* and y^*, we get $\partial x^*/\partial M = \frac{\alpha}{\alpha+\beta}\frac{1}{p_x} > 0$. As the income goes up, the individual can afford more of good x.

9. Using the envelope theorem, we compute $du(x^*(p_x,p_y,M), y^*(p_x,p_y,M))/dp_x = \partial(x^{\alpha}y^{\beta})/\partial p_x - \lambda\partial(p_x x + p_y y - M)/\partial p_x$. Notice that the utility function does not depend directly on p_x, and therefore the first term is zero. We therefore get $du(x^*(p_x,p_y,M), y^*(p_x,p_y,M))/dp_x = -\lambda^* x^*$. We know the expression for x^* and it is clear by equation (1) that $\lambda^* > 0$. Therefore, the effect on utility is negative. This is right. If one good becomes more expensive, the consumer can afford less, and the utility has to go down.

10. Similarly using the envelope theorem, we get $du(x^*(p_x,p_y,M), y^*(p_x,p_y,M))/dM = \partial(x^{\alpha}y^{\beta})/\partial M - \lambda\partial(p_x x + p_y y - M)/\partial M = +\lambda^*$. We know by equation (1) that $\lambda^* > 0$. Therefore, the effect on utility of an increase in income is positive. This is also right. If the individual becomes richer, he or she is able to afford more and is happier. It is interesting to note that λ^* captures exactly the effect of increased income on utility at the optimum.

Problem 4. Rationality of preferences (5 points) Prove the following statements:

- if $\succeq$ is rational, then $\sim$ is transitive, that is, $x \sim y$ and $y \sim z$ implies $x \sim z$ (3 points)
- if $\succeq$ is rational, then $\succeq$ has the reflexive property, that is, $x \succeq x$ for all x. (2 points)

Solution of Problem 4.1. We want to prove that $x \sim y$ and $y \sim z$ implies $x \sim z$ if $\succeq$ is rational. First, remember that $x \sim y$ is equivalent to $x \succeq y$ and $y \succeq x$. Similarly, $y \sim z$ is equivalent to $y \succeq z$ and $z \succeq y$. We therefore know $x \succeq y$ and $y \succeq z$. Using the transitivity of $\succeq$, this implies $x \succeq z$. But we also know $z \succeq y$ and $y \succeq x$. Using again transitivity of $\succeq$, this implies $z \succeq x$. But $x \succeq z$ and $z \succeq x$ is exactly $x \sim z$, which is what we wanted to prove.

Solution of Problem 4.2. We want to prove that for all x in X, $x \succeq x$ if $\succeq$ is rational. The completeness property of $\succeq$ says that for all z in X and y in X, either $z \succeq y$ or $y \succeq z$ or both. Rewrite this property with $z = x$ and $y = z$. Completeness then implies that $x \succeq x$ or $x \succeq x$ for all x in X. This proves the claim.