Endogenous Public Signals and Coordination*

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Abstract

If agents have rich higher order beliefs (multidimensional private signals), then (one dimensional) public signals will not usually generate approximate common knowledge. But in a noisy rational expectations equilibrium, as noise disappears, one dimensional prices reveal a sufficient statistic for two dimensional uncertainty.

1. Introduction

Coordination games famously give rise to multiple equilibria. But Carlsson and van Damme (1993) noted that relaxing the common knowledge of payoffs assumption, by assuming that players observe the true payoffs with a small amount of noise, would remove the multiplicity of equilibria. Allowing private information to be considerably more accurate that public information contained in the prior about payoffs removed common knowledge or approximate common knowledge of payoffs. More generally, uniqueness will arise if there is not "too much public information".

Consider the following simple illustration of this principle. A continuum population must invest or not invest. The return to investing is 0. The cost of investing

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is $c \in (0,1)$. The return to investing is 1 if the proportion investing is at least θ, 0 otherwise. If θ were common knowledge, there would be multiple Nash equilibria (all invest, all not invest) as long as θ were between zero and one. But suppose instead we assume that θ is normally distributed with mean y and precision α, and that each agent i observes a private signal $x_i = \theta + \varepsilon_i$, where the noise terms ε_i are distributed normally in the population with mean 0 and precision β. If and only if

$$\alpha^2 \leq 2\pi\beta, \tag{1.1}$$

there is a unique equilibrium.[1]

A number of commentators have questioned the relevance of the uniqueness results, since in many economic environments where coordination is important, interactions endogenously generate public information that might lead the uniqueness condition to fail.[2] An especially important source of endogenous public information is prices, and papers by Tarashev (2003), Hellwig, Mukherji and Tsyvinski (2005) and Angeletos and Werning (2005) have pursued various methods of combining endogenous public information with the type of coordination game described in the previous paragraph.[3] In particular, Angeletos and Werning (2005) consider a two period model where, in the first period, agents with private information about θ engage in a noisy rational expectations equilibrium of a CARA normal model of the economy of the type pioneered by Grossman (1976); the private information is transformed into public information about θ embodied in the price; in the second period, they engage in a coordination game of the type described in the previous paragraph, depending on the parameter θ. More private information entering the first period leads to more public revelation and thus more public information entering the second period. The net result is that increased

[1] This model and result first appeared in our 1998 working paper on "Coordination Risk and The Price of Debt," eventually published as Morris and Shin (2004). A similar condition on precisions is necessary and sufficient for uniqueness in global games where payoffs are linear in the unknown parameter θ, the class of global games used in our expository pieces, Morris and Shin (1998, 2000, 2003). Related conditions can be derived for more general global games, although in this case there is some gap between the known necessary and sufficient conditions, see Morris and Shin (2004, 2005). Appendix B of Carlsson and van Damme (1993) already worked out explicit uniqueness conditions for a normal signal "private value global game," where private shocks represent idiosyncratic payoffs. Morris and Shin (2005) analyze the differences between the uniqueness conditions for common value and private value global games.

[2] E.g., Atkeson (2000) and Hellwig (2002).

[3] Angeletos, Hellwig and Pavan (2003, 2004) note (inter alia) how other sources of endogenous public information may lead to multiplicity in such coordination games.

private information entering the first period makes uniqueness less not more likely in the second period game. Thus there is a sense in which the comparative statics from the static global game analysis is reversed.[4]

Prices may be essentially public in financial markets.[5] But prices do not (as an empirical matter) seem to fully reveal relevant information in financial markets. Why do prices reveal so much information in the work described above? One reason is that it is assumed that learning the average of private beliefs about θ automatically generates common knowledge. This counterfactual assumption is a consequence of the simple model of private and public information employed in Morris and Shin (2004) and later works using that model of coordination. This model may be rich enough to allow higher order beliefs to play an interesting role in static coordination games, but not rich enough to capture the role of endogenous public information. Based on this intuition, we examine a richer model of higher order beliefs that remains tractable but allows us to explore the role of endogenous public information when common knowledge of average beliefs does not automatically imply common knowledge of everything.

Our initial results are mixed. We show that if the agents' average belief about θ is publicly announced, common knowledge about θ is not generated. This remains true even as agents' signals about θ become arbitrarily accurate. The intuition here is that if there are at least two dimensions of aggregate uncertainty about θ in the population, then a one dimensional price will not reveal the relevant aggregate uncertainty and thus create common knowledge.

However, we also show that in noisy rational expectations equilibrium, prices do generate common knowledge if the noise goes to zero (or the precision of private signals becomes large). Thus the result in Angeletos and Werning (2005) is surprisingly robust to the information structure. However, we argue later that this finding relies on rather special features of the CARA normal environment, in particular, the common knowledge of no gains from trade that holds in either of the limits mentioned. We discuss the relation of this result to the literatures on the existence of fully revealing equilibria in rational expectations equilibria and

[4]The logic of this reversal is the same as that in Hellwig, Mukherji and Tsyvinski (2005), who examine the impact of endogenous public information in a model of currency crises that combines a coordination game and noisy price revelation. We focus on the more stylised model of Angeletos and Werning (2005) in our presentation, as it fits better with our methodological point.

[5]But market participants observe prices at different times and cannot in fact simultaneously execute unlimited trades at the quoted price even they did observe prices at the same time, so there may be limits on this claim.

on information aggregation in strategic common value trading mechanisms in the concluding section 4. In section 2, we review the role of endogenous public signals in the one dimensional signal model. The model with two dimensional private signals is introduced and analyzed in section 3.

2. One dimensional private signals

Let θ be normally distributed with mean y and precision α. Suppose that a continuum of agents observe private signals: agent i observes $x_i = \theta + \varepsilon_i$, where ε_i is normally distributed with mean 0 and precision β.

2.1. Higher Order Beliefs and Exogenous Public Signals

We review some straightforward properties of higher order beliefs in this setting. We have

$$E_i(\theta) = \frac{\alpha y + \beta x_i}{\alpha + \beta}$$

and thus

$$\begin{aligned}
\overline{E}(\theta) &= \frac{\alpha y + \beta \theta}{\alpha + \beta} \\
E_i\left(\overline{E}(\theta)\right) &= \frac{\alpha y + \beta E_i(\theta)}{\alpha + \beta} \\
&= \frac{\alpha y + \beta\left(\frac{\alpha y + \beta x_i}{\alpha + \beta}\right)}{\alpha + \beta} \\
&= \left(\frac{\beta}{\alpha + \beta}\right)^2 x_i + \left(1 - \left(\frac{\beta}{\alpha + \beta}\right)^2\right) y \\
\overline{E}\left(\overline{E}(\theta)\right) &= \left(\frac{\beta}{\alpha + \beta}\right)^2 \theta + \left(1 - \left(\frac{\beta}{\alpha + \beta}\right)^2\right) y \\
\overline{E}^k(\theta) &= \left(\frac{\beta}{\alpha + \beta}\right)^k \theta + \left(1 - \left(\frac{\beta}{\alpha + \beta}\right)^k\right) y
\end{aligned}$$

An important point to notice is that all higher order expectations are pinned down by θ and $\overline{E}(\theta)$. That is,

$$\overline{E}^k(\theta) = \overline{E}(\theta) + \frac{\beta}{\alpha}\left(1 - \left(\frac{\beta}{\alpha+\beta}\right)^{k-1}\right)\left(\overline{E}(\theta) - \theta\right).$$

2.2. Endogenous Public Signals: Observing Opinion Polls

Now suppose that there is no exogenous public signal, but that the average opinion $\overline{E}(\theta)$ is observed with noise. Thus everyone observes a second noisy signal $p = \overline{E}(\theta) + \xi$, where ξ is normally distributed with mean 0 and precision α. Note that $E_i(\theta) = x_i$ and thus $\overline{E}(\theta) = \theta$. Thus each agent observes

$$p = \theta + \xi$$

So agent i's new expectation of θ is

$$E_i^*(\theta) = \frac{\alpha p + \beta x_i}{\alpha + \beta},$$

with precision $\alpha + \beta$. Thus

$$\overline{E}_i^*(\theta) = \alpha p + \beta\theta.$$

Notice that if the public opinion poll consisted of sampling n individuals, we would have $\alpha = n\beta$. In the limit as $n \to \infty$ or $\alpha \to \infty$, y and $\overline{E}(\theta)$ are both publicly observed, it now becomes common knowledge that the true value of θ is

$$\overline{E}(\theta).$$

2.3. Endogenous Public Signals: Observing Prices

Endogenous learning about agents' beliefs may occur through prices rather than opinion polls. So consider a noisy CARA normal rational expectations equilibrium of the type developed by Grossman (1976), Grossman and Stiglitz (1980), Hellwig (1980) and Diamond and Verrecchia (1981). Thus assume the continuum of agents have the information structure described above and each agent has constant absolute risk aversion utility over wealth w with risk tolerance τ:

$$u(w) = e^{-\frac{w}{\tau}}.$$

Also assume that there is a noisy supply of an asset s with mean 0 and precision ν.

We solve for the noisy rational expectations equilibrium. Suppose that in equilibrium the price is normally distributed with mean θ and precision α. Then agent i's demand for the asset is

$$\begin{aligned} & \tau(\alpha+\beta)\left(\frac{\alpha p+\beta x_i}{\alpha+\beta}-p\right) \\ = & \ \tau\beta(x_i-p) \end{aligned}$$

Total demand is then $\tau\beta(\theta-p)+s$. Thus

$$p=\theta-\frac{1}{\tau\beta}s$$

and p is normally distributed with mean and precision $\tau^2\beta^2\nu$. Thus we obtain an endogenous expression for α:

$$\alpha=\tau^2\beta^2\nu.$$

This is the key expression in Angeletos and Werning (2005). Note that if either $\nu\to\infty$ or $\beta\to\infty$, then the endogenous α will also tend to infinite, and the uniqueness condition (1.1) will be violated.

3. Two dimensional private signals

Let θ be normally distributed with mean y and precision α. Suppose that a continuum of agents observe two private signals each: agent i observes $x_{ik}=\theta+\eta_k+\varepsilon_{ik}$, for $k=1,2$; each η_k is independently normally distributed with mean 0 and precision γ_1 and γ_2 respectively. The idiosyncratic noise terms ε_{ik} are each independently normally distributed in the population with mean 0 and precision β_k. Thus each agent has one public signal (y) and two private signals (x_{i1} and x_{i2}) about θ.

3.1. Higher Order Beliefs with Exogenous Public Signals

Observe that each x_{ik} is thus normally distributed with mean θ and precision $\frac{\gamma_k+\beta_k}{\gamma_k\beta_k}$. Thus agent i's expectation of θ is

$$E_i(\theta)=\frac{\alpha y+\frac{\gamma_1+\beta_1}{\gamma_1\beta_1}x_{i1}+\frac{\gamma_2+\beta_2}{\gamma_2\beta_2}x_{i2}}{\gamma+\frac{\gamma_1+\beta_1}{\gamma_1\beta_1}+\frac{\gamma_2+\beta_2}{\gamma_2\beta_2}}.$$

The average expectation of θ in the population is then

$$\overline{E}(\theta) = \frac{\gamma y + \frac{\gamma_1+\beta_1}{\gamma_1\beta_1}\eta_1 + \frac{\gamma_2+\beta_2}{\gamma_2\beta_2}\eta_2}{\gamma + \frac{\gamma_1+\beta_1}{\gamma_1\beta_1} + \frac{\gamma_2+\beta_2}{\gamma_2\beta_2}}.$$

Higher order average expectations $\overline{E}^k(\theta)$ will then differ from $\overline{E}(\theta)$ and will converge to y as $k \to \infty$.

3.2. Observing Opinion Polls

Now suppose that a public opinion poll is announced, and $p = \overline{E}(\theta)$ becomes common knowledge (for simplicity, we will abstract from noise in the public opinion poll). Now the agents will have observed two public signals (y and p) and two private signals (x_{i1} and x_{i2}) concerning θ. But despite the new information, their views do not become common knowledge.[6]

But also observe that an individual who observed all the agents' private information could aggregate it into

$$E^*(\theta) = \frac{\alpha y + \gamma_1\eta_1 + \gamma_2\eta_2}{\alpha + \gamma_1 + \gamma_2}.$$

Although this is a one dimensional variable, it is a sufficient statistic for (η_1, η_2), and a public announcement of this single statistic *would* create common knowledge beliefs about θ. However, it seems non-generic that the public signal would reflect aggregate noise in exactly the right proportion.

3.3. Observing Prices

We now examine what information is revealed by prices in the noisy rational expectations equilibrium. We assume the exogenous noisy supply of the asset, normally distributed with mean 0 and precision ν.

We adopt the usual trick of solving a noisy rational expectations equilibrium by assuming a linear price function and then solving for parameters that make this linear function an equilibrium price function. We assume that prices are equal to

$$p = a + (c_1 + c_2)\xi + c_1\eta_1 + c_2\eta_2 - ds.$$

[6] It would be interesting to check if there was uniqueness in the original coordination game both before and after the announcement of the opinion. Unfortunately, we do not know how to solve these models analytically.

We will show that as $\nu \to \infty$, there is an equilibrium of this form with $c_1/c_2 = \gamma_1/\gamma_2$. This exactly the condition required for prices to generate common knowledge of beliefs about θ.

Consider an agent who observes public signals y and p and private signals x_{i1} and x_{i2}. We show in the appendix that his expected value of θ conditional on x_{i1}, x_{i2}, y and p is

$$E_i(\theta) = y + \lambda_1 (x_{i1} - y) + \lambda_2 (x_{i2} - y) + \lambda_p (p - y)$$

where

$$\begin{aligned}
\lambda_1 &= \frac{d^2 (\gamma_1\gamma_2\beta_1 + \gamma_1\beta_1\beta_2) + c_2^2\nu\gamma_1\beta_1 - c_1c_2\nu\gamma_2\beta_1}{\Delta} \\
\lambda_2 &= \frac{d^2 (\gamma_1\gamma_2\beta_2 + \gamma_2\beta_1\beta_2) + c_1^2\nu\gamma_2\beta_2 - c_1c_2\nu\gamma_1\beta_2}{\Delta} \\
\lambda_p &= \frac{\nu (c_1\gamma_1 (\gamma_2 + \beta_2) + \gamma_2 c_2 (\gamma_1 + \beta_1))}{\Delta}
\end{aligned}$$

and

$$\Delta = \left(\begin{array}{l} d^2 \left(\begin{array}{l} \alpha\gamma_1\gamma_2 + \alpha\gamma_1\beta_2 + \alpha\gamma_2\beta_1 + \alpha\beta_1\beta_2 + \gamma_1\gamma_2\beta_1 \\ +\gamma_1\gamma_2\beta_2 + \gamma_1\beta_1\beta_2 + \gamma_2\beta_1\beta_2 \end{array} \right) \\ 2c_1c_2\nu\gamma_1\gamma_2 \\ +c_1^2\nu (\alpha\gamma_2 + \alpha\beta_2 + \gamma_1\gamma_2 + \gamma_1\beta_2 + \gamma_2\beta_2) \\ +c_2^2\nu (\alpha\gamma_1 + \alpha\beta_1 + \gamma_1\gamma_2 + \gamma_1\beta_1 + \gamma_2\beta_1) \end{array} \right).$$

His variance of θ conditional on x_{i1}, x_{i2} and p is

$$V = \frac{d^2 (\gamma_1\gamma_2 + \gamma_1\beta_2 + \gamma_2\beta_1 + \beta_1\beta_2) + c_1^2\nu (\gamma_2 + \beta_2) + c_2^2\nu (\gamma_1 + \beta_1)}{\Delta}$$

Now by the standard CARA formula, agent i's demand for the asset is

$$\frac{\tau}{V} ([y + \lambda_1 (x_{i1} - y) + \lambda_2 (x_{i2} - y) + \lambda_p (p - y)] - p).$$

Total demand is

$$\frac{\tau}{V} ([y + \lambda_1 (\theta_1 + \eta_1 - y) + \lambda_2 (\theta + \eta_2 - y) + \lambda_p (p - y)] - p).$$

Market clearing implies

$$\begin{aligned}
s &= \frac{\tau}{V} ([y + \lambda_1 (\theta_1 + \eta_1 - y) + \lambda_2 (\theta + \eta_2 - y) + \lambda_p (p - y)] - p) \\
&= \frac{\tau}{V} ([y + (\lambda_1 + \lambda_2) \xi + \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_p (p - y)] - p)
\end{aligned}$$

So

$$y + (\lambda_1 + \lambda_2)\,\xi + \lambda_1\eta_1 + \lambda_2\eta_2 + \lambda_p\,(p - y) - p - \frac{Vs}{\tau} = 0$$

and

$$p = \frac{1}{1-\lambda_p}\left(y + (\lambda_1 + \lambda_2)\,\xi + \lambda_1\eta_1 + \lambda_2\eta_2 - \lambda_p y - \frac{Vs}{\tau}\right)$$

Recall that we assumed

$$p = y + (c_1 + c_2)\,\xi + c_1\eta_1 + c_2\eta_2 - ds$$

Matching coefficients, we require

$$c_1 = \frac{\lambda_1}{1-\lambda_p},\ c_2 = \frac{\lambda_2}{1-\lambda_p} \text{ and } d = \frac{V}{\tau}\left(\frac{1}{1-\lambda_p}\right).$$

We will use a change of variables:

$$\rho = \frac{c_1}{c_1 + c_2} \text{ and } \kappa = \frac{c_1 + c_2}{d};$$

this implies that

$$c_1 = \rho\kappa d \text{ and } c_2 = (1-\rho)\,\kappa d.$$

Now we obtain three equations determining ρ, κ and d. From $c_1/c_2 = \lambda_1/\lambda_2$, we obtain

$$\begin{aligned}\frac{\rho}{1-\rho} &= \frac{\gamma_1\gamma_2\beta_1 + \gamma_1\beta_1\beta_2 + (1-\rho)^2\kappa^2\nu\gamma_1\beta_1 - \rho\,(1-\rho)\,\kappa^2\nu\gamma_2\beta_1}{\gamma_1\gamma_2\beta_2 + \gamma_2\beta_1\beta_2 + \rho^2\kappa^2\nu\gamma_2\beta_2 - \rho\,(1-\rho)\,\kappa^2\nu\gamma_1\beta_2} \\ &= \frac{\gamma_1\beta_1\,(\gamma_2 + \beta_2) + \kappa^2\nu\,(1-\rho)\,\beta_1\,((1-\rho)\,\gamma_1 - \rho\gamma_2)}{\gamma_2\beta_2\,(\gamma_1 + \beta_1) - \kappa^2\nu\rho\beta_2\,((1-\rho)\,\gamma_1 - \rho\gamma_2)};\end{aligned} \tag{3.1}$$

from $\frac{c_1+c_2}{d} = \frac{\tau(\lambda_1+\lambda_2)}{V}$, we obtain

$$\begin{aligned}\kappa &= \frac{\tau\left(\begin{array}{c}\gamma_1\gamma_2\beta_1 + \gamma_1\beta_1\beta_2 + \gamma_1\gamma_2\beta_2 + \gamma_2\beta_1\beta_2 \\ +\kappa^2\nu\left((1-\rho)^2\gamma_1\beta_1 - \rho\,(1-\rho)\,\gamma_2\beta_1 + \rho^2\gamma_2\beta_2 - \rho\,(1-\rho)\,\gamma_1\beta_2\right)\end{array}\right)}{\gamma_1\gamma_2 + \gamma_1\beta_2 + \gamma_2\beta_1 + \beta_1\beta_2 + \kappa^2\nu\left(\rho^2\,(\gamma_2 + \beta_2) + (1-\rho)^2\,(\gamma_1 + \beta_1)\right)} \qquad (3.2) \\ &= \frac{\tau\,(\gamma_1\gamma_2\,(\beta_1 + \beta_2) + (\gamma_1 + \gamma_2)\,\beta_1\beta_2 + \kappa^2\nu\,((1-\rho)\,\gamma_1 - \rho\gamma_2)\,((1-\rho)\,\beta_1 - \rho\beta_2))}{(\gamma_1 + \beta_1)\,(\gamma_2 + \beta_2) + \kappa^2\nu\left(\rho^2\,(\gamma_2 + \beta_2) + (1-\rho)^2\,(\gamma_1 + \beta_1)\right)};\end{aligned}$$

from $d = \frac{V}{\tau}\left(\frac{1}{1-\lambda_p}\right)$, we obtain

$$d = \frac{\left(\begin{array}{c}(\gamma_1+\beta_1)(\gamma_2+\beta_2)+\kappa^2\nu\left(\rho^2(\gamma_2+\beta_2)+(1-\rho)^2(\gamma_1+\beta_1)\right)\\ -\kappa\nu\tau\left((\rho\gamma_1(\gamma_2+\beta_2)+(1-\rho)\gamma_2(\gamma_1+\beta_1))\right)\end{array}\right)}{\tau\left(\begin{array}{l}\alpha(\gamma_1+\beta_1)(\gamma_2+\beta_2)+\gamma_1\gamma_2(\beta_1+\beta_2)+(\gamma_1+\gamma_2)\beta_1\beta_2\\ +\kappa^2\nu\left(\gamma_1\gamma_2+\rho^2(\alpha\gamma_2+\alpha\beta_2+\gamma_1\beta_2+\gamma_2\beta_2)+(1-\rho)^2(\alpha\gamma_1+\alpha\beta_1+\gamma_1\beta_1+\gamma_2\beta_1)\right)\end{array}\right)} \tag{3.3}$$

We are interested in solutions of this system of equations (3.1), (3.2) and (3.3) as $\nu \to \infty$. To explore this, consider one more change of variables, letting $z = (1-\rho)\gamma_1 - \rho\gamma_2$ and $\psi = \kappa^2\nu$. Now equation (3.1) can be re-written as

$$\begin{aligned}\frac{\gamma_1 - z}{\gamma_2 + z} &= \frac{\gamma_1\beta_1(\gamma_2+\beta_2)+\psi\beta_1(1-\rho)z}{\gamma_2\beta_2(\gamma_1+\beta_1)-\psi\beta_2\rho z}\\ &= \frac{\gamma_1\beta_1(\gamma_2+\beta_2)(\gamma_1+\gamma_2)+\psi\beta_1(\gamma_2+z)z}{\gamma_2\beta_2(\gamma_1+\beta_1)(\gamma_1+\gamma_2)-\psi\beta_2(\gamma_1-z)z}\end{aligned}$$

This gives the following cubic equation in z:

$$\left\{\begin{array}{l}\psi(\beta_1+\beta_2)z^3\\ +2\psi(\gamma_2\beta_1-\gamma_1\beta_2)z^2\\ +\left[\begin{array}{l}\psi(\beta_1\gamma_2^2+\beta_2\gamma_1^2)\\ +(\gamma_1\beta_1(\alpha_2+\beta_2)+\gamma_2\beta_2(\alpha_1+\beta_1))\end{array}\right]z\\ +(\gamma_1+\gamma_2)\gamma_1\gamma_2(\beta_1\gamma_2-\beta_2\gamma_1)\end{array}\right\} = 0$$

As $\psi \to \infty$, this equation has a root with

$$z \approx \frac{(\gamma_1+\gamma_2)\gamma_1\gamma_2(\beta_2\gamma_1-\beta_1\gamma_2)}{\psi(\beta_1\gamma_2^2+\beta_2\gamma_1^2)+(\gamma_1\beta_1(\alpha_2+\beta_2)+\gamma_2\beta_2(\alpha_1+\beta_1))}$$

and thus $z \to 0$ as $\psi \to \infty$. Now equation (3.2) can be re-written (for large $\kappa^2\nu$) as

$$\kappa \approx \frac{\tau\left(\gamma_1\gamma_2(\beta_1+\beta_2)+(\gamma_1+\gamma_2)\beta_1\beta_2+\frac{\kappa^2\nu(\gamma_1+\gamma_2)\gamma_1\gamma_2(\beta_2\gamma_1-\beta_1\gamma_2)}{\kappa^2\nu\left(\beta_1\gamma_2^2+\beta_2\gamma_1^2\right)+(\gamma_1\beta_1(\alpha_2+\beta_2)+\gamma_2\beta_2(\alpha_1+\beta_1))}\frac{\gamma_2\beta_1-\gamma_1\beta_2}{\gamma_1+\gamma_2}\right)}{(\gamma_1+\beta_1)(\gamma_2+\beta_2)+\kappa^2\nu\left(\left(\frac{\gamma_1}{\gamma_1+\gamma_2}\right)^2(\gamma_2+\beta_2)+\left(\frac{\gamma_2}{\gamma_1+\gamma_2}\right)^2(\gamma_1+\beta_1)\right)};$$

Thus as $\nu \to \infty$, κ solves

$$\kappa \approx \frac{\tau\left(\gamma_1\gamma_2(\beta_1+\beta_2)+(\gamma_1+\gamma_2)\beta_1\beta_2+\frac{(\gamma_1+\gamma_2)\gamma_1\gamma_2(\beta_2\gamma_1-\beta_1\gamma_2)}{\beta_1\gamma_2^2+\beta_2\gamma_1^2}\frac{\gamma_2\beta_1-\gamma_1\beta_2}{\gamma_1+\gamma_2}\right)}{\kappa^2\nu\left(\left(\frac{\gamma_1}{\gamma_1+\gamma_2}\right)^2(\gamma_2+\beta_2)+\left(\frac{\gamma_2}{\gamma_1+\gamma_2}\right)^2(\gamma_1+\beta_1)\right)};$$

and thus

$$\kappa \approx \left(\frac{\tau \left(\gamma_1\gamma_2\left(\beta_1+\beta_2\right) + \left(\gamma_1+\gamma_2\right)\beta_1\beta_2 + \frac{(\gamma_1+\gamma_2)\gamma_1\gamma_2(\beta_2\gamma_1-\beta_1\gamma_2)}{\beta_1\gamma_2^2+\beta_2\gamma_1^2}\frac{\gamma_2\beta_1-\gamma_1\beta_2}{\gamma_1+\gamma_2} \right)}{\nu\left(\left(\frac{\gamma_1}{\gamma_1+\gamma_2}\right)^2\left(\gamma_2+\beta_2\right)+\left(\frac{\gamma_2}{\gamma_1+\gamma_2}\right)^2\left(\gamma_1+\beta_1\right)\right)} \right)^{\frac{1}{3}}.$$

Note that this confirms our assumption that $\kappa^2\nu \to \infty$ as $\nu \to \infty$. Finally, this implies that

$$d \to \frac{-\tau\left(\left(\left(\frac{\gamma_1}{\gamma_1+\gamma_2}\right)\gamma_1\left(\gamma_2+\beta_2\right)+\left(\frac{\gamma_2}{\gamma_1+\gamma_2}\right)\gamma_2\left(\gamma_1+\beta_1\right)\right)\right)}{\tau\kappa\left(\gamma_1\gamma_2+\left(\frac{\gamma_1}{\gamma_1+\gamma_2}\right)^2\left(\alpha\gamma_2+\alpha\beta_2+\gamma_1\beta_2+\gamma_2\beta_2\right)+\left(\frac{\gamma_2}{\gamma_1+\gamma_2}\right)^2\left(\alpha\gamma_1+\alpha\beta_1+\gamma_1\beta_1+\gamma_2\beta_1\right)\right)}. \tag{3.4}$$

4. Discussion and Conclusion

Here is one explanation of the limit common knowledge of the previous section. As $\nu \to \infty$, there is common knowledge that there are no gains from trade. In this simple environment, every agent would trade on any private information not revealed by prices. But by the limit no trade theorem, no agent trades. So all private information must be revealed.

In an environment without this limit no trade property, we expect that the limit common knowledge result will not survive. This is consistent with the foundational literature on rational expectations equilibria: fully revealing equilibria exist if the dimension of prices exceeds the dimension of uncertainty, but not otherwise (e.g., Allen (1981)).

The solution concept of rational expectations equilibrium has the well-known flaw that agents are expected to simultaneously learn from prices and treat them parametrically in their trading decisions (see, e.g., Dubey, Geanakoplos and Shubik (1987)). The information aggregation literature has examined to what extent that explicit well-defined trading mechanisms generate the information aggregation of fully revealing rational expectations equilibria (key early contributions are Wilson (1977) and Milgrom (1979, 1981)). This literature is not close to addressing the problem of multi-dimensional types (see Pesendorfer and Swinkels (2000) for one attempt). In any case, there is a non-existence problem for rational expectations equilibria when the dimension of uncertainty exceeds the dimension of prices (Allen (1982) and Anderson and Sonnenschein (1982)), so a complete analysis of what will be revealed when agents with rich (multidimensional) higher order beliefs seems a long way off.

The fact that prices may themselves play the role of public signals is an economically important phenomenon that deserves the attention that it has received. On the uniqueness question, it is hard to relate the abstract uniqueness conditions of the static theory to real world applications. The formal, a priori, arguments of Angeletos and Werning (2005) and Hellwig, Mukherji and Tsyvinski (2005) that we should expect the uniqueness conditions to fail because of endogenous information revelation may not be robust to richer higher order beliefs: it is quite possible to observe public signals of average beliefs and have signals become arbitrarily accurate without generating common knowledge. The particular model of Angeletos and Werning (2005) does turn out to be robust to richer higher order beliefs, but it is not clear how robust this robustness is to other special features of the CARA normal model. A general theory of what prices reveal when agents have rich higher order beliefs is sadly lacking.

5. Appendix

We have

$$\begin{pmatrix} \xi \\ \eta_1 \\ \eta_2 \\ \varepsilon_{i1} \\ \varepsilon_{i2} \\ s \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\beta_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\nu} \end{pmatrix}\right).$$

Now

$$\begin{pmatrix} \theta \\ x_{i1} \\ x_{i2} \\ p \end{pmatrix} = \begin{pmatrix} y \\ y \\ y \\ a \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ c_1 + c_2 & c_1 & c_2 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta_1 \\ \eta_2 \\ \varepsilon_{i1} \\ \varepsilon_{i2} \\ s \end{pmatrix}.$$

So

$$\begin{pmatrix} \theta \\ x_{i1} \\ x_{i2} \\ p \end{pmatrix}$$

is normally distributed with mean

$$\begin{pmatrix} y \\ y \\ y \\ a \end{pmatrix}$$

and variance matrix Σ, where

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ c_1+c_2 & c_1 & c_2 & 0 & 0 & d \end{pmatrix} \begin{pmatrix} \frac{1}{\alpha} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\gamma_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\beta_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\beta_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\nu} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & c_1+c_2 \\ 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & c_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha}(c_1+c_2) \\ \frac{1}{\alpha} & \frac{1}{\alpha}+\frac{1}{\beta_1}+\frac{1}{\gamma_1} & \frac{1}{\alpha} & \frac{c_1}{\gamma_1}+\frac{1}{\alpha}(c_1+c_2) \\ \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{1}{\alpha}+\frac{1}{\beta_2}+\frac{1}{\gamma_2} & \frac{c_2}{\gamma_2}+\frac{1}{\alpha}(c_1+c_2) \\ \frac{1}{\alpha}(c_1+c_2) & \frac{c_1}{\gamma_1}+\frac{1}{\alpha}(c_1+c_2) & \frac{c_2}{\gamma_2}+\frac{1}{\alpha}(c_1+c_2) & \frac{d^2}{\nu}+\frac{c_1^2}{\gamma_1}+\frac{c_2^2}{\gamma_2}+\frac{1}{\alpha}(c_1+c_2)^2 \end{pmatrix}$$

Now the expected value of θ conditional on x_{i1}, x_{i2} and p is

$$y + \begin{pmatrix} \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{c_1+c_2}{\alpha} \end{pmatrix} \widehat{\Sigma}^{-1} \begin{pmatrix} x_{i1}-y \\ x_{i2}-y \\ p-a \end{pmatrix} = y + (\lambda_1, \lambda_2, \lambda_p) \begin{pmatrix} x_{i1}-y \\ x_{i2}-y \\ p-a \end{pmatrix}$$

where

$$\widehat{\Sigma} = \begin{pmatrix} \frac{1}{\alpha}+\frac{1}{\beta_1}+\frac{1}{\gamma_1} & \frac{1}{\alpha} & \frac{c_1}{\gamma_1}+\frac{1}{\alpha}(c_1+c_2) \\ \frac{1}{\alpha} & \frac{1}{\alpha}+\frac{1}{\beta_2}+\frac{1}{\gamma_2} & \frac{c_2}{\gamma_2}+\frac{1}{\alpha}(c_1+c_2) \\ \frac{c_1}{\gamma_1}+\frac{1}{\alpha}(c_1+c_2) & \frac{c_2}{\gamma_2}+\frac{1}{\alpha}(c_1+c_2) & \frac{d^2}{\nu}+\frac{c_1^2}{\gamma_1}+\frac{c_2^2}{\gamma_2}+\frac{1}{\alpha}(c_1+c_2)^2 \end{pmatrix}$$

Letting

$$\Delta = \begin{pmatrix} d^2 \begin{pmatrix} \alpha\gamma_1\gamma_2 + \alpha\gamma_1\beta_2 + \alpha\gamma_2\beta_1 + \alpha\beta_1\beta_2 + \gamma_1\gamma_2\beta_1 \\ +\gamma_1\gamma_2\beta_2 + \gamma_1\beta_1\beta_2 + \gamma_2\beta_1\beta_2 \end{pmatrix} \\ 2c_1c_2\nu\gamma_1\gamma_2 \\ +c_1^2\nu\left(\alpha\gamma_2 + \alpha\beta_2 + \gamma_1\gamma_2 + \gamma_1\beta_2 + \gamma_2\beta_2\right) \\ +c_2^2\nu\left(\alpha\gamma_1 + \alpha\beta_1 + \gamma_1\gamma_2 + \gamma_1\beta_1 + \gamma_2\beta_1\right) \end{pmatrix},$$

one can show that

$$\begin{aligned} \lambda_1 &= \frac{d^2\left(\gamma_1\gamma_2\beta_1 + \gamma_1\beta_1\beta_2\right) + c_2^2\nu\gamma_1\beta_1 - c_1c_2\nu\gamma_2\beta_1}{\Delta} \\ \lambda_2 &= \frac{d^2\left(\gamma_1\gamma_2\beta_2 + \gamma_2\beta_1\beta_2\right) + c_1^2\nu\gamma_2\beta_2 - c_1c_2\nu\gamma_1\beta_2}{\Delta} \\ \lambda_p &= \frac{\nu\left(c_1\gamma_1\left(\gamma_2 + \beta_2\right) + \gamma_2c_2\left(\gamma_1 + \beta_1\right)\right)}{\Delta} \end{aligned}$$

The variance of θ conditional on x_{i1}, x_{i2} and p is

$$V = \frac{1}{\alpha} - \begin{pmatrix} \frac{1}{\alpha} & \frac{1}{\alpha} & \frac{c_1+c_2}{\alpha} \end{pmatrix} \widehat{\Sigma}^{-1} \begin{pmatrix} \frac{1}{\alpha} \\ \frac{1}{\alpha} \\ \frac{1}{\alpha}\left(c_1 + c_2\right) \end{pmatrix}$$

One can show that this expression equals

$$\frac{d^2\left(\gamma_1\gamma_2 + \gamma_1\beta_2 + \gamma_2\beta_1 + \beta_1\beta_2\right) + c_1^2\nu\left(\gamma_2 + \beta_2\right) + c_2^2\nu\left(\gamma_1 + \beta_1\right)}{\Delta}$$

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