Environment and Directed Technical Change

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Abstract

This paper introduces endogenous directed technical change in a growth model with environmental constraints and limited resources. A unique final good is produced by combining the inputs produced by two sectors. One of these sectors uses “dirty” machines and thus creates environmental degradation. Profit-maximizing researchers direct their research to improving the technology of machines in the two sectors. We characterize dynamic tax policies that achieve sustainable growth or maximize intertemporal welfare, as a function of structural characteristics of the economy, in particular the degree of substitutability or complementarity between clean and dirty inputs, environmental and resource stocks, and cross-country technological spillovers. First, we find that factoring in directed technical change: (i) increases the cost of delaying intervention, particularly in the substitutability case; (ii) calls for the use of profit taxes or other instruments to direct innovation, in addition to the input tax emphasized so far by the literature. Second, we show that: (i) in the case where the clean and dirty inputs are substitutes, one can achieve sustainable long run growth with temporary taxation of dirty innovation and production; (ii) the sooner and stronger the policy response, the shorter the slow growth transition phase; (iii) the use of an exhaustible resource in dirty goods production helps the switch to clean innovation under laissez-faire when the two inputs are substitute, but the opposite holds when the two inputs are complements. Finally, in a two-country extension of the baseline model where: (a) the two inputs are substitutable in both countries; (b) dirty input production in both countries depletes the global environmental stock; (c) the South imitates technologies invented in the North, we show that taxing dirty innovation and production in the North only may be sufficient to avoid a global disaster, though international trade increases the need for global policy coordination.

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Very Preliminary and Incomplete.

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1 Introduction

How to control and limit climate change caused by our growing consumption of fossil fuels and to develop alternative energy sources to these fossil fuels are among the most pressing policy challenges facing the world today. While climate scientists have focused on various aspects of the damage that our current energy consumption causes to the environment, economists have emphasized both the benefits—in terms of limiting environmental degradation—and costs—in terms of reducing economic growth—of different policy proposals. More importantly, while a large part of the discussion among climate scientists focuses on the effect of various policies on development of alternative—and more “environmentally friendly”—energy sources, the response of technological change to environmental policy has until very recently been all but ignored by leading economic analyses of environment the policy, which have focused on computable general equilibrium models with exogenous technology. This omission is despite the fact that existing empirical evidence indicates that changes in the relative price of energy inputs have an important effect on the types of technologies that are developed and adopted. For example, Newell, Jaffe and Stavins (1999) show that between 1960 and 1980, when energy prices were stable, innovations in air-conditioning have reduced the prices faced by consumers, following the oil price hikes, these air conditioners became more energy efficient. Popp (2002) provides more systematic evidence on the same point by using patent data from 1970 to 1994; he documents the impact of energy prices on patents for energy-saving innovations.

This paper is motivated by our belief that a satisfactory framework for the study of the costs and benefits of different environmental policies must include, at its centerpiece, the endogenous response of different types of technologies to proposed policies. Our purpose is to take a first step towards the development of such a framework. We propose a simple two-sector endogenous model of directed technical change. The unique final good is produced by combining the inputs

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2 The analysis on the use of price or quantity instruments pioneered in Weitzman (1974), has been recently applied to the study of climate change policy and the choice between taxes and quotas, such as in Hepburn (2006) and Pizer (2002). In addition, several studies address the importance of internationally coordinated policy, such as Stern (2006) and Watson (2001). Aldy et al. (2003) provide a comparison of the different architectures for global climate policy.


4 Acemoglu and Finkelstein (2008) provide evidence from the health-care sector, suggesting that capital investments and technology adoption are highly responsive to changes in relative prices caused by regulation.
produced by these two sectors. One of them uses “dirty” machines and creates environmental
degradation. Profit-maximizing researchers direct their research to improving the quality of
machines in one or the other sector. We first focus on a single closed economy.

Our framework highlights the central roles played by the market size and the price effects on
the direction of technical change (see Acemoglu, 1998, 2002). The market size effect encourages
innovation towards the larger input sector, while the price effect makes innovation directed
towards the input sector with higher price. The relative magnitudes of these effects in our
framework are, in turn, determined by three factors:

1. The elasticity of substitution between the two sectors.

2. The relative levels of development of the technologies of the two sectors.

3. Whether dirty inputs are produced using an exhaustible resource.

Because of the environmental externality, the decentralized equilibrium is not optimal.
Nevertheless, this does not imply that economic growth without policy intervention will lead
to an “environmental disaster,” where the quality of the environment falls below a critical
threshold. Whether it does or not depends on how relatively backward the environmentally-
friendly technologies are, on the elasticity of substitution, and on whether dirty inputs use an
exhaustible resource such as oil.

More interesting are the results on what types of policies can prevent such disasters and how
costly delaying their introduction is. These also depend on the market size and price effects,
and thus on the three factors highlighted above. In particular, an environmental disaster is
more likely without policy intervention when the two sectors are highly substitutable. Broadly
speaking, in this case the market size effect implies that an initial productivity advantage in
dirty inputs will induce researchers to target innovation on these machines (as the expected
profit from innovating in a sector is increasing in employment and productivity in this sector),
in spite of the fact that more productive dirty machines also result in a lower relative price of
the dirty input (the price effect). However, in this case a temporary policy intervention (for
example, a temporary tax on the use of dirty inputs, reminiscent to a carbon tax, or simply a
profit tax/subsidy) can prevent an environmental disaster, as it will push the economy to shift
away from using the dirty input. Once the clean inputs become developed, the tax is no longer
required. But this case also shows that delaying the introduction of such policies is potentially quite costly. Delay will increase the gap between clean and dirty sectors and thus call for higher taxes in the future in order to avoid a disaster. In contrast, when the two sectors are highly complementary, an environmental disaster is less likely in the decentralized equilibrium, since in that case an initial productivity advantage in dirty machines induces innovation in the more backward clean sector, as the price effect now dominates. But preventing such a disaster, when it exists, now requires a permanent policy intervention, which is, naturally, more costly in the long run. Interestingly, delay is not as costly in this case.

The intuitions for these results are informative about the economic forces in our framework: when the two sectors are highly complementary, an equilibrium path where a large fraction of output is produced in the clean sector will increase the price of the output of the dirty sector more than proportionately. This will create a price effect increasing the incentives to produce in that sector, which would maintain the environmental damage unless policy is adjusted to further discourage this outcome. In the substitutes case, once the clean sector is sufficiently advanced, the size of the clean sector will grow and the relative price of the dirty sector will fall less than proportionately, and consequently, there will be no more incentive to produce in the dirty sector.

The degree of substitution in the model has a clear empirical counterpart. For example, renewable energy, provided it can be stored and transported as efficiently as gasoline, would correspond to a high degree of substitution. In contrast, if the “clean alternative” is to reduce our consumption of energy permanently, for example by using less effective transport technologies, this would correspond to a low degree of substitution, since greater consumption of non-energy commodities would increase the demand for energy.

The framework also shows the role of the exhaustibility of resources. An environmental disaster is less likely when the dirty sector uses an exhaustible resource (and the two sectors have a high degree of substitution), since this will create a tendency for its costs to increase as this resource is depleted. The greater cost of the resource will induce a contraction of the sector, while simultaneously increasing its price. The implications again depend on the market size and price effects. When the elasticity of substitution between the two sectors is high, the market size effect implies that technology for the clean sector will develop rapidly because as the resource is exhausted, the dirty sector shrinks and the clean sector expands. In this
case, therefore, the market will induce the development of clean technologies even without intervention. Once again, the contrast between exhaustible and non-exhaustible resources in the model has a clear empirical counterpart.

We also show how our framework can be used to analyze cross-country linkages in technology. Key questions in this case include: (i) whether environmental degradation (and in the extreme case, a disaster) can be avoided by policies in the “North” alone, that is, without imposing similar environmental regulations in the South, (i.e in developing countries such as India and China); (ii) what the effects of international trade are on the development of environmental technologies and on environmental sustainability. Our framework provides new answers to these questions. Again, the three factors emphasized above turn out to be central. For example, when the two sectors are highly substitutable (and there are international technology linkages), policies only in the North are sufficient to stave off an environmental disaster, because once these policies induce a sufficient improvement in the technology of the clean sector, the South will also adjust its pattern of production. International trade, on the other hand, allows the South to specialize in and increase the production of the dirty sector, which is not taxed there, and thus increases the need for global policy coordination for avoiding environmental disaster.

Our paper relates to the substantial and growing literature on growth, resources, and the environment. Nordhaus’s (1994) pioneering study proposed a dynamic integrated model of climate change and the economy (the DICE model), which extends the neoclassical Ramsey model with equations representing geophysical relationships (emissions equations, concentrations equations, climate-change equations, climate-damage equations), and their relationship with economic outcomes. More recent papers have focused on climate’s impact on different types of outcomes, such as health, agriculture, conflict, and economic growth. Other papers have contributed to the measurement of the costs of climate change, particularly focusing on issues related to risk, uncertainty and discounting. Based on the assessment of discounting

\[5\] Nordhaus and Boyer (2000) introduce a version of the previous model with eight regions making decisions independently, with limited interaction between them (the “RICE” model, or Regional Dynamic Integrated model of Climate and the Economy). The analysis of economic activity and its consequences in terms of climate change using this type of approach has been the subject of an extensive report conducted by Stern (2006).

\[6\] See, for example, Deschenes and Moretti (2007), Sachs and Malaney (2002), Deschenes and Greenstone (2007), Miguel et al. (2004), and Dell et al. (2008).

and related issues, this literature has prescribed either decisive and immediate governmental action (for example, Stern, 2006) or a more gradualist approach, with modest control in the short-run followed by sharper emissions reduction in the median and long run. Finally, some authors have built on Weitzman (1974)'s analysis on the use of price or quantity instruments to study climate change policy and the choice between taxes and quotas.

The response of technology to environmental degradation and environmental policy has received much less attention in the economics literature. Early work by Stokey (1998) highlighted the tension between growth and the environment and showed that degradation of the environment can create an endogenous limit to growth. Aghion and Howitt (1998, Chapter 5) introduced environmental constraints in a Schumpeterian growth model and emphasized that environmental constraints may not prevent sustainable long-run growth when environment-saving innovations are allowed. Neither of these early contributions allowed technological change to be directed to clean or dirty technologies.

Subsequent work by Popp (2004) allowed for directed innovation in the energy sector. Popp presents a calibration exercise and establishes that models that ignore the direct technological change effects can significantly overstate the cost of environmental regulation. While Popp's highly complementary to ours, neither his work nor others develop a systematic framework for analysis of the impact of environmental regulations on the direction of technological change. We develop a general and tractable framework that allows: (i) carrying out comparative analyses of outcomes under the presence or absence of directed technical change; (ii) studying implications of exhaustible resources being used by the dirty sector, (iii) shedding light on policy optimality, and (iv) the understanding of how the effects of environmental reg-

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9 See for example Hepburn (2006) and Pizer (2002). In addition, several studies address the importance of internationally coordinated policy, such as Stern (2006) and Watson (2001). Aldy et al. (2003) provide a comparison of the different architectures for global climate policy.

10 First attempts at introducing endogenous directed technical change in models of growth and the environment, include Grubler and Messner (1998), Manne and Richels (2002), Messner (1997), Buonanno et al (2003), Nordhaus (2002), Saint-Paul (2002), and Wing (2003). Finally, Aghion and Howitt (2009, Chapter 16) analyze a stripped down version of the model presented in this paper, where the two inputs are perfect substitutes.

11 Nordhaus (2002) also extends the R&DICE model by including a simple form of induced technical change. In particular, uses a variant of his previous framework with fixed proportions, in which R&D is modeled as shifting the minimum level of carbon/energy inputs required for production. However, since factor substitution is not allowed in the model, it is not possible to compare the role of induced innovation with that of factor substitution in reducing greenhouse emissions. Popp's (2004) ENTICE model allows for both endogenous technological change and factor substitution.
ulations depend on international interactions. Our general framework builds on and extends the directed technical change models developed in Acemoglu (1998, 2002).

More recently, Gans (2009) develops a two-period model based on Acemoglu (2009) to discuss the Porter hypothesis, that environmental regulation can lead to faster technological progress. In particular he shows that this would require a high degree of substitutability between clean and dirty energy. We abstract from this channel in the current paper by assuming that the total R&D resources in the economy are constant, and we focus instead on long-run growth sustainability and the characterization of dynamic optimal policies.

The remaining part of the paper is organized as follows. Section 2 lays out the basic framework. Section 3 analyzes the equilibrium of the laissez-faire economy, and shows the possibility that this equilibrium lead to a disaster path with unsustainable growth. Section 4 analyzes the social planner’s problem and characterizes the optimal dynamic tax schedule as a function of the environmental stock and the degree of substitutability between the two input baskets. Section 5 introduces a resource constraint in the production of the dirty input basket. Section 6 extends the analysis to the two-country case. And Section 7 concludes.

2 Baseline Model: Non-Exhaustible Resource

In this section, we introduce the baseline framework (without exhaustible resources). We identify the market size and price effects on the direction of technical change and characterize the equilibrium of the economy under laissez-faire. We then discuss how policy interventions may be necessary to avoid “environmental disasters”, and the costs of delayed intervention. Finally, we fully characterize the optimal policy in this environment.

2.1 Preferences, Production and the Environment

We consider an infinite horizon discrete time economy inhabited by a continuum of households of mass 1 (making up the workers, entrepreneurs and scientists). All households have preferences given by

$$\sum_{t=0}^{\infty} \frac{1}{(1 + \rho)^t} u(C_t, S_t)$$

where $C_t$ is consumption of the unique final good at time $t$, $S_t$ denotes the quality of the environment at time $t$, and $\rho > 0$ is the discount rate. We introduce the role of the environment in the simplest possible way by assuming that the representative household directly cares
about the quality of the environment, so the instantaneous utility function $u$ is assumed to be increasing in its first argument and decreasing in the second, we also assume that $u$ is concave in $(C, S)$. Moreover, we also impose that the following Inada-type assumptions:

$$
\lim_{S \to 0} \frac{\partial u(C, S)}{\partial S} = \infty, \quad \lim_{C \to 0} \frac{\partial u(C, S)}{\partial C} = \infty,
$$

and we also assume that there exists $\bar{S} > 0$ such that

$$
\frac{\partial u(C, S)}{\partial S} = 0 \text{ for all } S \geq \bar{S},
$$

which implies that that utility is satiated in the quality of the environment. This last assumption is introduced to simplify the full characterization of optimal policy below.

There is a unique final good, produced competitively from the output of two intermediate sectors, according to the aggregate production function

$$
Y_t = \left( \frac{Y_{ct}}{\varepsilon} + \frac{Y_{dt}}{\varepsilon} \right)^{\frac{\varepsilon}{\varepsilon-1}},
$$

where $\varepsilon \in (0, +\infty)$ is the elasticity of substitution between the two sectors. Throughout, we say that the two sectors are substitutes when $\varepsilon > 1$ and complements when $\varepsilon < 1$. Both $Y_{ct}$ and $Y_{dt}$ are produced using labor and a continuum of sector-specific machines (intermediates) according to the production functions

$$
Y_{ct} = L_{ct}^{1-\alpha} \int_0^1 A_{cit}^{1-\alpha} x_{cit}^\alpha di \quad \text{and} \quad Y_{dt} = L_{dt}^{1-\alpha} \int_0^1 A_{dit}^{1-\alpha} x_{dit}^\alpha di,
$$

where $\alpha \in (0, 1)$, $A_{jit}$ is the quality of machine of type $i$ used in sector $j \in \{c, d\}$ at time $t$ and $x_{jit}$ is the quantity of this machine. This setup is identical to that used in Acemoglu (1998), except that distribution parameters have been dropped in (1) to simplify the algebra. We also define

$$
A_{jt} \equiv \int_0^1 A_{jit} di
$$

as the aggregate productivity in sector $j \in \{c, d\}$.

Market clearing for labor requires that

$$
L_{ct} + L_{dt} \leq 1.
$$

An example of substitutable dirty and clean inputs would be cars using gasoline versus cars using clean energy sources. Examples with less substitutable inputs would be cars versus green public transports, or fuel efficiency on SUVs.
In line with the literature on endogenous technical change, machines (for both sectors) are supplied by monopolistically competitive firms. Regardless of the quality of the machines and the sector for which they are designed, producing one unit of any machine costs $\psi$ units of the final good. Without loss of generality, we normalize $\psi \equiv \alpha^2$.

We also normalize the measure of scientists $s$ to 1. The technology of innovation is as follows. At the beginning of every period, each scientist decides whether to direct his research to clean or dirty technology. She is then randomly allocated to at most one machine (without any congestion; so that each machine is also allocated to at most one scientist) and is successful in innovation with probability $\eta_j < 1$ in sector $j \in \{c, d\}$, where innovation increases the quality of a machine by a factor $1 + \gamma$. A successful scientist (who has invented a better version of machine $i$ in sector $j \in \{c, d\}$) obtains a one-period patent and becomes the entrepreneur for the current period in the production of machine $i$. In sectors where innovation was not successful, monopoly rights are attributed randomly to an entrepreneur drawn from a pool of measure 2 who can use the old technology. This technology for innovation where scientists decide only to allocate to a specific sector (and not to a specific machine) ensures that scientists are allocated across the different machines in a sector.\textsuperscript{13} We denote scientists working on machines in sector $j \in \{c, d\}$ at time $t$ by $s_{jt}$, and thus market clearing for scientists takes the form

$$s_{ct} + s_{dt} \leq 1. \quad (5)$$

Finally, the quality of the environment, $S_t$, evolves according to the difference equation

$$S_{t+1} = \max \{-\xi Y_{dt} + (1 + \delta) S_t, 0\}, \quad (6)$$

where $\xi$ measures the rate of deterioration and $\delta$ measures the rate at which the environment is (partly) regenerated. This equation introduces the major externality in our model, from the output of the dirty intermediate to environmental degradation. Note also that, with this formulation, if $S_t = 0$, then $S_{\tau}$ will remain at zero for all $\tau > t$.

### 2.2 Laissez-faire equilibrium

In this subsection we characterize the laissez-faire equilibrium outcome, that is, the decentralized equilibrium without any policy intervention. We first characterize the equilibrium

\textsuperscript{13}The assumptions here are adopted to simplify the exposition and mimic the structure of equilibrium in continuous time models (see, for example, Acemoglu, 2002). We adopt a discrete time setup throughout to simplify the analysis of dynamics. Appendix C shows that the qualitative results are identical in an alternative formulation with patents and free entry.
production and labor decisions for given productivity parameters; then we move back one step and analyze the directed innovation decisions of intermediate producers and establish sufficient conditions under which the equilibrium leads to an environmental disaster.

An equilibrium is sequences of wages \((w_t)\), price for inputs \((p_{jt})\), price for machines \((p_{jit})\), demands for machines \((x_{jit})\), demands for inputs \((Y_{jt})\), labor demands \((L_{jt})\) by input producers \(j \in \{c, d\}\) and research allocations \((s_{dt}, s_{ct})\) such that, in each period \(t\): (i) \((p_{jit}, x_{jit})\) maximizes profits by the producer of machine \(i\) in sector \(j\); (ii) \(L_{jt}\) maximizes profits by producers of input \(j\); (iii) \(Y_{jt}\) maximizes the profits of final good producers; (iv) \((s_{dt}, s_{ct})\) maximizes the expected profit of a researcher at date \(t\); (v) the wage \(w_t\) and the prices \(p_{jt}\) clear the labor and input markets respectively.

To simplify the algebra, we define \(\varphi \equiv (1 - \alpha) (1 - \varepsilon)\), and impose the following assumption:

**Assumption 1**

\[
\frac{A_{c_0}}{A_{d_0}} < \min \left( \left(1 + \gamma_1 \eta_c \right)^{-\frac{\gamma}{\theta}}, \left(1 + \gamma_2 \eta_d \right)^{\frac{\gamma + 1}{\theta}} \right).
\]

\[
b \eta_d \geq \eta_c.
\]

Part a) states that initially the clean sector is sufficiently backward, whereas part b) states that research efforts directed at the dirty sector are at least as productive as R&D efforts directed at the clean sector. Together, these imply that under laissez-faire (that is, in the absence of tax or subsidies), the economy starts innovating in the dirty input in the substitutability case \(\varepsilon > 1\).

### 2.2.1 Laissez-faire equilibrium: static part and the price effect

Here we consider the equilibrium at time \(t\), once the innovation process has taken place, and technological levels, \(A_{cit}\) and \(A_{dit}\), are given. For this particular part we then drop the subscripts \(t\).

As the final good is produced competitively the ratio of relative price satisfies

\[
\frac{p_c}{p_d} = \left(\frac{Y_c}{Y_d}\right)^{\frac{1}{\varepsilon}}
\]

(7)

The greater the supply of clean good relative to dirty good, the lower will be its price, the elasticity of the relative prices response is the inverse of the elasticity of substitution between
the two intermediary inputs. We normalize the price of the final good at 1, this translates into the condition

\[ \left[ p_c^{1-\varepsilon} + p_d^{1-\varepsilon} \right]^{1/(1-\varepsilon)} = 1. \]

Profit maximization by producers of intermediate input \( j \) can be written as

\[
\max_{x_{ji}, L_j} p_j L_j^{1-\alpha} \int_0^1 A_{ji}^{1-\alpha} x_{ji}^\alpha di - wL_j - \int_0^1 p_{ji} x_{ji} di
\]

and leads to the following inverse demand curve for the producer of machine \( i \) in sector \( j \):

\[
x_{ij} = \left( \frac{\alpha p_{ji}}{p_j} \right)^{-\frac{1}{1-\alpha}} A_{ji} L_{ji}.
\]

Thus the demand for machines \( i \) in sector \( j \) increases with the price \( p_j \) of input \( j \), employment \( L_j \) in that sector (both increase the profitability of all types of machines used in that sector, thereby encouraging producers to hire more of them), and it is also increasing in the quality of such machines \( A_{ji} \) and decreases in their price \( p_{ji} \).

The monopolistic producer of machines \( i \) in sector \( j \) chooses \( p_{ji} \) and \( x_{ji} \) so as to maximize profits \( \pi_{ji} = (p_{ji} - \psi) x_{ji} \), subject to the inverse demand curve (8). Recalling the normalization \( \psi \equiv \alpha^2 \), equilibrium demand for machines is

\[
x_{ji} = p_j^{\frac{1}{1-\alpha}} L_j A_{ji},
\]

and equilibrium profits for monopolists are

\[
\pi_{ji} = (1 - \alpha) \alpha p_j^{\frac{1}{1-\alpha}} L_j A_{ji}.
\]

In addition, relative prices of clean and dirty inputs are given by:\(^{14}\)

\[
\frac{p_c}{p_d} = \left( \frac{A_c}{A_d} \right)^{-(1-\alpha)}.
\]

This equation formalizes the natural idea that the input produced with more productive machines, will be relatively cheaper.

\(^{14}\)In particular, use the first-order condition \( (1 - \alpha) p_j L_j^{\alpha} \int_0^1 A_{ji}^{1-\alpha} x_{ji}^\alpha di = w \) together with (9).
2.2.2 Laissez-faire equilibrium: directed innovation

We next endogenize productivity by linking productivity growth to R&D investments in clean versus dirty technologies (for clarity, we reintroduce the time subscripts $t$).

If a scientist succeeds in innovation, she discovers a new machine that is $(1 + \gamma)$ times more productive than its previous vintage, $A_{jit-1}$. Therefore, denoting the mass of scientists directing their effort to sector $j$ by $s_{jt}$, and recalling that scientists targeting sector $j$ are randomly allocated across machines in that sector, the average productivity in sector $j$ at time $t$ evolves over time according to

$$A_{jt} = (1 + \gamma \eta_j s_{jt}) A_{jt-1}.$$  \hspace{1cm} (12)

where market clearing for scientists implies that at any period $t$:

$$s_{ct} + s_{dt} = 1.$$ \hspace{1cm} (13)

The expected profit $\Pi_{jt}$ for a scientist engaging in research in the sector $j$ is therefore given by

$$\Pi_{jt} = \eta_j \int (1 - \alpha) \alpha p_{jt}^{1/\alpha} L_{jt} (1 + \gamma) A_{jt-1} \, di$$

$$= \eta_j (1 + \gamma) (1 - \alpha) \alpha p_{jt}^{1/\alpha} L_{jt} A_{jt-1}.$$ \hspace{1cm} (14)

Consequently, the relative benefit from undertaking research in sector $c$ relative to sector $d$ is governed by the ratio:

$$\frac{\Pi_{ct}}{\Pi_{dt}} = \frac{\eta_c}{\eta_d} \left( \frac{p_{ct}}{p_{dt}} \right)^{1/\alpha} \frac{L_{ct}}{L_{dt}} \frac{A_{ct-1}}{A_{dt-1}}.$$ \hspace{1cm} (15)

Thus a scientist’s incentive to innovate in the clean versus the dirty sector machines, is shaped by three forces: (i) a direct productivity effect (captured by the term $A_{ct}/A_{dt}$), which pushes towards innovating in the sector with higher productivity; (ii) the price effect (captured by the term $(p_{ct}/p_{dt})^{1/\alpha}$), which encourages innovation towards the sector with higher prices, which from (11) is the relatively backward sector; (iii) a market size effect (captured by the term $L_{ct}/L_{dt}$), which pushes towards innovating in the sector which employs more labor in equilibrium.
In other words, the market size effect directs innovation towards the largest market. But the largest market is in turn determined by relative productivities and the elasticity of substitution between the two inputs. The price effect will direct innovation towards the more scarce and therefore more expensive input. The more substitutable the two inputs are, the more important the market size effect compared to the relative price effect.

More formally, plugging the expression for the equilibrium production of machines gives that the equilibrium production of intermediary input $j$ as:

$$Y_{jt} = (p_{jt})^{\frac{\alpha}{\alpha}} A_{jt} L_{jt}$$  \hspace{1cm} (16)

Using this last expression in the ratio of relative price and the definition of $\varphi$ (recall that $\varphi \equiv (1 - \alpha) (1 - \varepsilon)$) gives us the relationship between relative productivities and relative employment as:

$$\frac{L_{ct}}{L_{dt}} = \left( \frac{p_{ct}}{p_{dt}} \right)^{-\frac{\varphi - 1}{\alpha}} \frac{A_{dt}}{A_{ct}} = \left( \frac{A_{ct}}{A_{dt}} \right)^{-\varphi}.$$  \hspace{1cm} (17)

This in turn implies that the market size effect creates a force towards innovation in the more backward sector when $\varepsilon < 1$, and in the more advanced sector when $\varepsilon > 1$. More specifically, combining (11), (15) and (17), we obtain

$$\frac{\Pi_{ct}}{\Pi_{dt}} = \frac{\eta_{c}}{\eta_{d}} \left( 1 + \gamma \eta_{c} s_{ct} \right)^{-\varphi - 1} \left( \frac{A_{ct}}{A_{dt}} \right)^{-\varphi},$$

which yields the following lemma:

**Lemma 1** In the laissez-faire equilibrium, innovation at time $t$ can occur in the clean sector only when $\eta_{c} A_{ct-1}^{-\varphi} > \eta_{d} (1 + \gamma \eta_{c} s_{ct})^{\varphi + 1} A_{dt-1}^{-\varphi}$, in the dirty sector only when $\eta_{c} (1 + \gamma \eta_{d} s_{dt})^{\varphi + 1} A_{ct-1}^{-\varphi} < \eta_{d} A_{dt-1}^{-\varphi}$, and can occur in both sectors when $\eta_{c} (1 + \gamma \eta_{d} s_{dt})^{\varphi + 1} A_{ct-1}^{-\varphi} = \eta_{d} (1 + \gamma \eta_{c} s_{ct})^{\varphi + 1} A_{dt-1}^{-\varphi}$.

The proof of this lemma is provided in Appendix A, which also provides a complete characterization of the allocation of scientists and innovation levels.

The important result is that innovation will favor the more advanced sector when $\varepsilon > 1$. In particular, in this case $\varphi < 0$ and the direct productivity and market size effects are stronger than the price effect. In contrast, it will favor the most backward sector when $\varepsilon < 1$; in this case $\varphi > 0$ and the direct productivity effect is weaker than the price effect and the market size effect (which now reinforce each other). Using Lemma 1 we can establish:
Proposition 1  Under laissez-faire, if Assumption 1 is satisfied, there is a unique equilibrium. If \( \varepsilon > 1 \), innovation always occurs in the dirty sector only. If \( \varepsilon < 1 \) innovation first occurs in the clean sector, then occurs in both sectors and asymptotically the share of scientists devoted to the clean sector is given by \( s_c = \frac{\eta_d}{(\eta_c + \eta_d)} \).

Proof. See Appendix B. ■

The intuition for this proposition follows from Lemma 1. When the two inputs are substitutes (\( \varepsilon > 1 \)), innovation starts in the dirty sector, which is more advanced initially (Assumption 1). This increases the gap between the dirty and clean sectors and the initial pattern of equilibrium is reinforced. In contrast, when the two inputs are complements (\( \varepsilon < 1 \)), because the price effect dominates, innovation initially takes place in the more backward clean sector. This, however, reduces the technology gap between the two sectors and ultimately the equilibrium involves innovation in both sectors. Once this point is reached, the equivalent involves \( s_c = \frac{\eta_c}{(\eta_c + \eta_d)} \), which ensures that both sectors grow at the same rate (see Appendix B). In particular, in this case average quality levels in both sectors, \( A_c \) and \( A_d \), grow at the same asymptotic rate \( \gamma \tilde{\eta} \), where \( \tilde{\eta} \equiv \frac{\eta_c \eta_d}{(\eta_c + \eta_d)} \).

2.3 Directed technical change and environmental disaster

A main fear by climatologists is that the environment may deteriorate so much over time that it reaches a point of no return. In our environment equation (6), this notion is captured by the fact that if environmental quality \( S_t \) reaches 0 in finite time, it remains at 0 forever after. Motivated by this feature, we define a notion of environmental disaster, which will be useful for developing the main intuitions, before we provide a more complete characterization of optimal environmental policy.

Definition 1 An environmental disaster occurs if \( S_t = 0 \) for some \( t < \infty \).

An environmental disaster is highly detrimental to welfare and there is an infinite willingness to avoid it (i.e., \( \frac{\partial u}{\partial S}(c,0) = \infty \)). In this subsection, we focus on how a simple policy of “redirecting technical change” can avoid an environmental disaster (when it would otherwise take place in the laissez-faire equilibrium). We will then highlight the role of directed technical change by comparing the results to a model in which scientists cannot direct their research to different sectors.
2.3.1 Disaster under laissez-faire

Output of the two inputs and the final good in the laissez-faire equilibrium can be written as (again dropping time subscripts simplify notation):

\[ Y_c = (A_c^\varphi + A_d^\varphi) - \frac{\varphi}{\varphi} A_c A_d^{\alpha + \varphi}, \]
\[ Y_d = (A_c^\varphi + A_d^\varphi) - \frac{\varphi}{\varphi} A_c^{\alpha + \varphi} A_d, \]
\[ Y = (A_c^\varphi + A_d^\varphi) - \frac{1}{\gamma} A_c A_d. \]

(18)

These expressions, together with Proposition 1, give the long-run (asymptotic) growth rate of the dirt input. Whether an environmental disaster takes place in the laissez-faire equilibrium then depends on whether this growth rate is greater than the “regeneration rate” \( \delta \) of the environment. In particular, when \( \varepsilon > 1 \), innovation always occurs in the dirty sector, and \( Y_d \) and \( A_d \) grow at the rate \( \gamma \eta_d \). When \( \varepsilon < 1 \), both inputs and final output grow at the rate \( \gamma \eta \).

Therefore:

**Proposition 2** The laissez-faire equilibrium will lead to an environmental disaster if the regeneration rate of the environment, \( \delta \), is sufficiently low, namely, if \( \delta < \gamma \eta_d \) when \( \varepsilon > 1 \), and if \( \delta < \gamma \eta \) when \( \varepsilon < 1 \).

This proposition states that an environmental disaster is more likely (it occurs for a wider range of parameters) when the two inputs are substitutes, because in this case the market does not generate any incentive for research into clean sector. We next discuss how simple government policies can prevent environmental disasters and how the effects of these policies differ depending on the degree of substitution between the two inputs.

Suppose first that the government can impose a profit tax \( q_t \) on the dirty input production, with the proceeds redistributed lump-sum to the representative household (which is equivalent to a subsidy to scientists working on clean inputs with lump-sum taxes). The expected profit from undertaking research in sector \( d \) then becomes

\[ \Pi_{dt} = (1 - q_t) \eta_d (1 + \gamma) \frac{1 - \frac{1}{\alpha} p_{dt}^{-1} \sigma L_{dt} A_{dt-1}}{\alpha}. \]
while \( \Pi_{ct} \) is still given by (14). This immediately implies that a sufficiently high profit tax can divert innovation away from the dirty sector.\(^{15}\) Moreover, while this tax is implemented, the ratio \( A_{ct}/A_{dt} \) will grow at a rate \( \gamma \eta_c \). The implications of the tax rate then depend on the degree of substitutability between the inputs.

When the two inputs are substitutes (\( \varepsilon > 1 \)), a temporary profit tax (maintained for the necessary number of periods) will be sufficient to redirect all research to the clean sector. More specifically, while the profit taxes being implemented, the ratio \( A_{ct}/A_{dt} \) will increase, and when it has become sufficiently high, it will be profitable for scientists to direct their research to the clean sector even without the tax.\(^{16}\) Then (18) implies that \( Y_d \) will grow asymptotically at the same rate as \( A_c^{\alpha+\varphi} \), and thus if \( \varepsilon \geq 1/(1 - \alpha) \) (or \( \alpha + \varphi \leq 0 \)), \( Y_d \) will not grow in the long-run and therefore, as long as the initial environmental quality is sufficiently high (so that an environmental disaster does not happen during the implementation of the temporary tax), a temporary profit tax policy will be sufficient to avoid an environmental disaster. In contrast, if \( \varepsilon < 1/(1 - \alpha) \) (or \( \alpha + \varphi > 0 \)), (18) implies that even after all research is directed to the clean inputs, \( Y_d \) will keep growing at rate \( (1 + \gamma \eta_c)^{\alpha+\varphi} - 1 \).\(^{17}\) In this case, whether environmental disaster is avoided depends on whether this growth rate is greater than \( \delta \).

When the two inputs are complements (\( \varepsilon < 1 \)), the more backward sector will always catch up with the more advanced sector and in the long run equilibrium, innovation will take place in both sectors, ensuring equal growth rate at the rate \( \gamma \bar{\eta} \) (regardless of the presence of a profit tax). But this is exactly the same growth rate as in Proposition 1, implying that a profit tax

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\(^{15}\)In particular, following the analysis in Appendix A, to implement a unique equilibrium where all scientists direct their research to the clean sector, the profit tax \( q_t \) must satisfy

\[
q_t > 1 - (1 + \gamma \eta_d)^{\varepsilon+1} \frac{\eta_c}{\eta_d} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{-\varphi} \quad \text{if } \varepsilon \geq \frac{2 - \alpha}{1 - \alpha}
\]

\[
q_t \geq 1 - (1 + \gamma \eta_c)^{-1} \frac{\eta_c}{\eta_d} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{-\varphi} \quad \text{if } \varepsilon < \frac{2 - \alpha}{1 - \alpha}
\]

\(^{16}\)In particular, the temporary tax needs to be imposed for \( D \) periods where \( D \) is the smallest integer such that:

\[
\frac{A_{ct+D-1}}{A_{dt+D-1}} > (1 + \gamma \eta_d)^{\varphi+1} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varepsilon}} \quad \text{if } \varepsilon \geq \frac{2 - \alpha}{1 - \alpha}
\]

\[
\frac{A_{ct+D-1}}{A_{dt+D-1}} \geq (1 + \gamma \eta_c)^{-1} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varepsilon}} \quad \text{if } 1 < \varepsilon < \frac{2 - \alpha}{1 - \alpha}
\]

\(^{17}\)Sustained growth of dirty inputs with all technical change taking place in the clean sector is due to the fact that improvements in the quality of machines in the clean sector increases the efficiency in the production of the final good, which is an input into the production of the dirty input.
cannot prevent an environmental disaster\textsuperscript{18}.

This discussion establishes the following proposition:

**Proposition 3** When the two inputs are complements, a profit tax cannot prevent an environmental disaster. When the two inputs are substitutes, a temporary profit tax will prevent a disaster when the initial environmental quality is sufficiently high and in addition either if 
\( \varepsilon \geq 1 / (1 - \alpha) \) or if 
\( 1 < \varepsilon < 1 / (1 - \alpha) \) and 
\( (1 + \gamma \eta_e)^{\alpha + \varphi} - 1 < \delta \).

This proposition therefore shows that when the two inputs are substitutes, redirecting technical change using a temporary policy intervention can be sufficient to avoid a disaster. Nevertheless, the policy intervention still has economic costs because during the period of adjustment (while productivity in the clean sector is catching up with that in the dirty sector), final output increases more slowly than if innovation were directed towards the dirty sector. To see this, let us define a simple measure of the cost of intervention as number of periods 
\( T \) necessary for the economy under the policy intervention to reach the same level of output as it would have done without intervention (more specifically, this is the number of periods of "delay" in output growth). This measure \( T \) (starting at time \( t \)) is then the smallest integer such that:

\[
\frac{(1 + \gamma \eta_e)^T}{((1 + \gamma \eta_e)^T \varphi A^{\varphi}_{dt-1} + A^{\varphi}_{ct-1})^{1/\varphi}} \geq \frac{(1 + \gamma \eta_d)}{(A^{\varphi}_{ct-1} + (1 + \gamma \eta_d)^{\varphi} A^{\varphi}_{dt-1})^{1/\varphi}}
\]

or equivalently,

\[
T = \left\lfloor \ln \left( \frac{((1 + \gamma \eta_d)^{-\varphi} - 1) A^{\varphi}_{dt-1}}{A^{\varphi}_{ct-1} + 1} \right) \right\rfloor^{-\varphi \ln (1 + \gamma \eta_e)} \quad (19)
\]

One can first show that \( T \geq 1 \) if \( A_{dt-1}/A_{ct-1} \geq 1 \). In other words, as conjectured above, we indeed need more than one period once innovation is directed towards the clean sector in order to achieve the same output growth as would be achieved during one period if innovation had been maintained in the dirty sector.

The above equation (19) also shows that \( T \) is increasing in \( A_{dt-1}/A_{ct-1} \), so that the larger gap between the initial quality of dirty and clean machines the longer the lower growth intervention phase. It can also be verified that \( T \) is increasing in the elasticity of substitution between

\textsuperscript{18}The formal reasoning goes along the same line as the proof of 1, namely: no matter in which sector innovation is directed first, innovation must end up occurring in both sectors, which in turn implies that the only possible asymptotic growth rate is equal to \( \gamma \bar{\eta} \).
the two inputs \( \varepsilon \): the more substitutable the two inputs, i.e. the larger \( \varepsilon \), the more growth is reduced during the intervention phase. Intuitively, if the two inputs are close substitutes, final output production relies mostly on the most productive input, and therefore productivity improvement in the laggard sector (which is what happens the intervention phase) will have less of an effect on the overall productivity.

Moreover, the higher is \( \varepsilon \), the more \( T \) increases with \( A_{d,t-1}/A_{c,t-1} \). This implies that delaying the starting date of the intervention is costly when the two inputs are substitutes and this cost increases with the degree of substitution. The longer is the intervention delayed, not only is there more environmental degradation, but the gap between the quality of machines and clean and dirty sectors widens, so the temporary profit tax needs to be imposed for longer and the economy will suffer slower growth for a greater number of periods. We next illustrate this cost by computing the value of \( T \) for different values of the elasticity of substitution \( \varepsilon \) and different starting dates of intervention.\(^{19}\) For illustration purposes, we to the following parameter values: \( \eta_c = \eta_d = 0.025 \) and \( \gamma = 1 \) so that the growth rate \( \gamma \eta_j \) is equal to 2.5%; \( \psi = \alpha = 1/3 \) (same as the capital coefficient in aggregate production functions), and we choose the initial productivity values \( A_{c0} = 1, A_{d0} = 3 \). Table 1 below shows that \( T \) increases fast with the delay and/or the substitutability between the two inputs.

| delay \( \varepsilon \) | 2 | 5 | 10 | 20 | 5 | 8 | 21 | 43 | 52 | 10 | 13 | 29 | 53 | 62 | 15 | 18 | 38 | 63 | 72 | 20 | 23 | 47 | 73 | 82 |
|----------------------|---|---|----|----|---|---|-----|-----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| 0                    | 3 | 13| 33 | 42 | 5 | 8 | 21 | 43 | 52 | 10 | 13 | 29 | 53 | 62 | 15 | 18 | 38 | 63 | 72 | 20 | 23 | 47 | 73 | 82 |

Table 1

To summarize our discussion in this section, we have shown that a simple policy intervention that consists on “redirecting” technical change towards environment friendly technologies can help prevent an environmental disaster. Our analysis also highlights the idea that delaying intervention can actually be quite costly, not only because it further destroys the environment but also because it widens the gap between the dirty and clean technologies, thereby inducing a longer period of catch-up with slower growth.

\(^{19}\)We discuss the cost of delayed intervention more systematically in terms of welfare losses in the next section.
2.3.2 Comparison with a model without directed technical change

Let us next consider a version of the models studied so far but without directed technical change, to highlight how the endogeneity of the direction of technical change is crucial for the effects of temporary policy interventions. In particular, consider the same environment, but suppose that scientists are randomly allocated across sectors so as to ensure equal growth in the qualities of clean and dirty machines (at the rate $\gamma \bar{\eta}$). This implies that a fraction $s_c = \eta_d/(\eta_c + \eta_d)$ of scientists are allocated to the clean sector.\textsuperscript{20} Consequently, the production of the dirty input will always grow at rate $\gamma \bar{\eta}$, and an environmental disaster will occur whenever $\gamma \bar{\eta} > \delta$. Comparing this with 3 then gives the following result:

**Proposition 4** If $\varepsilon < 1$, the range of $\delta$ for which an environmental disaster will occur for any initial environmental quality is the same with and without directed technical change, with or without a profit tax.

If $\varepsilon \geq 1/(1 - \alpha)$, the range of $\delta$ for which a disaster will occur under laissez-faire for any initial environmental stock is greater with directed technical change than without (equal to $\delta < \gamma \eta_d$ with directed technical change instead of $\delta < \gamma \bar{\eta}$ without directed technical change); but allowing for a temporary profit tax makes it becomes smaller with directed technical change (a disaster is avoided for all $\delta$ with directed technical change whereas it is avoided only if $\delta < \gamma \bar{\eta}$ without directed technical change).

It can also be verified that if $1 < \varepsilon < 1/(1 - \alpha)$, the growth rate of the dirty input can be lowered down to $(1 + \gamma \eta_c)^{\alpha + \varepsilon} - 1$ with directed technical change but is always equal to $\gamma \bar{\eta}$ without directed technical change. Which of these two rates is higher, depends upon the parameters of the model. Therefore, for sufficiently high elasticities of substitutions, directed technical change makes environmental disasters more likely because it encourages research to be directed to the more advanced (dirty) sectors, but it also facilitates the prevention of such disasters.

2.4 Optimal policy design

We have so far studied the behavior of the laissez-faire equilibrium and discussed how environmental disaster may be avoided. In this subsection, we characterize the optimal allocation

\textsuperscript{20}If we assume that half of the scientists are allocated to the clean sector, the qualitative results would be similar, though the expressions would become more complicated.
of resources in this economy and discuss how it can be decentralized by using profit and input taxes. The socially planned (optimal) allocation will “correct” for two externalities: the environmental externality exerted by dirty input producers, and the knowledge externalities coming from the R&D sector: namely, in the decentralized equilibrium innovators care only about profits next period, they do not internalize the effects of their research on the productivity of their own machines next periods. In addition the planner will correct for the traditional static monopoly distortion in the price of machines.

2.4.1 Solving the social planner’s problem

The social planner’s problem is one of choosing a dynamic path of final good production $Y_t$, consumption $C_t$, intermediary input productions $Y_{jt}$, expected machines production $x_{jit}$, labor share allocation $L_{jt}$, scientists allocation $s_{jt}$, environmental quality $S_t$, and quality of machines $A_{jit}$, that maximizes the intertemporal utility of the representative consumer

$$\sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} u(C_t, S_t)$$

subject to (1), (2), (4), (5), (6), (12), (13), and

$$C_t = Y_t - \psi \left( \int_0^1 x_{cit} di + \int_0^1 x_{dit} di \right).$$  \hspace{1cm} (20)

Letting $\lambda_t$ denote the Lagrange multiplier for (1), i.e the shadow value of one unit of final good production. Taking first order conditions with respect to $Y_t$ we immediately see that this shadow value is also equal to the Lagrange multiplier for (20), i.e to the shadow value of one unit of consumption. Now, taking first order condition with respect to $C_t$ yields

$$\lambda_t = \frac{1}{(1+\rho)^t} \frac{\partial u}{\partial C} (C_t, S_t),$$  \hspace{1cm} (21)

so that the shadow value of the final good is simply equal to its marginal utility.

The ratio $\lambda_{jt}/\lambda_t$ can then be interpreted as the shadow price of input $j$ at time $t$ (relative to the price of the final good). To emphasize this interpretation, we will denote this ratio by $p_{jt}$.

Taking first order condition on $x_{jit}$,\(^{21}\) then plugging the equilibrium expressions for $x_{ji}$ in

\(^{21}\)Taking first order condition on $x_{ji}$ then leads to a production of machines $j,i$ given by

$$x_{jit} = \left( \frac{\alpha}{\psi} p_{jt} \right)^{\frac{1}{1-\alpha}} A_{jit} L_{jt}$$
the expression for the production of intermediary input $j$, leads to

$$Y_{jt} = \left( \frac{\alpha}{\psi p_j} \right)^{\frac{\alpha}{1-\alpha}} A_{jt} L_{jt}$$

so that for given price, average technology and labor allocation, the production of the intermediary input is just scaled up by a factor $\alpha^{\frac{\alpha}{1-\alpha}}$ compared to the decentralized equilibrium analyzed in the previous subsection (this simply results from the subsidy on the use of machines).

Letting $\omega_t$ denote the Lagrange multiplier for the environmental equation (6), the first order condition with respect to $S_t$ gives

$$\omega_t = \frac{1}{(1 + \rho)^t} \frac{\partial u}{\partial S}(C_t, S_t) + (1 + \delta) \omega_{t+1},$$

which just states that the price of one unit of environmental quality at time $t$ is equal to the marginal utility that it generates in this period plus the price of $(1 + \delta)$ units of environmental quality at time $t + 1$ (as one unit of environmental quality at time $t$ generates $1 + \delta$ units at time $t + 1$), so that the price of unit of environmental quality is equal to the marginal utility this unit generates in all subsequent periods. In particular, this implies that if for all $\tau > T$, $S_\tau > \bar{S}$ then $\omega_t = 0$ for all $t > T$.

Taking first order conditions with respect to $Y_{ct}$ and $Y_{dt}$ gives

$$Y_{ct}^{-\frac{1}{\varepsilon}} \left( Y_{ct}^{1-\varepsilon} + Y_{dt}^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}} = p_{ct},$$

$$Y_{dt}^{-\frac{1}{\varepsilon}} \left( Y_{ct}^{1-\varepsilon} + Y_{dt}^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}} - \frac{\omega_{t+1} \xi}{\lambda_t} = p_{dt}.$$

Thus compared to the laissez-faire equilibrium, the social planner introduces a wedge $-\frac{\omega_{t+1} \xi}{\lambda_t}$ between the marginal product of the dirty input in the production and its price. $-\frac{\omega_{t+1} \xi}{\lambda_t}$ is the environmental cost of the production of one unit of the dirty input evaluated in terms of units of the final good at time $t$: since one unit of dirty production at time $t$ destroys $\xi$ units of environmental quality at time $t + 1$. This wedge is equivalent to a tax

$$\tau_t = \frac{\omega_{t+1} \xi}{\lambda_t \rho_{dt}}$$

which is equivalent to introducing a subsidy $1 - \alpha$, in the use of monopolistically produced machines (so that the price that intermediary inputs producers is equal to $(1 - (1 - \alpha)) \frac{\xi}{\alpha} = \psi$ the cost of producing one unit of machine.
on the use of dirty input by the final good producer. This tax will be higher for a higher price of environmental quality, a lower marginal utility of consumption at time \( t \) and a lower price of dirty input at time \( t \).\(^{22}\)

Finally, the social planner must correct for the knowledge externality. In Appendix D we show that the social planner’s problem is unchanged if we replace the growth equations (12) by the average growth equation

\[
A_{jt} = \left(1 + \gamma_j s_{jt}\right) A_{jt-1}
\]  

(25)

for \( j = c, d \). Then, let \( \mu_{jt} \) denote the corresponding Lagrange multiplier, i.e the shadow value for one unit of average productivity in sector \( j \) at time \( t \).

Taking first order conditions with \( A_{jt} \) in the modified social planner’s problem, we then obtain:

\[
\mu_{jt} = \lambda_t \left( \frac{\alpha}{\psi_j} \right) \frac{\alpha}{\psi_j} (1 - \alpha) p_{jt}^{\frac{1}{\psi}} L_{jt} + \left(1 + \gamma_j s_{jt+1}\right) \mu_{j,t+1}.
\]  

(26)

In words: the shadow value of a unit of \( j \)-productivity is equal to its marginal contribution to time-\( t \) utility plus its shadow value at time \( t + 1 \) times \( 1 + \gamma_j s_{jt+1} \) (the number of units of productivity created out of it at time \( t + 1 \)): this term captures the intertemporal knowledge externality.

At the optimum, scientists will be allocated towards the sector with higher social gain \( \gamma_j \mu_{jt} A_{jt-1} \) from innovation. Using (26), we then have that the social planner will allocate scientists to clean sector whenever the ratio:

\[
\frac{\eta_c (1 + \gamma_c s_{ct})^{-1} \sum_{\tau \geq t} \lambda_{ct}^{1/\omega} L_{ct} A_{ct}}{\eta_d (1 + \gamma_d s_{dt})^{-1} \sum_{\tau \geq t} \lambda_{dt}^{1/\omega} L_{dt} A_{dt}}
\]  

(27)

is higher than 1. This contrast with the decentralized outcome where scientists are allocated according to the private value of innovation, that is to the ratio of the first term in the two sums.\(^{23}\) In Appendix D we show that the optimal allocation of scientists can then be

\[^{22}\text{In Appendix D we show that the tax is implicitly uniquely defined by} \]  

\[
\tau_t^{1-\epsilon} = \left( \frac{\omega_{t+1}^{1/\gamma_t}}{\lambda_t} \right)^{1-\epsilon} \left( 1 + \left( \frac{A_{dt}^{1-\alpha}}{(1 + \tau_t A_{ct})^{1-\alpha}} \right)^{1-\epsilon} \right) .
\]

In particular this optimal tax is increasing in the ratio of relative quality \( \frac{A_d}{A_c} \).

\[^{23}\text{This knowledge externality is emphasized here in our model where scientists capture the profits for only one period, but it would still be present in any Schumpetarian model, where creative destruction creates a gap between social and private value of innovation.} \]
implemented through an appropriate profit tax. Thus overall, we can establish:

**Proposition 5** The government can implement the social optimum through a tax on the use of the dirty input, a tax/subvention on the profits realized in the dirty sector and a subsidy in the use of all machines (all taxes/subsidies are reversed/financed in a lump sum way to the consumers)

That we need both, an input tax and a profit tax to implement the social optimum, can be explained as follows: the profit tax deals with future environmental externalities by directing innovation and therefore technical progress towards the clean sector, whereas the input tax deals more directly with the current environmental externality by reducing current production of the dirty input which causes this externality. True, by reducing input production the input tax also negatively affects scientists’ incentives to innovate in the dirty sector. However, using the sole input tax to deal with both, current and future environmental externalities, would force the social planner to set a higher input tax than if both the input and the profit tax were allowed. This in turn would inhibit current production to an excessive extent.

This is a main result of the paper: in particular, it implies that it is not optimal to only use a carbon tax to deal with global warming; one should also use instruments (profit tax, R&D subsidies,...) that direct innovation investments across sectors.

### 2.4.2 Social optimum: characterization of the solution

In subsection 2.3 above we showed that a temporary profit tax could prevent a disaster in the substitutable case but not in the complementary case. Here we shift attention to the optimal tax schedule, and show that it inherits the same property, namely that of being temporary if only if the two inputs are sufficiently close substitutes.

More formally, recall that the optimal input tax schedule is given by \( \tau_t = \frac{\omega_{t+1}}{\lambda \rho dt} \), where \( \omega_{t+1} \), the shadow value of one unit of environmental quality at time \( t + 1 \) is equal to the discounted marginal utility of environmental quality as of period \( t + 1 \), namely

\[
\omega_{t+1} = \sum_{v=t+1}^{\infty} (1 + \delta)^{v-(t+1)} \frac{1}{(1 + \rho)^v} \frac{\partial u}{\partial S} (C_{v+1}, S_{v+1}),
\]

\(^{24}\)We must have: \( \lim_{t \to \infty} \omega_t < \infty \), otherwise all \( \omega_t \) would be infinite.
so that, using (21), we get

\[ \tau_t = \frac{\xi}{1+\rho} \sum_{v=t+1}^{\infty} \left( \frac{1+\delta}{1+\rho} \right)^{v-(t+1)} \frac{\partial u}{\partial S} (C_{v+1}, S_{v+1}) \frac{\partial u}{\partial C} (C_t, S_t). \]  

(28)

This expression shows in particular that when \( S_t \) becomes sufficiently large, the optimal tax on dirty input falls down to zero: this follows from the fact that \( \frac{\partial u}{\partial S} (C_{t+1}, S_{t+1}) \) is equal to zero for \( S_{t+1} \) sufficiently large. This, in turn, has implications on how the optimal dynamics tax schedule depends upon the degree of substitutability between the clean and the dirty input basket.

When the two input baskets are sufficiently close substitutes in producing final output (\( \varepsilon \) is sufficiently large) we know that taxing the dirty input activities induces a permanent shift of production and innovation towards clean activities. As a result, \( S_t \) goes to infinity over time. In that case, the optimal tax will be equal to zero in finite time, and so will the profit tax when the dirty technology has become sufficiently backward. In contrast, when \( \varepsilon \) is small, production of the dirty input will grow over time which calls for permanent taxation of the dirty input when the regeneration rate of the environment is low.

In Appendix E we establish:

**Proposition 6:** If \( \varepsilon \geq 1/(1-\alpha) \), the discount rate is sufficiently small and \( \delta < \gamma \eta_c \), then the optimal input tax \( \tau_t \) and the optimal profit tax are both temporary. In the long run innovation occurs only in the clean sector and the economy grows at a rate \( \gamma \eta_c \); if \( \varepsilon < 1 \), and \( \delta < \gamma \tilde{\eta} \), then the optimal input tax and the optimal profit tax are both permanent. Innovation occurs always in both sectors and the economy’s growth rate is bounded above by the regeneration rate of the environment \( \delta \).

**Sketch of the proof** The detailed proof is developed in Appendix E, here we just provide a sketch of the proof. Consider first the substitutability case. The temporarity of the optimal input tax in that case has been already discussed above. As for the optimal profit tax, the proof proceeds in four steps: (i) initially, a sufficient tax on the dirty sector profits will divert innovation towards the clean sector; (ii) thus at some point \( A_c \) will grow above \( A_d \) at which point innovation will occur in the clean sector with no more need for a profit tax; (iii) for \( \varepsilon > 1/(1-\alpha) \) production of the dirty sector will consequently decrease (our above discussion) so that eventually \( S \) will become higher than \( \overline{S} \). From then on the optimal input tax becomes
0 and eventually, the economy will mainly rely on the clean sector, thus generating a long-run growth rate which will be the same as the growth rate of $A_c$, namely $\gamma \eta_c$. This policy is growth maximizing for $\delta$ sufficiently low, and therefore it will also be welfare maximizing for $\delta$ sufficiently low and for sufficiently low discount rate.

In the complementarity case, the long-run growth rate of final output is the minimum of the long-run growth rates of the two input productions whenever that minimum is less than $\delta$. As innovation in clean technology increases the incentive to both produce the dirty input and to innovate in dirty technologies, a combination of the input tax and the profit tax should be maintained permanently if $\delta$ is too low for growth to be sustainable under laissez-faire.25

### 2.4.3 Some simulations

To illustrate Proposition 5 we simulate the optimal dynamic solution and outcome in Figure 1 below, we use a Bernoulli utility function of the form26

$$u(C_t, S_t) = \frac{\left(2\sqrt{5 \min(S_t, 5)} - \min(S_t, 5)\right) C_t^{1-\psi}}{1-\psi}.$$  

We set parameter values $\rho = 0.01$ (a low discount rate); $\varepsilon = 5$; $\xi = 0.03$, $\delta = 0.011$ (we choose $\delta$ sufficiently small that the laissez-faire equilibrium leads to a disaster); and $S_1 = 4$. And we keep fixing $\eta_c = \eta_d = 0.025$, $\gamma = 1$, and $\psi = \alpha = \frac{1}{5}$; $A_c0 = 1$, $A_d0 = 3$.

25In Appendix E, we also show that: (i) if $\varepsilon \in (1, \frac{1}{1+\beta})$, the optimal input tax will be temporary if $(1 + \gamma \eta_c)^{\alpha + \psi} - 1 < \delta$. The economy still grows at $\gamma \eta_c$; (ii) if $\gamma \eta_c < \delta < \gamma \eta_d$ and $\varepsilon > 1$ then the government may not want to induce a permanent switch to clean innovation (in this case switching to innovation in clean technologies is not growth maximizing and hence for low discount rate not necessarily welfare maximizing).

26This is a CRRA function with a relative risk aversion parameter of 1.4 (within the range of values used in the macro literature), but which also factors in the environmental externality. Its being multiplicative with respect to consumption $C_t$ makes our model equivalent to one where environmental depletion directly affects net final production, or to a model where environmental damages need to be compensated for by devoting some fraction of final output to cleaning the environment. We set $N = 5$ - so the initial environmental quality is not too low- but the square roots imply that as $S$ depletes the extent of environmental damages increases fast.
Figure 1A shows that it is optimal to immediately allocate all scientists to clean intermediate sectors and to maintain this allocation forever after. Figure 1B shows that while $A_d$ remains constant, $A_c$ grows steadily over time. The environment stock first decreases (as dirty input production is initially high), however it eventually initiates a recovery. Finally, final good consumption grows smoothly over time. Figure 1C shows that the optimal input tax schedule decreases rapidly towards zero over time, whereas Figure 1D shows that the optimal profit tax also decreases rapidly over time, and reaches zero after 28 periods.

2.4.4 Social optimum: comparison with a world without directed technical change and cost of delaying policy

Taking into account the possibility of directed technical dramatically affects the optimal policy and the resulting optimal consumption pattern particularly in the substitutability case. As
in subsection 2.3, consider a world where technical change does not respond to economic incentives, and where a fixed fraction $s_c = \eta_c / (\eta_c + \eta_d)$ of scientists is always allocated to innovation in clean sector’s machines, the dirty technology $A_c$ will keep growing at rate $\gamma \bar{\eta}$. If this rate is higher than the regeneration rate of the environment, the only way to avoid a disaster is to implement a permanent tax on the dirty input, and to have this tax increase over time. In any case, the growth rate of the economy will be bounded above by $\gamma \bar{\eta} < \gamma \eta_c$ (in the substitutability case) or by $\delta$ (in the complementarity case). This establishes:

**Proposition 7** Without directed technical change, the growth rate of the economy is bounded above by $\gamma \bar{\eta}$. Moreover, the optimal tax on dirty input is permanent and increasing over time if $\gamma \bar{\eta} > \delta$.

Note that in the substitutability case, factoring in directed technical change (DTC) pushes towards higher taxes on dirty input and innovation in the short run: the sooner we induce the switch towards clean technology, the smaller the "transition cost" from a dirty to a clean economy. Moreover, since the tax on dirty input and innovation maintains $A_d$ constant under DTC while pushing $A_c$ to grow faster over time, the optimal input tax is decreasing over time whereas it is increasing in the absence of DTC.

To illustrate how the optimal response would look like if the allocation of scientists did not respond to economic incentives, we simulate the previous economy but now assuming that a constant fraction $\eta_c / (\eta_c + \eta_d)$ of scientists is allocated each period to the clean sector. By contrast with the case with DTC (Figure 1 above), here the clean average productivity $A_c$ does not catch up with the dirty average productivity $A_d$; moreover the environmental quality $S$ keeps decreasing over time; finally, the optimal input tax must now increase over time and always be significantly higher than before, the reason being that one can no longer rely on the profit tax and the resulting reallocation of innovation effort towards clean technologies, in order to preserve the environmental quality.
We can push forward this idea that taking into account DTC pushes towards strong policy responses in the short run by computing the cost of delaying intervention. In the following we simulate the same economy but letting the profit tax and the input tax be 0 for the first 20 periods.\footnote{In order to make the comparison more meaningful we keep the same optimal subsidy on intermediary inputs that relieves the monopolist distortion in the first 20 periods.} Figure 3A shows that all scientists allocate to clean sectors only from period 21 onwards. Figure 3B shows that $A_d$ increases only during the first twenty periods, and $A_c$ increases only after twenty periods. More interestingly, the environmental quality decreases fast during the first twenty periods and then recovers only very slowly over the subsequent time interval. Finally, consumption must be substantially cut back at date 20 and then at all subsequent times remains always strictly lower than its level if policy intervention had not been delayed. Figure 3C shows that delaying policy intervention leads to setting a much higher (but still decreasing over time) input tax schedule; and finally Figure 3D shows that here the input tax is sufficient for making sure that all scientists allocate to clean intermediate sectors,
so that the optimal profit tax is zero.

To evaluate what welfare cost this delay represent, we compute by how much consumption without delay should be reduced per period, to replicate the welfare resulting from delaying policy intervention. Table 2 below shows the corresponding percentage reductions in consumption for different values of the substitutability parameter $\varepsilon$ and also for different delays. Not surprisingly, delay costs increase with the duration of the delay.

<table>
<thead>
<tr>
<th>delay</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
</tr>
<tr>
<td>30</td>
<td>20</td>
</tr>
<tr>
<td>40</td>
<td>disaster</td>
</tr>
</tbody>
</table>

Table 2
3 Production of dirty input uses an exhaustible resource

Polluting activities often make use of exhaustible resources such as oil or coal. In this section we analyze a variant of our basic model where dirty input production uses an exhaustible resource. In particular, we show that adding a limited resource constraint may sometimes dispense from having to tax dirty input production or innovation in order to avoid a disaster: the idea is that the increased scarcity of the dirty input (due to the resource constraint) may be sufficient to induce clean input production and innovation on clean machines only. However, we also show that whether this latter effect dominates, hinges upon the substitutability between the two inputs. More precisely, we show that the presence of an exhaustible resource will help avoid a disaster when the two inputs are sufficiently substitute; on the other hand when the two inputs are complementary, the resource constraint tilts innovation towards the dirty technology, and at the same time it prevents long-run sustainable growth.

More formally, we amend our basic model by assuming that the dirty input is now produced according to the technology:

$$Y_d = R^{\alpha_2} L_d^{1-\alpha} \int_0^1 A_{d_1}^{1-\alpha_1} x_{d_1}^{\alpha_1} \, dt,$$

(29)

where $R$ is the flow consumption of the exhaustible resource, and $\alpha_1 + \alpha_2 = \alpha$ (so the labor share in the production of intermediary input remains $1 - \alpha$). The basic model is then simply a subcase of this one with $\alpha_2 = 0$. We assume that the exhaustible resource can be directly extracted at a cost $c(Q_t)$ in term of units of final good, where $Q_t$ denotes the resource stock at date $t$, and $c$ is a decreasing function of $Q$. The dynamic evolution of the resource stock is then simply described by the difference equation:

$$Q_{t+1} = Q_t - R_t$$

(30)

In the first subsection we analyze the decentralized equilibrium of the augmented model, and in the second subsection we derive the socially optimal policy.

3.1 Decentralized equilibrium

While the description of clean sectors remains exactly as before, profit maximization by dirty machine producers will now let to the equilibrium output levels

$$x_{dit} = \left( \frac{(\alpha_1)^2 p_{dt} R^{\alpha_2} L_d^{1-\alpha}}{\psi} \right)^{\frac{1}{1-\alpha_1}} A_{dit}$$

29
with corresponding equilibrium profits equal to
\[ \pi_{dit} = \frac{(1 - \alpha_1) \beta_1^{1+\alpha_1} \gamma_1^{1-\alpha_1}}{\psi_1^{1+\alpha_1}} p_{dt}^{1-\alpha_1} R_t^{\alpha_2} L_{dt}^{1-\alpha_1} A_{dit}. \]

Whether innovation occurs in the clean or in the dirty sector will result from the same three effects as before: the direct productivity effect, the price effect and the market size effect identified above. The only change relative to the baseline model, is that the resource stock will affect the magnitude of the price effect and market size effects: namely, the more resources have already been extracted, the lower the effective productivity of the dirty input and therefore the higher its relative price. Thus, we show in Appendix F that the price ratio of dirty to clean input is given by:

\[ \frac{p_{ct}}{p_{dt}} = \frac{\psi_2^{\alpha_2} (\alpha_1)^{2\alpha_1} (\alpha_2)^{\alpha_2} A_{dt}^{\alpha_1}}{c(Q)^{\alpha_2} A_{ct}^{\alpha_1}}. \] (31)

But this in turn will impact on the allocation of labor between the two sectors, in other words it will also affect the relative size of the two sectors. Thus in Appendix F we show that

\[ \frac{L_{ct}}{L_{dt}} = \left( \frac{c(Q_t)^{\alpha_2} \gamma_1^{2\alpha_1} (\alpha_2)^{\alpha_2}}{\psi_2^{\alpha_2} \alpha_1^{2\alpha_1} (\alpha_2)^{\alpha_2}} \right)^{(\varepsilon-1)} \frac{A_{ct}^{-\phi}}{A_{dt}^{-\phi_1}}. \] (32)

(where \( \phi_1 \equiv (1 - \alpha_1) (1 - \varepsilon) \)) so that the share of labor allocated to the dirty sector decreases with the extraction cost only when the two inputs are substitute.

Overall, the ratio of expected profits from undertaking research in the clean versus the dirty sector is now given by (see Appendix F)

\[ \frac{\Pi_{ct}}{\Pi_{dt}} = \kappa \frac{\eta_c c(Q_t)^{\alpha_2} (\varepsilon-1)}{\eta_d} \frac{(1 + \gamma \eta_c s_{ct})^{-\phi_1} A_{ct}^{-\phi}}{(1 + \gamma \eta_d s_{dt})^{-\phi_1} A_{dt}^{-\phi_1}}. \] (33)

where \( \kappa \equiv \frac{(1 - \alpha_1) \beta_1^{1+\alpha_1} \gamma_1^{1-\alpha_1}}{(1 - \alpha_1) \beta_1^{1+\alpha_1} \gamma_1^{1-\alpha_1}} \left( \psi_2^{\alpha_2} \alpha_1^{2\alpha_1} (\alpha_2)^{\alpha_2} \right)^{(\varepsilon-1)}. \)

A main difference with the profit ratio in the baseline model, is the term \( c(Q_t)^{\alpha_2} (\varepsilon-1) \) in the RHS of (33). This new term, together with the assumption that \( c(Q_t) \) decreases with \( Q_t \), implies the following proposition:

**Proposition 8** Resource depletion and the resulting increase in the cost of extracting natural resource tilts innovation towards clean machines if the two inputs are substitute (\( \varepsilon > 1 \)) and towards dirty machines otherwise.
What happens is that resource depletion increases the relative price of the dirty input, and thus reduces the market for the dirty input. In the substitutability case this encourages innovation in the clean sector. However, in the complementarity case the increase in the relative price of the dirty input encourages innovation in the dirty sector. In fact one can show that in the substitutability case, innovation under laissez-faire will always end up occurring in the clean sector only: either because the extraction cost increases too rapidly compared to the productivity ratio between clean and dirty inputs, or because the resource stock gets fully depleted in finite time. In the substitutability case, the dirty input is not essential to final production and therefore, whenever initial environmental quality is sufficiently high, an environmental disaster will always be avoided whereas the economy will converge to a positive long run growth rate equal to $\gamma \eta_c$. In the complementary case instead, the dirty input remains essential for final production. Thus positive growth requires an ever increasing rate of extraction, which in turn leads to the exhaustion of the natural resource in finite time. But this in turn prevents positive long-run growth. This establishes:

**Proposition 9**: (i) in the substitutability case where $\varepsilon > 1$, in the long run innovation will always end up occurring in the clean sector only, the economy will grow at a rate $\gamma \eta_c$ and avoid a disaster for a sufficiently high initial environmental quality; (ii) in the complementarity case where $\varepsilon < 1$, economic growth is not possible in the long run.

**Proof.** See Appendix G

### 3.2 Optimal taxation policy

In the presence of an exhaustible resource, the social planner’s problem is modified as follows: she will maximize over $R_t$ and $Q_t$ in addition to the previous variables the intertemporal utility of the representative consumer

$$\sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} u(C_t, S_t)$$

under the same constraints as in the baseline model, except that dirty input production now satisfies (29), consumption is defined by

$$C_t = Y_t - \psi(X_{ct} + X_{dt}) - c(Q_t)R_t, \chi_t$$

and the evolution of the resource stock satisfies (30).
As in section 2.3, the social planner will correct for the monopolist distortion and she will also impose a wedge between the shadow price of the dirty input and its marginal product in the production of the final good, equivalent to a tax \( \tau_t = \frac{\alpha_1 + \xi}{\lambda_t p_d t} \) on dirty input production. In addition, while private firms do not internalize the effect of their current decisions on the future availability of the resource, the social planner will internalize this resource externality by putting a wedge between the cost of extraction and the marginal product of resource in the production of the dirty input.

Letting \( m_t \) denote the Lagrange multiplier for the resource equation, one can solve for this wedge by taking the first order condition with respect to \( R_t \):

\[
\alpha_2 p_d R^{\alpha_2 - 1} L^{1-\alpha}_d \int_0^1 A^{1-\alpha_1} d x^{\alpha_1} d i = \frac{m_t}{\lambda_t} + c(Q)
\]

(recall that \( p_{jt} = \frac{\lambda_t}{\lambda_t} \)): the wedge \( \frac{m_t}{\lambda_t} \) is the value, in time \( t \) units of final good, of one unit of resource at time \( t \).

The shadow value of one unit of natural resource at time \( t \), is in turn determined by taking the first order condition with respect to \( Q_t \):

\[
m_t = m_{t-1} + \lambda_t c' (Q_t) R_t
\]

so that:

\[
m_t = m_\infty + \sum_{v=t+1}^{\infty} \lambda_v \left( -c' (Q_v) \right) R_v.
\]

(where \( m_\infty \) is the limit of \( m_t \) when \( t \to \infty \)).

Thus achieving the social optimum requires a resource tax equal to

\[
\theta_t = \frac{m_t}{\lambda_t c (Q_t)} = \frac{m_\infty + \sum_{v=t+1}^{\infty} \frac{1}{(1+\rho)^v} \frac{\partial u}{\partial C} (C_v, S_v) (-c' (Q_v)) R_v}{\frac{\partial u}{\partial C} (C_t, S_t) c (Q_t)}.
\]

(35)

In particular, the optimal resource tax is always positive. This establishes:

**Proposition 10** The government can implement the social optimum through a tax on the use of the dirty input, a tax on the profits realized in the dirty sector, a subsidy on the use of all machines and a resource tax (all taxes/subsidies are imposed as a lump sum way to the corresponding agents). The resource tax must be maintained forever.
Proof. The proof is similar to that of Proposition 5.

To illustrate the social optimum solution we simulate an economy. We use the same utility function as in the above simulations on the basic model. In the baseline case we set parameter values $\rho = 0.01; \psi = \alpha = \frac{1}{3}, \varepsilon = 5; \xi = 0.03, \delta = 0.011; \eta_c = \eta_d = 0.025, \gamma = 1, \alpha_2 = 0.05$ and $\sigma = 1.4$. We set $c(Q_t) = \frac{10^{10}}{\max(Q_t^{0.00001})}$. We also assume that initially $A_c(0) = 1, A_d(0) = 3, S_1 = 4, Q_1 = 1$ and simulate the economy over 150 periods.

This baseline case is depicted in Figure 4 below.

Figure 4

---

Figure 4A shows that under these parameter values it is optimal to immediately allocate all scientists to clean machines and to maintain this research allocation forever after. Figure 4B shows that, as a result, while $A_d$ remains constant, $A_c$ grows steadily over time. Consequently the environment stock also increases, and faster than before due to the increasing extraction cost. Finally, final good consumption grows smoothly over time. Figure 4C shows the optimal dirty input production tax is now much lower than before, and the reason for this is simple:

---

28 This cost function together with the initial values $A_{c0} = 1, A_{d0} = 3, Q_1 = 1$, ensures that the extraction cost is economically significant at time $t = 1$, and increases fast as the resource depletes.
the increasing extraction cost already discourages dirty input production and innovation, and
the introduction of the resource tax only reinforces this. With these parameters, the profit tax
needs to be quite high. In Figure 4D, we observe that the exhaustible resource stock decreases
over time and is kept positive at the end of the simulation horizon.

Finally we can perform the same exercise of computing delay costs in terms of the foregone
welfare-maximizing consumption as we did for the baseline model. Table 3 below keeps fixing
$\varepsilon$ at $\varepsilon = 5$, but lets the delay time and the share ($\alpha_2$) of exhaustible resource in the production
function vary.

<table>
<thead>
<tr>
<th>delay \ $\alpha_2$</th>
<th>0</th>
<th>0.05</th>
<th>0.1</th>
<th>0.23</th>
</tr>
</thead>
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<tr>
<td>5</td>
<td>2.64</td>
<td>2.6</td>
<td>4.71</td>
<td>0.02</td>
</tr>
<tr>
<td>10</td>
<td>6.19</td>
<td>4.53</td>
<td>7.98</td>
<td>0.03</td>
</tr>
<tr>
<td>20</td>
<td>17.24</td>
<td>8.66</td>
<td>11.74</td>
<td>0.04</td>
</tr>
<tr>
<td>30</td>
<td>40.61</td>
<td>15.79</td>
<td>14.35</td>
<td>0.04</td>
</tr>
<tr>
<td>40</td>
<td>disaster</td>
<td>30.11</td>
<td>18.73</td>
<td>0.04</td>
</tr>
</tbody>
</table>

Table 3

In particular, we see that while higher delay times again increase delay costs, this is mostly
ture for low values of $\alpha_2$ but not for $\alpha_2 = 0.23$. What happens is that delaying taxation of dirty
production speeds up resource depletion. This in turn accelerates the increase in extraction cost
over time, which then deters dirty input production. Obviously, this latter effect is stronger
the higher the importance of the resource in dirty input production, that is the higher $\alpha_2$. This
explains why the economy can now avoid a disaster after a 40 period delay when the resource
share $\alpha_2$ is equal to 0.05 or higher. The higher that share, the higher the relative expected
profits from innovating in clean instead of dirty technologies. In particular for $\alpha_2 = 0.23$ or
higher, scientists choose the clean sector without the need for any tax on dirty production or
innovation: the role for environmental policy is then substantially reduced and so is the cost
of delaying policy intervention.

4 Two-country case

In a multi-country world with both, environmental and knowledge externalities between the
two countries, can it be enough to enforce environmental policy in only a subset of countries,
in order to avoid a global environmental disaster? And is this facilitated by liberalizing cross-
country trade?

In this section we consider two countries, a North and a South, and we index all variables
(but the quality of the environment) with a superscript $k$, where $k = N$ denotes a variable in the North and $k = S$ denotes a variable in the South. The environment stock $S$ affects the two countries in exactly the same way, and it is depleted by dirty input production in both countries, so that:

$$S_{t+1} = -\xi \left( Y_{dt}^N + Y_{dt}^S \right) + (1 + \delta) S_t. \quad (36)$$

The North is identical to the economy described in the baseline model of Section 2. The South is almost identical to it, except that the South cannot innovate but only imitate the technology in the North. Thus, there is a mass 1 of scientists in the South who choose to imitate the North either in the clean or in the dirty technology. Once they have made their decision, scientists in the South are randomly allocated to a single machine in the sector of their choice (without congestion, so there is at most one scientist per machine).

Imitation in sector $j$ in the South succeeds with a probability $\kappa_j$. When successful scientists are granted monopoly rights over the machine for one period using the current technology in the North ($A_{jt}^N$)\(^{29}\), otherwise monopoly rights are attributed for one period to an entrepreneur drawn at random, who uses the technology that was already used in the previous period in the South ($A_{jt-1}^S$).

We first analyze the case where there is no trade between North and South, then we allow for trade in the two inputs $j = d, c$.

### 4.1 No trade case: (when) can we avoid a disaster without taxing the South?

Imitation in the South is such that if $s_{jt}^S$ scientists undertake research in sector $j$ at time $t$, we have

$$A_{jt}^S = \kappa_j s_{jt}^S A_{jt}^N + (1 - \kappa_j s_{jt}^S) A_{jt-1}^S \quad (37)$$

Now, suppose that the North follows some environmental policy $(\tau_t^N, q_t^N)$ where $\tau_t^N$ is a tax on the production of the dirty input and $q_t^N$ is the profit tax on dirty innovation in the North, but that the South remains under pure laissez-faire. As before we can show that the expected flow of profits $\Pi_{jt}^S$ generated by scientists who start imitating in sector $j$ at time $t$ is given by

\(^{29}\)This technology is in fact the one already used in the South in the previous period when there has been no successful innovation in the North since the South imitated the North successfully for that particular machines for the last time.
\[ \Pi^S_{jt} = \kappa_j (1 - \alpha) \alpha (p^a_{jt})^{1-\sigma} L^S_{jt} A^N_{jt} \]

(as successful imitation will ensure the monopolist the right to use the corresponding technology of the North!). Thus in the South the incentive to imitate the clean technology rather than the dirty one will be determined by the ratio

\[ \frac{\Pi^S_{ct}}{\Pi^S_{dt}} = \frac{\kappa_c (p^S_{ct})^{1-\sigma} L^S_{ct} A^N_{ct}}{\kappa_d (p^S_{dt})^{1-\sigma} L^S_{dt} A^N_{dt}} = \frac{\kappa_c (A^S_{ct})^{1-\varphi^{-1}} A^N_{ct}}{\kappa_d (A^S_{dt})^{1-\varphi^{-1}} A^N_{dt}} \]  

(38)

If this ratio is greater than one imitation occurs in the clean sector only, if it is smaller than one innovation occurs in the dirty sector only and if it is equal to 1 imitation occurs in both sectors simultaneously.\(^{30}\)

Overall, the relative returns from imitation by the South on both types of activities are shaped by the same market size and price effect as before, but now there is also a knowledge spillover effect, captured by the factor \((A^N_{ct}/A^N_{dt})\) on the RHS of (38)). As profits from imitation are proportional to the target productivity level, which here is the technology in the North, this knowledge spillover effects favors imitation in the sector which is the most advanced in the North. In particular if the quality of clean machines becomes much higher than the quality of dirty machines in the North this will create an incentive for the South to imitate in the clean sector.

In fact, if \(\varepsilon > 1\) and if the North devotes all its research effort to innovation on clean machines, firms in the South will eventually switch to clean imitation activities,\(^{31}\) and \(A^S_{ct}\)

\(^{30}\)The expression in term of time-\(t-1\) productivity levels is

\[ \frac{\Pi^S_{ct}}{\Pi^S_{dt}} = \frac{\kappa_c (1 - \kappa_c s^S_{ct}) A^S_{ct-1} + \kappa_c s^N_{ct} A^N_{ct})^{-\varphi^{-1}} A^N_{ct}}{\kappa_d (1 - \kappa_d s^S_{dt}) A^S_{dt-1} + \kappa_d s^N_{dt} A^N_{dt})^{-\varphi^{-1}} A^N_{dt}} \]

\(^{31}\)Indeed, if \(A^N_{ct}\) becomes arbitrarily large compared to \(A^N_{dt}\) it is not possible to keep

\[ \kappa_c \left( A^S_{ct-1})^{-\varphi^{-1}} A^N_{ct} \leq \kappa_d \left( (1 - \kappa_c) A^S_{dt-1} + \kappa_d A^N_{dt})^{-\varphi^{-1}} A^N_{dt} \]

as would be required to have imitation only in dirty technology, indeed \((1 - \kappa_c) A^S_{ct-1} + \kappa_d A^N_{dt}\) would approach \(A^N_{dt}\) which doesn’t grow, whereas \(A^N_{ct}\) keeps growing and \(A^S_{ct-1}\) is constant as long as imitation only takes place in the dirty technology.

Similarly, keeping imitation in both sectors would require

\[ \kappa_c \left( (1 - \kappa_c s^S_{ct}) A^S_{ct-1} + \kappa_c s^N_{ct} A^N_{ct})^{-\varphi^{-1}} A^N_{ct} = \kappa_d \left( (1 - \kappa_d s^S_{dt}) A^S_{dt-1} + \kappa_d s^N_{dt} A^N_{dt})^{-\varphi^{-1}} A^N_{dt} \]

the RHS approaches a constant whereas the LHS is of order \((A^N_{ct})^\varphi\) which grows when \(\varepsilon > 1\).

So it must be the case that eventually imitation occurs in the clean sector only.
will grow in the long-run as the same rate as $A_{ct}^N$, namely $\gamma \eta_c$. Now, suppose that indeed the North undertakes an environmental policy that redirects all innovation towards the clean sector, whereas the South remains under laissez-faire. To analyze the conditions under which a disaster can be avoided, we can look at long-run growth rates of dirty input production in the South (meanwhile an input tax can reduce production of the dirty input in the North as much as necessary). Using the fact that the equilibrium production of dirty input in the South is given as in subsection 2.3 by:

$$Y_{dt}^S = \frac{(A_{ct}^S)^{\phi + \alpha} A_{dt}^S}{(A_{ct}^{S})^{\phi} + (A_{dt}^{S})^{\alpha + \phi}},$$

then in the long run we have

$$Y_{dt}^S \approx \left(\frac{A_{ct}^S}{A_{dt}^S}\right)^{\phi + \alpha},$$

which does not grow if $\phi + \alpha < 0$ (that is if $\varepsilon \geq 1/(1 - \alpha)$) and otherwise grows at rate $(1 + \gamma \eta_c)^{\phi + \alpha} - 1$. If this growth rate is smaller than $\delta$ then a disaster can be avoided without policy intervention in the South when the initial environmental stock is sufficiently large.

This establishes:

**Proposition 11** In the two-country case when $\varepsilon > 1$, a policy in the North that would direct innovation towards clean technologies only, is sufficient to avoid a disaster without taxation in the South provided that the initial environmental quality is sufficiently high if either (i) $\varepsilon \geq 1/(1 - \alpha)$ or (ii) $1 < \varepsilon < 1/(1 - \alpha)$ and $(1 + \gamma \eta_c)^{\phi + \alpha} - 1 < \delta$.

### 4.2 No trade case: global optimum

We now characterize the optimal policy from the point of view of a global social planner that would maximize the sum of the utilities of individuals in the two countries. This social planner will choose a dynamic path of final good production $Y_{kt}$, consumption $C_{kt}$, intermediary input productions $Y_{jkt}$, machines production $x_{jkt}$, labor share allocation $L_{jkt}$, scientists allocation $s_{jkt}$ and quality of machines $A_{jkt}$ for each country $k = N, S$ and environmental quality $S_t$ to maximize the Social Welfare Function

$$\sum_{t=0}^{\infty} \frac{1}{(1 + \rho)^t} \left( L^N u \left( \frac{C_{tn}}{L^N}, S_t \right) + L^S u \left( \frac{C_{ts}}{L^S}, S_t \right) \right)$$

37
under the same constraints as for the baseline model, except that the equation (6) becomes (36), and the productivity growth in the South equation (37).

Thus the maximization problem is very similar to that analyzed in subsection 3.2. One difference is that the shadow value of an environmental unit, which is identical in the two countries, now includes the marginal benefit to the utility of individuals in both countries so that:

$$\omega_t = \frac{1}{(1+\rho)^t} \left( L^N \frac{\partial u^N}{\partial S} + L^S \frac{\partial u^S}{\partial S} \right) + (1+\delta) \omega_{t+1}.$$ 

The social planner will still introduce a wedge $\frac{\omega_{t+1} L^k}{\lambda^k_{t+1}}$ between the price of the dirty input and its marginal product in the production of the final good. This wedge has the same interpretation as in the one country case, it will be the higher (in absolute value) in the country with the lowest value $\lambda^k_t$, that is, the country with the lowest marginal utility of consumption (the rich country). This wedge translates into an optimal tax on the dirty input in country $k$:

$$\tau^k_t = \frac{\omega_{t+1} L^k}{\lambda^k_{t+1}} = \frac{\omega_{t+1} \xi^k}{\lambda^k_{t+1} p^k_{dt}}.$$ 

This expression is identical to that in the one-country case, and therefore one can similarly establish that the optimal input tax will be temporary if the clean and dirty baskets are sufficiently close substitutes. But in addition, we get the result that the optimal input tax in the North is higher than in the absence of the South, as $\omega$ is increased by the existence of a South.

Now, the comparison between the optimal input tax in the North and the South, is governed by the implicit expression already given above and derived in Appendix D, namely:

**Proposition 12** The global optimal input tax schedule $(\tau^N_t, \tau^S_t)$ satisfies:

$$\left( \frac{\lambda^k_t}{\omega_{t+1} \xi^k \tau^k_t} \right)^{1-\varepsilon} = 1 + \left( \frac{A^k_{dt}}{(1+\tau^k_t)} \left( A^k_{dt} \right)^{1-\alpha} \right)^{1-\varepsilon}.$$ 

In particular an increase in the relative productivity in the dirty sector in country $k$ $(\frac{A^k_{dt}}{A^k_{ct}})$, a decrease in the marginal value of consumption $\lambda^k_t$ or an increase in the shadow value of environment $\omega_{t+1}$ increases the tax $\tau^k_t$ in country $k$. The second effect will push towards a higher tax in the North, whereas (as long as the dirty sector is more advanced relative to the clean sector in the South than in the North) the first effect will push for a higher tax in the South. Without further assumptions either of these two effects may dominate, and in
particular if the South lags far behind with respect to productivity in the clean sector, the dirty input tax may end up being higher in the South.

Define $\mu_{jt}^k$ as the Lagrange parameter at time $t$ for the growth equation for sector $j$ in country $k$. The first order condition with respect to $A_{jt}^N$ now gives:

$$\mu_{jt}^N = \lambda_t^N \left( \frac{\alpha}{\psi} \right)^{1-\alpha} (1-\alpha) \left( p_{jt}^N \right)^{1-\alpha} L_{jt}^N + (1 + \gamma \eta_j s_{jt+1}^N) \mu_{jt+1}^N + \kappa_j s_{jt}^S S_{jt}^S$$ (39)

In words: the shadow value of one more unit of clean productivity is equal to its marginal product at time $t$ (first term), plus its shadow value at time $t+1$ times $(1 + \gamma \eta_j s_{jt+1}^N)$ (the rate of productivity growth in the North between $t$ and $t+1$), plus an additional term which we did not have in the one-country case, namely $\kappa_c s_{jt}^S$ times the value of one unit of clean productivity in the South, as each additional unit of productivity in sector $j$ in country $N$ creates $\kappa_c s_{jt}^S$ units of productivity in sector $j$ in country $S$: this term captures the international knowledge spillover effect, which pushes towards inducing innovation in technologies that the South will then subsequently imitate.32

The optimal allocation of scientists in the South will be governed by the comparison between the social gains from imitation in clean versus dirty technologies, namely $\mu_{ct}^S \kappa_c A_{ct}^N$ versus $\mu_{dt}^S \kappa_d A_{dt}^N$, and in the North it will be governed by the comparison between $\mu_{ct}^N \eta_c A_{ct-1}^N$ and $\mu_{dt}^N \eta_d A_{dt-1}^N$.

Along the same lines as for the one-country case, one can establish:

**Proposition 13** The social optimum can be implemented through a combination of profits and input taxes both in the North and in the South, and a subsidy to machine consumers (to relieve the monopoly distortion). If $\varepsilon > \frac{1}{1-\alpha}$, the optimal environmental taxes are temporary.

32The shadow value $\mu_{jt}^S$ is itself determined by first order conditions with respect to $A_{jt}^S$: this yields

$$\mu_{jt}^S = \lambda_t^S \left( \frac{\alpha}{\psi} \right)^{1-\alpha} (1-\alpha) \left( p_{jt}^S \right)^{1-\alpha} L_{jt}^S + (1 - \kappa_j s_{jt+1}^S) \mu_{jt+1}^S$$ (40)

The interpretation is basically the same as for $\mu_{jt}^N$: the shadow value of a unit of clean productivity is equal to its marginal product at time $t$, plus $1 - \kappa_j s_{jt+1}^S$ times its shadow value at time $t+1$ ($\kappa_j s_{jt+1}^S$ machines will adopt the technology in the North at time $t+1$: thus, the decision to allocate scientists to imitate in clean technologies in the South, is more "short sighted", that is with a higher weight on current profits, than if the North did not exist and the South had to innovate without benefitting from knowledge spillovers from the North), here, there is no technological spillover from South to North, hence the absence of a third term on the RHS of this equation (unlike in the previous equation for $\mu_{jt}^N$).
4.3 Free trade case: (when) can we avoid a disaster without taxing the South?

The argument that knowledge spillovers should induce the South to follow the North’s switch to clean technologies, may be counteracted by trade considerations: namely, liberalizing trade between North and South, may foster the comparative advantage in dirty input production in the South and thereby encourage the South to become a "pollution heaven". To analyze this more formally, we now allow for trade in the two inputs between North and South. Our model will behave as a Ricardian model within each period, and productivity growth between periods again involves the South imitating technologies in the North. We shall focus attention on the case where $\varepsilon > 1$. As in subsection 4.1, we assume (i) that the North follows a given environmental policy $(\tau_t^N, q_t^N)$ where $\tau_t^N$ is a positive tax on the production of the dirty input in the North, and $q_t^N$ is a tax on profits in the dirty sector in the North, such that the government in the North redirects innovation towards the clean technology only; (ii) that the South remains under pure laissez-faire.

Similarly to (16) in the one country case, equilibrium input production levels are given by:

$$Y_{jt}^k = \left( p_{jt}^k \right)^{\frac{1}{1-\alpha}} A_{jt}^k L_{jt}^k,$$

where $j \in \{c, d\}, k \in \{N, S\}$, and $p_{jt}^k$ is the pre-tax price of input $j$ in country $k$.

The ratio of marginal products of labor in sectors $c$ and $d$ in country $k$ is then equal to.

$$\frac{MPL_c^k}{MPL_d^k} = \left( \frac{p_c^k}{p_d^k} \right)^{\frac{1}{1-\alpha}} \frac{A_c^k}{A_d^k}$$

(42)

The country $k$ for which this ratio is higher, will have a comparative advantage in the clean sector, whereas the other country will have a comparative advantage in the dirty sector.

But now free-trade imposes that the post-tax price for each input $j = c, d$, be equalized across countries, so that:

$$p_{jt}^N = p_{jt}^S \text{ and } (1 + \tau_t^N)p_{jt}^N = p_{jt}^S.$$

(43)

Thus the South will have a comparative advantage in producing the dirty input as long as

$$\left( 1 + \tau_t^N \right) \frac{A_{jt}^N}{A_{jt}^S} > \frac{A_{jt}^N}{A_{jt}^S}.$$

(44)

As before we rule out the case where the North would innovate in dirty technology to reduce the growth rate of the dirty input in the South.
In particular we immediately see that a higher input tax rate $\tau_i^N$ in the North, or an increase in the quality of clean machines in the North, both reinforce the South’s comparative advantage in dirty input production: this is the pollution heaven hypothesis. Thus an environmental policy which is implemented in the North only, will induce dirty activities to move to the South, which will can export part of its dirty input production back to the North. This in turn will favor the occurrence of a disaster.

Particularly illustrating is the case where both countries fully specialize: namely, the North specializes in clean input production and the South specializes in dirty input production. Equilibrium equilibrium input production levels are then given by:

$$Y_d^S = (p_d^S)^{1-\alpha} A_d^S L^S$$ and $Y_d^N = 0$; \hspace{1cm} (45)

and

$$Y_c^N = (p_c)^{1-\alpha} A_c^N L^N$$ and $Y_c^S = 0$. \hspace{1cm} (46)

Thus, in the absence of any tax/subsidy scheme in the South, scientists in the South will always target machines in the dirty sector since there is no local demand for clean machines by potential input producers in the South.

To see under which conditions a disaster can be avoided, all we need to do at this stage is to compute the long-run growth rate of dirty input production. Since only $A_c^N$ grows in the long run at rate $\gamma\eta_c$ (the South imitates and the North innovates in clean technologies only), we show in Appendix H that in the long-run:

$$Y_d^S \approx \left(\frac{\alpha^2}{\psi}\right)^{1-\alpha} (A_c^N L^N)^{\frac{\alpha}{(1-\alpha)+\alpha}} (A_d^S L^S)^{\frac{\varphi}{(1-\alpha)+\alpha}},$$

which increases at rate $(1 + \gamma\eta_c)^{(1-\alpha)/\alpha} - 1$. This rate is strictly larger than the long run growth rate of $Y_d^S$ in the no-trade case, namely 0 when $\alpha + \varphi \geq 0$ and $(1 + \gamma\eta_c)^{\frac{\alpha+\varphi}{\psi}} - 1$ otherwise.

Consequently, a disaster is more likely to occur in the free trade case than in the no-trade case if the South is not taxed. Intuitively: (i) both under free trade and no-trade, inducing the North to innovate more on clean inputs, has the perverse effect of making the final good cheaper as an input for dirty basket production in the South: (ii) in the no-trade case this effect

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34We show in Appendix H that $A_d^N$ is a sufficient condition to ensure that there is an equilibrium with complete specialization in the long-run.
was mitigated by a labor supply response: namely, workers in the South were moving away from dirty input production to clean input production, thereby reducing the production of dirty input in the South; under free trade, instead, the labor force in the South keeps working in the dirty sector; (iii) consequently, under free trade, dirty production in the South will keep growing at a higher rate than in the no trade case.

More generally, we can establish the following proposition:

**Proposition 14** In the two-country case, with free trade in the two inputs \( j = c, d \) between the two countries and when these two inputs are substitutes \((\varepsilon > 1)\), the range of regeneration rate \( \delta \) under which a disaster can never be prevented through an environmental policy in the North while the South remains in laissez-faire is at least as large under free trade than under autarky; if \( A^S_{\delta_0} \) is sufficiently low it is strictly larger.

**Proof.** See Appendix H ■

5 Conclusion

In this paper we have introduced endogenous directed technical change in a growth model with environmental constraints and limited resources. Then we have characterized dynamic tax policies that achieve sustainable growth or maximize intertemporal welfare, as a function of structural characteristics of the economy, in particular the degree of substitutability or complementarity between clean and dirty inputs, environmental and resource stocks, and cross-country technological spillovers. A first conclusion from our analysis, is that factoring in directed technical change: (i) increases the cost of delaying intervention, particularly in the substitutability case; (ii) calls for the use of profit taxes or other instruments to direct innovation, in addition to the input tax emphasized so far by the literature. Moreover we showed that: (i) in the case where the clean and dirty inputs are substitutes, one can achieve sustainable long run growth with temporary taxation of dirty innovation and production; (ii) the sooner and stronger the policy response, the shorter the slow growth transition phase; (iii) the use of an exhaustible resource in dirty goods production helps the switch to clean innovation under laissez-faire when the two inputs are substitute, but the opposite holds when the two inputs are complements (iv) in a two-country extension where: (a) the two inputs are substitutable in both countries; (b) dirty input production in both countries depletes the global environmental
stock; (c) the South imitates technologies invented in the North, then taxing dirty innovation and production in the North only, may be sufficient to avoid a global disaster; however, this is less likely to be true if free trade is allowed between North and South, since in that case taxing the North only may induce full specialization by the South in dirty input production.

This research can be extended in several interesting directions. A first extension would be to embed this model into a more complete model of climate change that could be calibrated and then simulated, in order to reassess the costs and benefits of tax or cap devices for limiting CO2 emissions. Another extension would be to develop a multi-country version of our model, which could then be used to discuss the extent to which trade policies should be made contingent upon environmental criteria.
References


6 Appendix A: Equilibrium allocations of scientists

In this Appendix, we systematically describe the unique or multiple equilibrium allocation(s) of innovation effort across the two sectors, without making Assumption 1. Recall that at time \( t \), the ratio of expected profits from undertaking research in clean technologies over the expected profits from undertaking research in dirty technologies is given by

\[
\frac{\Pi_c}{\Pi_d} = \frac{\eta_c}{\eta_d} \left( \frac{1 + \gamma \eta_c s_{ct}}{1 + \gamma \eta_d s_{dt}} \right)^{-\varphi - 1} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{-\varphi}
\]

Three case scenarios must then be considered:

1. Innovation can occur in the clean sector only whenever

\[
\frac{\eta_c}{\eta_d} (1 + \gamma \eta_c)^{-\varphi - 1} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{-\varphi} \geq 1
\]

or equivalently:

\[
\frac{A_{ct-1}}{A_{dt-1}} \geq \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varphi}} (1 + \gamma \eta_c)^{-\varphi \left( \frac{\varphi + 1}{\varphi} \right)} \quad \text{when } \varepsilon > 1
\]

\[
\frac{A_{ct-1}}{A_{dt-1}} \leq \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varphi}} (1 + \gamma \eta_c)^{-\varphi \left( \frac{\varphi + 1}{\varphi} \right)} \quad \text{when } \varepsilon < 1
\]

2. Innovation can occur in the dirty sector only whenever

\[
\frac{\eta_c}{\eta_d} \left( \frac{1}{1 + \gamma \eta_d s_{dt}} \right)^{-\varphi - 1} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{-\varphi} \leq 1
\]

or equivalently

\[
\frac{A_{ct-1}}{A_{dt-1}} \leq \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varphi}} (1 + \gamma \eta_d)^{\varphi \frac{\varphi + 1}{\varphi}} \quad \text{when } \varepsilon > 1
\]

\[
\frac{A_{ct-1}}{A_{dt-1}} \geq \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varphi}} (1 + \gamma \eta_d)^{\varphi \frac{\varphi + 1}{\varphi}} \quad \text{when } \varepsilon < 1
\]

3. Finally innovation can occur in both sectors if

\[
\frac{\eta_c}{\eta_d} \left( \frac{1 + \gamma \eta_c s_{ct}}{1 + \gamma \eta_d s_{dt}} \right)^{-\varphi - 1} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{-\varphi} = 1
\]

If \( \varepsilon = \frac{2 - \alpha}{1 - \alpha} \) (that is if \( \varphi + 1 = 0 \)) this case occurs if and only if \( \frac{\eta_c}{\eta_d} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{-\varphi} = 1 \), in which case any \( s_{ct} \) is an equilibrium allocation of scientists.
If \( \varepsilon \neq \frac{2 - \alpha}{1 - \alpha} \), the only possible equilibrium with scientists allocating to both sectors, involves the following share of scientists being allocated to the clean sector:

\[
s_{ct} = \frac{\left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha+1}} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{\frac{\alpha}{\alpha+1}} (1 + \gamma \eta_d) - 1}{\left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha+1}} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{\frac{\alpha}{\alpha+1}} \gamma \eta_d + \gamma \eta_c}.
\]

This is indeed an equilibrium with scientists allocating to both sectors if this value strictly lies between 0 and 1, which in turn implies:

\[
(1 + \gamma \eta_c)^{-\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} < \frac{A_{ct-1}}{A_{dt-1}} < (1 + \gamma \eta_d)^{\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \text{ when } \varepsilon < 1 \text{ or } \varepsilon > \frac{2 - \alpha}{1 - \alpha}
\]

\[
(1 + \gamma \eta_d)^{-\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} < \frac{A_{ct-1}}{A_{dt-1}} < (1 + \gamma \eta_c)^{-\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \text{ when } 1 < \varepsilon < \frac{2 - \alpha}{1 - \alpha}
\]

This leads to following characterization of equilibrium innovation allocations between the two sectors:

- When \( \varepsilon > \frac{2 - \alpha}{1 - \alpha} \): If \( \frac{A_{ct-1}}{A_{dt-1}} > (1 + \gamma \eta_d)^{\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \) innovation at time \( t \) must occur only in the clean sector; if \( \frac{A_{ct-1}}{A_{dt-1}} = (1 + \gamma \eta_d)^{\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \) or if \( \frac{A_{ct-1}}{A_{dt-1}} = (1 + \gamma \eta_c)^{-\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \) innovation occurs only in the clean sector or only in the dirty sector; if \( (1 + \gamma \eta_c)^{-\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} < \frac{A_{ct-1}}{A_{dt-1}} < (1 + \gamma \eta_d)^{\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \) innovation occurs only in the clean sector or only in the dirty sector or in both sectors with \( s_{ct} = \frac{\left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha+1}} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{\frac{\alpha}{\alpha+1}} (1 + \gamma \eta_d) - 1}{\left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha+1}} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{\frac{\alpha}{\alpha+1}} \gamma \eta_d + \gamma \eta_c} \); and if \( \frac{A_{ct-1}}{A_{dt-1}} < (1 + \gamma \eta_c)^{-\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \) innovation at time \( t \) must occur only in the dirty sector.

- When \( \varepsilon = \frac{2 - \alpha}{1 - \alpha} \): If \( \frac{A_{ct-1}}{A_{dt-1}} > \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \) innovation at time \( t \) must occur only in the clean sector; if \( \frac{A_{ct-1}}{A_{dt-1}} = \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \) innovation can occur with any \( s_{ct} \); and if \( \frac{A_{ct-1}}{A_{dt-1}} < (1 + \gamma \eta_c)^{-\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \) innovation at time \( t \) must occur only in the dirty sector.

- When \( 1 < \varepsilon < \frac{2 - \alpha}{1 - \alpha} \): If \( \frac{A_{ct-1}}{A_{dt-1}} \geq (1 + \gamma \eta_c)^{-\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \) innovation at time \( t \) must occur only in the clean sector; if \( (1 + \gamma \eta_d)^{\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} < \frac{A_{ct-1}}{A_{dt-1}} < (1 + \gamma \eta_c)^{-\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \) innovation occurs in both sectors with \( s_{ct} = \frac{\left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha+1}} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{\frac{\alpha}{\alpha+1}} (1 + \gamma \eta_d) - 1}{\left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha+1}} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{\frac{\alpha}{\alpha+1}} \gamma \eta_d + \gamma \eta_c} \); and if \( \frac{A_{ct-1}}{A_{dt-1}} \leq (1 + \gamma \eta_d)^{\frac{\alpha+1}{\alpha}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\alpha}} \) innovation at time \( t \) must occur only in the dirty sector.
• When $\varepsilon < 1$: If $A^{ct+1}_{dt-1} \geq (1 + \gamma \eta_d)^{\varepsilon+1} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varepsilon}}$, innovation at time $t$ must occur only in the dirty sector; if $(1 + \gamma \eta_c)^{-\varepsilon^1} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varepsilon}} < A^{ct+1}_{dt-1} < (1 + \gamma \eta_d)^{\varepsilon+1} \left( \frac{\eta_d}{\eta_d} \right)^{\frac{1}{\varepsilon}}$, innovation must occur in both sectors $s^{ct} = \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varepsilon}} \left( \frac{A^{ct+1}_{dt-1}}{A^{ct+1}_{dt-1}} \right)^{\frac{1}{\varepsilon}} \left( 1 + \gamma \eta_d \right)^{-1}$; and if $A^{ct+1}_{dt-1} \leq (1 + \gamma \eta_c)^{-\varepsilon^1} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varepsilon}}$, innovation at time $t$ must occur only in the clean sector.

In particular, we obtain potentially multiple equilibria when $\varepsilon > \frac{2-\alpha}{1-\alpha}$ and a unique equilibrium when $\varepsilon < \frac{2-\alpha}{1-\alpha}$.

7 Appendix B: Proof of Proposition 1

Case $\varepsilon > 1$

Assumption 1 together with the characterization of allocation in Appendix A implies that initially innovation will occur in the dirty sector only. This in turn widens the gap between clean and dirty technologies so that innovation will keep on occurring in the dirty sector only in all subsequent periods. Thus in particular Assumption 1 guarantees uniqueness of the equilibrium allocation under laissez-faire.

Case $\varepsilon < 1$

In this case the result follows from the following lemma:

Lemma 2 When $\varepsilon < 1$, innovation will end up occurring in both sectors so that the equilibrium share of scientists in the clean sector converges to $\frac{\eta_d}{\eta_c + \eta_d} = \tilde{\eta}$.

Proof. Suppose that at time $t$ innovation occurred in both sectors so that $\frac{\Pi_{ct+1}}{\Pi_{dt+1}} = 1$. Then

$$
\frac{\Pi_{ct+1}}{\Pi_{dt+1}} = \frac{\eta_c}{\eta_d} \left( 1 + \gamma \eta_c s_{ct+1} \right)^{-\varepsilon} \left( 1 + \gamma \eta_d s_{dt+1} \right)^{-\varepsilon}$$

$$
= \left( \frac{1 + \gamma \eta_c s_{ct+1}}{1 + \gamma \eta_c s_{dt+1}} \right)^{-\varepsilon} \left( \frac{1 + \gamma \eta_d s_{ct+1}}{1 + \gamma \eta_d s_{dt+1}} \right)^{-\varepsilon}
$$

Innovation will then occur in both sectors at time $t+1$ whenever the equilibrium allocation of scientists $(s_{ct+1}, s_{dt+1})$ at time $t+1$ is such that

$$
\frac{1 + \gamma \eta_c s_{ct+1}}{1 + \gamma \eta_d s_{dt+1}} = \left( \frac{1 + \gamma \eta_c s_{ct+1}}{1 + \gamma \eta_d s_{dt+1}} \right)^{\frac{1}{\varepsilon+1}}
$$

(47)

50
This equation defines $s_{ct+1} = 1 - s_{dt+1}$ as a function of $s_{ct} = 1 - s_{dt}$. What we want to show, is that is has an interior solution in $s_{ct+1}$ when $s_{ct}$ is itself interior. The proof follows from the following remarks:

A first remark is that when $\varphi > 0$ (that is, $\varepsilon < 1$), the function $z(x) = x^{\frac{1}{1+\varepsilon}} - x$ is strictly decreasing for $x < 1$ and strictly increasing for $x > 1$. Therefore $x = 1$ is the unique positive solution to $z(x) = 0$.

A second remark is that the function

$$X(s_{ct}) = \frac{1 + \gamma \eta_c s_{ct}}{1 + \gamma \eta_d s_{dt}} = \frac{1 + \gamma \eta_c s_{ct}}{1 + \gamma \eta_d (1 - s_{ct})},$$

is a one-to-one mapping from $(0, 1)$ to $(\frac{1}{1+\gamma \eta_d}, 1 + \gamma \eta_c)$.

A third remark is that $X$ belongs to the interval $(\frac{1}{1+\gamma \eta_d}, 1 + \gamma \eta_c)$, the same is true for $X^{\frac{1}{1+\varepsilon}}$. This, together with (47), implies that if $s_{ct}$ is interior to $(0, 1)$, $s_{ct+1} = X^{-1}(X(s_{ct})^{\frac{1}{1+\varepsilon}})$ must lie strictly between 0 and 1.

But from Appendix A we know that when $\varphi > 0$, the equilibrium allocation of scientists is unique at each period $t$. When $t$ goes to infinity, this allocation must converge to the unique fixed point of the Lipschitzian function

$$Z(s) = X^{-1} \circ (X(s))^{\frac{1}{1+\varepsilon}},$$

namely

$$s = \frac{\eta_d}{\eta_c + \eta_d}.$$

This establishes the lemma.

Now given the characterization of Appendix A, under Assumption 1, innovation occurs initially in the clean sector only. Thus $\frac{4 \eta d}{\lambda dt}$ will grow at a rate $\gamma \eta_c$, it has then to cross the interval $\left( (1 + \gamma \eta_c)^{-\frac{\varepsilon + 1}{\varepsilon}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varepsilon}}, (\frac{\eta_c}{\eta_d})^{\frac{1}{\varepsilon}} (1 + \gamma \eta_d)^{-\frac{\varepsilon + 1}{\varepsilon}} \left( \frac{\eta_c}{\eta_d} \right)^{\frac{1}{\varepsilon}} \right)$, where innovation occurs in both sectors. From then on, as stated by the lemma above, innovation will occur in both sectors and the share of scientists devoted to the clean sector converges towards $\frac{\eta_d}{\eta_d + \eta_c}$. This completes the proof of Proposition 1.

8 Appendix C: Perfect competition in the absence of innovation

Here we show how our results are slightly modified if, instead of having monopoly rights randomly attributed to "entrepreneurs" when innovation does not occur, machines are produced
competitively. There are two types of machines. Those where innovation occurred at the beginning of the period are produced monopolistically with demand function $x_{ji} = x_{ji}^m = \left(\frac{\alpha p_j}{\psi}\right)^{1-\alpha} L_j A_{ji}$. Those for which innovation failed are produced competitively. In this case, machines are priced at marginal cost $\psi$, which leads to a demand for competitively produced machines equal to $x_{ji} = x_{ji}^c = \left(\frac{\alpha p_j}{\psi}\right)^{1-\alpha} L_j A_{ji}$. The number of machines produced under monopoly, is simply given by $\eta_j s_j$ (the number of successful innovation).

Hence the equilibrium production of input $j$ is given by

$$Y_j = L_j^{1-\alpha} \int_0^1 A_{ji}^{1-\alpha} \left(\eta_j s_j x_{ji,m}^\alpha + (1-\eta_j s_j) x_{ji,c}^\alpha\right) di$$

$$= \left(\frac{\alpha p_j}{\psi}\right)^{1-\alpha} \left(\eta_j s_j \left(\alpha^{1-\alpha} - 1\right) + 1\right) A_j L_j$$

$$= \left(\frac{\alpha p_j}{\psi}\right)^{1-\alpha} \tilde{A}_j L_j$$

where $s_j$ is the number of scientists employed in clean industries and $\tilde{A}_j = \left(\eta_j s_j \left(\alpha^{1-\alpha} - 1\right) + 1\right) A_j$ is the average corrected productivity level in sector $j$ (taking into account that some machines are produced by monopolists and others are not).

The equilibrium price ratio is now equal to:

$$\frac{p_c}{p_d} = \left(\frac{\tilde{A}_c}{\tilde{A}_d}\right)^{-(1-\alpha)}$$

and equilibrium labor ratio becomes:

$$\frac{L_c}{L_d} = \left(\frac{\tilde{A}_c}{\tilde{A}_d}\right)^{-\varphi}.$$ 

The ratio of expected profits from innovation in clean versus dirty sector now becomes

$$\frac{\Pi_{ct}}{\Pi_{dt}} = \frac{\eta_c}{\eta_d} \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\alpha} \frac{L_{ct} A_{ct-1}}{L_{dt} A_{dt-1}}$$

$$= \left(\frac{\eta_c s_{ct} \left(\alpha^{1-\alpha} - 1\right) + 1}{\eta_d s_{dt} \left(\alpha^{1-\alpha} - 1\right) + 1} \left(1 + \gamma \eta_c s_{ct}\right)\right)^{-\frac{1}{\varphi}} \left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-\varphi}$$

This yields the modified lemma:

**Lemma 3** In the decentralized equilibrium, innovation at time $t$ can occur in the clean sector only when $\eta_c A_{ct-1}^{-\varphi} > \eta_d \left(1 + \gamma \eta_c\left(\eta_c \left(\alpha \frac{1}{1-\alpha} - 1\right) + 1\right)\right)^{\varphi+1} A_{dt-1}^{-\varphi}$, in the dirty sector only

52
when \( \eta_c \left( (1 + \gamma \eta_d) \left( \eta_d \left( \alpha \frac{1}{\alpha} - 1 \right) + 1 \right) \right) \phi + 1 \) and can occur in both when 
\( \eta_c \left( \left( \eta_d \eta_{sd} \left( \alpha \frac{1}{\alpha} - 1 \right) + 1 \right) (1 + \gamma \eta_d \eta_{sd}) \right) \phi + 1 \), 
\( \eta_d \left( \left( \eta_c \eta_{sd} \left( \alpha \frac{1}{\alpha} - 1 \right) + 1 \right) (1 + \gamma \eta_c \eta_{sd}) \right) \phi + 1 \).

This modified lemma leads to the same Propositions 1, 2 and 3 as in the lead case. And similarly, the results in the section on exhaustible resource carry over to this modified set-up.

9 Appendix D: Proof of Proposition 5

First note that, as in the laissez-faire equilibrium, marginal product of labor is equalized across sectors, so that we still have 
\[ p_{ct}^{\frac{1}{\gamma}} A_{ct} = p_{dt}^{\frac{1}{\gamma}} A_{dt}. \] (48)

We first derive the expression for \( \tau_t \) in function of \( A_{ct}, A_{dt}, \omega_{t+1} \) and \( \lambda_t \). Using 24 leads to 
\[ p_{ct}^{1-\varepsilon} + (p_{dt} (1 + \tau_t))^{1-\varepsilon} = 1 \] (which is just stating that \( p_{ct} \) and \( p_{dt} \) are prices of the two inputs in units of the final good).

This together with 48, leads to the following expression for the equilibrium (pre-tax) price of the dirty input 
\[ p_{dt} = \frac{A_{ct}^{1-\alpha}}{A_{ct}^{\phi} (1 + \tau_t)^{1-\varepsilon} + A_{dt}^{1-\varepsilon}}, \] (49)

which decreases with the tax rate \( \tau_t \) (whereas the post-tax price increases), similarly the price of the clean input is given by 
\[ p_{ct} = \frac{A_{dt}^{1-\alpha}}{A_{ct}^{\phi} (1 + \tau_t)^{1-\varepsilon} + A_{dt}^{1-\varepsilon}}, \] (50)

The tax is then implicitly and uniquely determined by 
\[ \tau_t^{1-\varepsilon} = \left( \frac{\omega_{t+1} \xi}{\lambda_t} \right)^{1-\varepsilon} \left( 1 + \left( \frac{A_{dt}^{1-\alpha}}{(1 + \tau_t) A_{ct}^{1-\alpha}} \right)^{1-\varepsilon} \right), \]

Now we derive equation (26). Using the expression for the optimal demand for machines \( ji \), 
\[ x_{jit} = \left( \frac{\alpha}{p_{jt}} \right)^{\frac{1}{1-\alpha}} A_{jit} L_{jt} \] (51)
and the expression for optimal input productions (22), we can rewrite the social planner problem as one of maximizing intertemporal utility

$$\sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} u(C_t, S_t)$$

subject to (1), (22), (4), (5), (6), (12), (13), and

$$C_t = Y_t - \psi \left( \frac{\alpha}{\psi} \right) \left( \left( \frac{\lambda_{ct}}{\lambda_t} \right)^{1-\alpha} A_{ct}L_{ct} + \left( \frac{\lambda_{dt}}{\lambda_t} \right)^{1-\alpha} A_{dt}L_{dt} \right)$$

Since only the average quality $A_{jt}$ matters for equilibrium production of input $j$ and thereby for final production, the solution to the social planner’s program remains the same where we replace (12) by

$$A_{jt} = (1 + \gamma \eta_{jt} s_{jt}) A_{jt-1}.$$  

If $\mu_{jt}$ denotes the Lagrange multiplier for this equation, then taking first order condition with respect to $A_{jt}$ leads to equation (26).

In the text we already show how the monopoly distortion can be removed with a subsidy on the use of machines. To complete the proof of Proposition 5, we just need to show that the appropriate profit tax can implement the optimal allocation of scientists. Using 22 in 24, combined with 48, we find that

$$\frac{L_{ct}}{L_{dt}} = (1 + \tau_t)^{\varepsilon} \left( \frac{A_{ct}}{A_{dt}} \right)^{-\varphi}$$

(so a higher tax induces people to work more in the clean sector).

Now taking into account that the subsidy to the consumption of machines leads to demand functions 51, pre-tax profits can be expressed as:

$$\pi_{jit} = (1 - \alpha) \left( \frac{\alpha}{\psi} \right)^{\alpha \alpha} p_{jt}^{\alpha - \varphi} A_{jit} L_{jt}.$$  

Assuming that the government implements a tax $q_t$ on profits in sector $d$, the ratio of expected profits from innovation in sectors $c$ and $d$, is equal to:

$$\frac{\Pi_{ct}}{\Pi_{dt}} = \frac{\eta_{ct}}{\eta_{dt}} \left( \frac{1 + \gamma \eta_{ct} s_{ct}}{1 + \gamma \eta_{dt} s_{dt}} \right)^{-\varphi - 1} \left( \frac{1 + \tau_t}{1 - q_t} \right)^{\varepsilon} \left( \frac{A_{ct-1}}{A_{dt-1}} \right)^{-\varphi}$$

Using the same argument as in lemma 1, innovation at time $t$ occurs in the clean sector only when $\frac{\Pi_{ct}}{\Pi_{dt}} > 1$, in the dirty sector only when $\frac{\Pi_{ct}}{\Pi_{dt}} < 1$ and it may occur in both when
\[ \frac{q_{ct}}{q_{dt}} = 1. \] Clearly, by adjusting \( q_t \) adequately we can get \( s_{ct} = 1 \) or \( s_{ct} = 0 \). More generally, we can induce any specific value \( s_{ct} \) by setting:

\[
q_t = 1 - \frac{\eta_c}{\eta_d} \left( \frac{1 + \gamma d s_{dt}}{1 + \gamma c s_{ct}} \right)^{\varphi+1} (1 + \tau t)^{\varepsilon} \left( \frac{A_{dt}^{-1}}{A_{ct}^{-1}} \right)^{\varphi}
\]

This establishes Proposition 5.

10 Appendix E: Proof of Proposition 6

Using (52), (4), (50), (49) and (22), we get the optimal production of each input at time \( t \) as:

\[
Y_{ct} = \left( \frac{\alpha}{\psi} \right)^{1 - \frac{\alpha}{\varphi}} \frac{(1 + \tau t)^{\varepsilon} A_{ct}^{\alpha + \varphi} A_{dt}^{\varphi}}{(A_{dt}^{\varphi} + (1 + \tau t)^{1 - \varepsilon} A_{ct}^{\varphi})^{\frac{\varphi}{\varepsilon}} (A_{ct}^{\varphi} + (1 + \tau t)^{\varepsilon} A_{dt}^{\varphi})}
\]

\[
Y_{dt} = \left( \frac{\alpha}{\psi} \right)^{1 - \frac{\alpha}{\varphi}} \frac{A_{ct}^{\alpha + \varphi} A_{dt}^{\varphi}}{(A_{dt}^{\varphi} + (1 + \tau t)^{1 - \varepsilon} A_{ct}^{\varphi})^{\frac{\varphi}{\varepsilon}} (A_{ct}^{\varphi} + (1 + \tau t)^{\varepsilon} A_{dt}^{\varphi})}
\]

so that the production of dirty input is decreasing in \( \tau_t \) and converges to zero when the dirty input tax becomes arbitrarily large.

**Substitutability case**: For sufficiently low discount rate, achieving the social optimum requires avoiding a disaster. But this in turn requires that the asymptotic growth rate for the dirty input not be higher than the regeneration rate of the environment \( \delta \). In the substitutability case the asymptotic growth rate of final output is the maximum of the asymptotic growth rates of the clean and the dirty inputs.

Suppose that \( Y_c \) grows faster than \( Y_d \) in the long-run. Then, since

\[
\frac{Y_{ct}}{Y_{dt}} = \left( (1 + \tau t) \left( \frac{A_{ct}}{A_{dt}} \right)^{1 - \alpha} \right)^{\varepsilon},
\]

\( (1 + \tau_t) A_{ct}^{1 - \alpha} \) must grow faster than \( A_{dt}^{1 - \alpha} \), so that a fortiori \( A_{ct}^{1 - \alpha} \) grows faster than \( (1 + \tau t)^{\frac{\varepsilon}{1 - \varepsilon}} A_{dt}^{1 - \alpha} \).

But then, we asymptotically have:

\[ Y_{ct} = O(A_{ct}) \]

and

\[ Y_{dt} = O \left( \frac{A_{ct}^{\alpha + \varphi}}{(1 + \tau_t)^{\varepsilon} A_{dt}^{\varphi}} \right), \]
so that the long-run growth rate of the economy is maximized when \( s_{ct} = 1 \)\(^{35}\). Then \( Y_{ct} \) and \( Y_t \) grow at the same rate as \( A_{ct} \), namely at rate \( \gamma \eta_c \) which is greater than \( \delta \) by hypothesis.

To avoid a disaster, the social planner needs to ensure that \( Y_{dt} \) grows at a rate less than \( \delta \). If \( \alpha + \varphi \leq 0 \), or if \( \alpha + \varphi > 0 \) and \((1 + \gamma \eta_c)^{\alpha+\varphi} - 1 < \delta \), this will indeed be the case given the above asymptotic expressions for \( Y_{ct} \) and \( Y_{dt} \). But then the environmental quality \( S_t \) will end up being higher than \( \mathcal{S} \), ensuring that the optimal input tax \( \tau_t \) drops down to zero in finite time. If \( \alpha + \varphi > 0 \) and \((1 + \gamma \eta_c)^{\alpha+\varphi} - 1 > \delta \), however, the optimal input tax will have to increase over time, in order to ensure that \( \frac{A_{ct}^{\alpha+\varphi}}{(1+\tau_t)^{\varphi}} \) does not grow asymptotically faster than \( \delta \).

However, the optimal profit tax still drops down to zero in finite time as innovation will occur in the clean sector without the need for any taxes once \( A_{ct} \) becomes sufficiently larger than \( A_{dt} \). If \( \gamma \eta_c > \delta \) this policy yields the highest growth rate of final output that does not lead to a disaster. This is also the welfare maximizing policy as long as the discount rate is sufficiently small.

Note that the hypothesis \( \delta < \gamma \eta_c \) is crucial here: absent that hypothesis the government may want to avoid a disaster by increasing the tax rate without diverting technology towards the clean sector.

**Complementarity case:** If \( \varepsilon < 1 \), the growth rate of the economy is the minimum of the growth rates of the two inputs, so that it must be bounded above by \( \delta \) to avoid a disaster. Policy intervention must then be permanent (without policy intervention the growth rate of the economy would asymptotically be equal to \( \gamma \bar{\eta} \), which is greater than \( \delta \) by hypothesis)\(^{36}\).

This establishes Proposition 6.

\(^{35}\)In fact \( s_{ct} = 1 \) in finite time, because in the substitute case, once the clean sector is sufficiently advanced relative to the dirty sector, an additional unit of innovation in clean technology is more productive than an additional unit of innovation in dirty technology every period. So welfare will be higher when \( s_{ct} \) is exactly equal to 1.

\(^{36}\)In fact both the input tax and the profit tax must be maintained permanently. If the input tax was null above some \( T \), the ratio \( \frac{\eta_{x,t} A_{x,t+1}}{\eta_{x,t} A_{x,t+1}} \) would not internalize any environmental externality, and would lead to a pattern of innovation similar to what happens in the laissez-faire case with innovation in both sectors. But this leads to a growth rate of \( \gamma \bar{\eta} \) higher than \( \delta \) so we get a contradiction. The input tax cannot be null from some \( T \). But the input tax is determined only according to the environmental externality. To get the optimal allocation of scientists, the government needs to take into account the knowledge externality so a profit tax/subsidy must be implemented.
11 Appendix F: Equilibrium profit ratio in the exhaustible resource case, proving equation 33

We first analyze how the static equilibrium changes when we introduce the limited resource constraint. Thus here we drop subscript \( t \) for notational simplicity. The description of clean sectors remains exactly as before. Profit maximization by producers of machines in the dirty sector now leads to the equilibrium price \( p_{dit} = \frac{\psi}{\alpha_1} \) (as \( \alpha_1 \) is the share of machines in the production of dirty input). The equilibrium output level for machines is then given by:

\[
x_{di} = \left( \frac{(\alpha_1)^2 p_d r^{\alpha_2} L_d^{1-\alpha}}{\psi} \right)^{\frac{1}{1-\alpha_1}} A_{di} \tag{53}
\]

Profit maximization by the dirty input producer leads to the following demand for the resource:

\[
p_d^{\alpha_2} R^{\alpha_2-1} L_d^{1-\alpha} \int_0^1 A_{di}^{1-\alpha_1} x_{di}^{\alpha_1} di = c(Q)
\]

Plugging in the equilibrium output level of machines (53) yields:

\[
R = \left( \frac{(\alpha_1)^2}{\psi} \right)^{\frac{\alpha_1}{1-\alpha}} \left( \frac{\alpha_2 A_d}{c(Q)} \right)^{\frac{1-\alpha_1}{1-\alpha}} p_d^{\frac{1}{1-\alpha}} L_d \tag{54}
\]

which in turn, with (29), leads to the following expression for the equilibrium production of dirty input:

\[
Y_d = \left( \frac{(\alpha_1)^2}{\psi} \right)^{\frac{\alpha_1}{1-\alpha}} \left( \frac{\alpha_2 A_d}{c(Q)} \right)^{\frac{\alpha_2}{1-\alpha}} p_d^{\frac{\alpha}{1-\alpha}} L_d A_d.
\tag{55}

The equilibrium profits from producing machine \( di \) becomes:

\[
\pi_{di} = (1 - \alpha_1) \alpha_1^{\frac{\alpha_1}{1-\alpha_1}} \left( \frac{1}{\psi^{\alpha_1}} \right)^{\frac{1}{\alpha_1}} p_d^{\frac{1}{1-\alpha_1}} R^{\frac{\alpha_2}{1-\alpha}} L_d^{\frac{1-\alpha}{1-\alpha_1}} A_{di}.
\]

The production of the clean input and the profits of the producer of machine \( di \) are still given by (16), that is:

\[
Y_c = \left( \frac{\alpha_2}{\psi^{\alpha}} \right)^{\frac{\alpha}{1-\alpha}} L_c A_c
\tag{56}
\]

and profits from producing machines \( ci \) are

\[
\pi_{ci} = (1 - \alpha) \alpha^{\frac{\alpha}{1-\alpha}} \left( \frac{1}{\psi^{\alpha}} \right)^{\frac{1}{\alpha}} p_c^{\frac{1}{1-\alpha}} L_c A_{ci}.
\]
Next, labor market clearing requires that the marginal product of labor must be equalized across sectors, together with (55) and (56), leads to the equilibrium price ratio (31): thus a higher extraction cost will bid up the price of the dirty input. Profit maximization by the final good producer still yields (7) which, combined with (55), (56) and (31) yields the equilibrium labor share (equation (32)). Hence, the higher the extraction cost, the higher the amount of labor allocated to the clean industry when \( \varepsilon > 1 \), but the opposite holds when \( \varepsilon < 1 \) (the price effect is then stronger than the average productivity effect).

The ratio of expected profits from undertaking innovation at time \( t \) in the clean versus the dirty sector, is then equal to (we reintroduce the time subscript):

\[
\frac{\Pi_{ct}}{\Pi_{dt}} = \frac{\eta_c}{\eta_d} \frac{(1 - \alpha_1) \alpha_1^{1+\alpha_1} \left( \frac{1}{\psi_{1t}} \right)^{1-\alpha_1}}{(1 - \alpha) \alpha^{1+\alpha} \left( \frac{1}{\psi} \right)^{1-\alpha}} \frac{1}{p_{ct}^{1-\alpha} L_{ct}} A_{ct-1} \frac{1}{p_{dt}^{1-\alpha} R_{t}^{1-\alpha} L_{dt}} A_{dt-1} \]

\[
= \kappa \frac{\eta_c}{\eta_d} \frac{c(Q_t)^{\alpha_2 (1-\varepsilon)} (1 + \gamma \eta_\pi s_{ct})^{-\varphi-1} A_{ct-1}^{-\varphi-1}}{(1 + \gamma \eta_\pi s_{dt})^{-\varphi-1} A_{dt-1}^{-\varphi+1}}
\]

where we let \( \kappa \equiv \frac{(1-\alpha)\alpha}{(1-\alpha_1)\alpha_1} \left( \frac{\alpha_2 \alpha_1^{2\alpha_1 \alpha_2}}{\psi^{\alpha_1 \alpha_2}} \right)^{(\varepsilon-1)} \) This establishes (33).

12 Appendix G: Proof of Proposition 9

Using the fact that the final good is the numeraire and the expression for the equilibrium price ratio (31) in Appendix F, we get:

\[
p_c = \frac{\psi^{\alpha_2} (\alpha_1)^{2\alpha_1} (\alpha_2)^{\alpha_2} A_d^{1-\alpha} A_c^{\alpha}}{\left( (\alpha^{2\alpha} c(Q)^{\alpha_2})^{1-\varepsilon} A_c^{\varphi} + \left( \psi^{\alpha_2} (\alpha_1)^{2\alpha_1} (\alpha_2)^{\alpha_2} A_d^{\varphi_1} \right)^{1-\varepsilon} A_c^{\varphi_1} \right)^{1-\varepsilon}}
\]

\[
p_d = \frac{\alpha^{2\alpha} (c(Q))^{\alpha_2} A_c^{1-\alpha}}{\left( (\alpha^{2\alpha} c(Q)^{\alpha_2})^{1-\varepsilon} A_c^{\varphi} + \left( \psi^{\alpha_2} (\alpha_1)^{2\alpha_1} (\alpha_2)^{\alpha_2} A_d^{\varphi_1} \right)^{1-\varepsilon} A_c^{\varphi_1} \right)^{1-\varepsilon}}
\]

Similarly, using the expression for the equilibrium labor ratio (32) in Appendix F, and labor market clearing, we obtain:

\[
L_d = \frac{(c(Q)^{\alpha_2 \alpha^{2\alpha}})^{(1-\varepsilon)} A_c^{\varphi}}{(c(Q)^{\alpha_2 \alpha^{2\alpha}})^{(1-\varepsilon)} A_c^{\varphi} + \left( \psi^{\alpha_2 \alpha_1^{2\alpha_1} (\alpha_2)^{\alpha_2}} A_d^{\varphi_1} \right)^{(1-\varepsilon)} A_d^{\varphi_1}}
\]
Next, using the above expressions for equilibrium prices and labor allocation, and plugging them in (55) and (54), we obtain:

\[
Y_d = \frac{\left(\psi^{\alpha_2} \alpha_1^{2\alpha_1} (\alpha_2)^{\alpha_2} (1-\varepsilon)\right) A_d^{\varphi_1}}{(c(Q)^{\alpha_2} \alpha_2^{2\alpha_2} (1-\varepsilon) A_d^{\varphi} + \left(\psi^{\alpha_2} \alpha_1^{2\alpha_1} (\alpha_2)^{\alpha_2} (1-\varepsilon)\right) A_d^{\varphi_1})^{\frac{\alpha + \varphi}{\varphi}}}
\]

and

\[
R = \frac{\alpha_2^{2\alpha_2} (1-\alpha_2 + 1-\varepsilon) \alpha_1^{2\alpha_1} \alpha_2^{1-\alpha_2} \alpha_2^{-\alpha_2} (c(Q))^{\alpha_2 - 1 - \alpha_2 \varepsilon} A_c^{1+\varphi} A_d^{1-\alpha_1}}{\left((c(Q)^{\alpha_2} \alpha_2^{2\alpha_2} (1-\varepsilon) A_c^{\varphi} + \left(\psi^{\alpha_2} \alpha_1^{2\alpha_1} (\alpha_2)^{\alpha_2} (1-\varepsilon)\right) A_d^{\varphi_1})^{\frac{1+\varphi}{\varphi}}\right)}
\]

so that:

\[
\frac{R}{Y_d} = \frac{\alpha_2^{2\alpha_2} (c(Q))^{\alpha_2 - 1}}{\left((\alpha_2^{2\alpha_2} (c(Q)^{\alpha_2})^{1-\varepsilon} + \left(\psi^{\alpha_2} (\alpha_1)^{2\alpha_1} (\alpha_2)^{\alpha_2} \right)^{1-\varepsilon} A_d^{\varphi_1} A_c^{\varphi})\right)^{\frac{1}{1-\varepsilon}}}
\]

**Substitutability case**

In the substitutability case ($\varepsilon > 1$), production of the dirty input is not essential to final good production. Thus, even if the stock of exhaustible resource gets fully depleted, it is still possible to achieve positive long-run growth. For a disaster to occur for any initial value of the environmental quality, it is necessary that $Y_d$ grow at rate higher than $\delta$ while $R$ must converge to 0. This implies that $\frac{R}{Y_d}$ must converge to 0. This means that the expression

\[
(\alpha_2^{2\alpha_2} (c(Q)^{\alpha_2})^{1-\varepsilon} + \left(\psi^{\alpha_2} (\alpha_1)^{2\alpha_1} (\alpha_2)^{\alpha_2} \right)^{1-\varepsilon} A_d^{\varphi_1} A_c^{\varphi})^{-\frac{1}{1-\varepsilon}}
\]

must become infinite, but this cannot be since $c(Q)$ is bounded above. This in turn implies that for sufficiently high initial quality of the environment, a disaster is always going to be avoided.

Next, one can show that innovation will always end up occurring in the clean sector only. This is obvious if the resource gets depleted in finite time, so let us consider the case where it never gets depleted. The ratio of expected profits in clean versus dirty innovation is given by

\[
\frac{\Pi_c}{\Pi_d} = \frac{\eta_c c(Q)^{\alpha_2 (\varepsilon - 1)} (1 + \gamma \eta_d s_{dt})^{-\varphi - 1} A_c^{\varphi_1}}{\eta_d (1 + \gamma \eta_d s_{dt})^{-\varphi - 1} A_d^{\varphi_1}}
\]

59
so that to prevent innovation from occurring asymptotically in the clean sector only it must be the case that \( A_c^{-\varphi} \) does not grow faster than \( A_d^{-\varphi'} \). In this case \( R = O \left( A_d^{\frac{1-\alpha}{1-\alpha_1}} \right) \). But this grows at a positive rate over time, so that the resource gets depleted in finite time after all.

**Complementarity case**

In the complementarity case, \( Y_d \) is now essential for production and thus is the resource flow \( R \). Consequently, it is necessary that \( Q \) does not get depleted in finite time in order to get positive long-run growth. Recall that innovation takes place in both sectors if and only if

\[
\left( \frac{\eta_d}{\eta_c} \right)^{\alpha_1} \frac{c(Q_t)^{\alpha_2}(e-1)(1+\gamma_{e,sc})^{-\varphi'} A_d^{-\varphi}}{(1+\gamma_{d,sc})^{-\varphi_1-1} A_d^{-\varphi_1}} = 1,
\]

and positive long run growth requires positive growth of both dirty input and clean input productions. This requires that innovation occurs in both sectors, so \( A_d^{(1-\alpha_1)} \) and \( A_c^{(1-\alpha)} \) should be of same order.

But then:

\[
R = O \left( A_d^{\frac{1-\alpha}{1-\alpha_1}} \right),
\]

so that \( R \) grows over time. But this in turn leads to the resource stock being fully exhausted in finite time, thereby also shutting down the production of dirty input, which here prevents positive long-run growth!

This establishes Proposition 9.

### 13 Appendix H: Proof of Proposition 14

Assume that the South is under laissez-faire, whereas the North is subject to some environmental policy \((\tau^N_i, q^N_i)\) where \( \tau^N_i \geq 0 \), and the policy implies that at all point in time innovation occurs in the clean sector only in the North. North and South can freely trade the clean and dirty inputs (equation (43) is satisfied - which allows us to drop the superscript \( k \) for the price \( p^k \)). The global economy must satisfy labor market clearing in both countries

\[
L^k_c + L^k_d = L^k, \text{ for } k = N, S, \tag{57}
\]

as well as trade balance:

\[
p_c \left( Y^k_c - \bar{Y}^k_c \right) + p^S_d \left( Y^k_d - \bar{Y}^k_d \right) = 0, \tag{58}
\]

and global market clearing for both input markets:

\[
Y^N_j + Y^S_j = \bar{Y}^N_j + \bar{Y}^S_j. \tag{59}
\]
The proof proceeds in four steps:

1. We show that asymptotically, to avoid a disaster, it must be the case that the South has a comparative advantage in the production of the dirty input when \( \alpha + \varphi > 0 \) and 
   \[
   (1 + \gamma \eta_c)^{\alpha + \varphi} - 1 > \delta
   \]

2. We describe the set of possible equilibria when the South has a comparative advantage in the production of the dirty input

3. Using steps 1 and 2, we prove that in the range of parameters (that is \( \alpha + \varphi > 0 \) and 
   \[
   (1 + \gamma \eta_c)^{\alpha + \varphi} - 1 > \delta
   \]) for which redirecting innovation towards the clean sector in the North only, fails to prevent a disaster for any initial quality of the environment under autarky, also fails to prevent a disaster under free trade

4. We derive explicit conditions under which redirecting innovation towards the clean sector in the North only, prevents a disaster under autarky but no longer under free trade.

**STEP 1**

**Lemma 4** When \( \alpha + \varphi > 0 \) and \( (1 + \gamma \eta_c)^{\alpha + \varphi} - 1 > \delta \), any policy in the North aimed at avoiding a disaster would involve the South having a comparative advantage in dirty input production in the long-run

**Proof.** Suppose that the North had a comparative advantage in dirty input production in long-run. Then it would produce more dirty input for given taxes and technology under free-trade than in autarky. Under autarky, the production of dirty input in the North would become asymptotically proportional to \( (1 + \tau_t^N)^{-\varphi} \left( A_{ct}^N \right)^{\alpha + \varphi} \). However, by assumption, \( (A_{ct}^N)^{\alpha + \varphi} \) grows faster than \( \delta \), therefore in order to avoid a disaster the government must impose a tax \( \tau_t^N \) on dirty input production which increases over time. At the same time, that the North has a comparative advantage in dirty input production implies that

\[
(1 + \tau_t^N)^{-\frac{1}{\alpha}} \frac{A_{ct}^N}{A_{dt}^N} > \frac{A_{ct}^S}{A_{dt}^S}.
\]

However, the fact that \( \frac{A_{ct}^N}{A_{ct}^S} > 1 \) whereas \( \frac{A_{ct}^N}{A_{ct}^S} \) remains bounded above, makes it is impossible for this equality to keep being satisfied over time meanwhile the tax schedule \( \tau_t^N \) increases.
Now, suppose that there exists an infinite sequence of periods where the South would have a comparative advantage in clean input production and there exists an infinite sequence of periods where it has a comparative advantage in dirty input production. First, note that in the long run imitation must asymptotically occur in the clean sector only. Indeed if imitation in the dirty sector kept on occurring indefinitely over time, then $A_{dt}^S$ would tends towards $A_{dt}^N$ so it would be impossible to satisfy $\left(1 + \tau_t^N\right)^{-\alpha} \frac{A^N_{dt}}{A^S_{dt}} < \frac{A^S_{dt}}{A^N_{dt}}$ even when $\tau_t^N = 0$. Consequently, in the long run $A_{ct}^S$ should grow at the same rate as $A_{ct}^N$. In periods where the South has a comparative advantage in dirty input production, it produces more dirty input than it would under autarky for given technological levels. But with $A_{ct}^S$ growing exponentially while $A_{dt}^S$ is not, production under autarky in the long-run would be proportional to $(A_{ct}^S)^{\alpha+\varphi}$, which grows at rate $(1 + \gamma_t e)^{\alpha+\varphi} - 1$ which in turn is higher than the regeneration rate of the environment $\delta$. Therefore, in any period $t$ sufficiently large where the South has a comparative advantage in dirty input production, environmental quality decreases. This in turn would make it necessary to tax the North even more than before when the comparative advantage in dirty input production goes back to the North, in order to avoid a disaster. However, we saw that having a comparative advantage in dirty input production in the South is inconsistent with an increasing dirty input tax.

This establishes that to avoid a disaster it must be that the South ends up having a comparative advantage in dirty input production. ■

**STEP 2**

**Lemma 5** Assuming that the South has a comparative advantage in the production of dirty input, then in equilibrium the global economy will feature:

- If $\frac{L^N}{L^S} > \frac{(A^N_s)^{\varphi}(1+\tau^N)^{\alpha+\varphi}}{(A^N_d)^{\varphi}(1+\tau^N)^{\alpha+\varphi}} \frac{A^N_d}{A^N_s}$, non complete specialization in the North and complete specialization in dirty input production in the South
- If $\frac{(A^S_s)^{\varphi}}{(A^S_d)^{\varphi}} \frac{A^S_s}{A^S_d} \leq \frac{L^N}{L^S} \leq \frac{(A^N_s)^{\varphi}(1+\tau^N)^{\alpha+\varphi}}{(A^N_d)^{\varphi}(1+\tau^N)^{\alpha+\varphi}} \frac{A^N_d}{A^N_s}$ complete specialization in clean input production in the North and in dirty input production in the South
- If $\frac{L^N}{L^S} < \left(\frac{A^S_s}{A^S_d}\right)^{\varphi} \frac{A^S_s}{A^S_d}$ complete specialization in clean input production in the North and non complete specialization in the South

**Proof.** We consider each of those three cases in turn and derive necessary conditions for them to arise in equilibrium:
Case 1: non complete specialization in the North, complete specialization in the South

In this case all labor in the South is devoted to the production of dirty input. Equation (41) and equalization of the MPL across sectors in the North leads to:

\[
\frac{p_c}{p_d^N} = \left( \frac{A_d^N}{A_c^N} \right)^{1-\alpha}
\]

From this we can express the equilibrium price levels as:

\[
p_c = \frac{(A_d^N)^{(1-\alpha)}}{\left( (A_d^N)^{\phi} + (1 + \tau^N)^{1-\varepsilon} (A_c^N)^{\phi} \right)^{\frac{1}{1-\varepsilon}}}
\]

\[
p_d^S = (1 + \tau^N) p_d^N = \frac{(1 + \tau^N) (A_c^N)^{(1-\alpha)}}{\left( (A_d^N)^{\phi} + (1 + \tau^N)^{1-\varepsilon} (A_c^N)^{\phi} \right)^{\frac{1}{1-\varepsilon}}}
\]

Final good producer maximization then implies:

\[
\frac{\tilde{Y}_c^N}{Y_c^N} = \frac{\tilde{Y}_c^S}{Y_c^S} = \left( \frac{p_c}{p_d^N} \right)^{-\varepsilon} = (1 + \tau^N)^{-\varepsilon} \left( \frac{A_c^N}{A_d^N} \right)^{(1-\alpha)\varepsilon}
\]

(60)
Finally, (58), (59) and (57) yield the equilibrium allocation of labor between the two sectors

\[
\begin{align*}
L^N_c &= \frac{(1 + \tau^N)^\frac{\alpha}{\epsilon} \left( A^N_d \right)^{\frac{\alpha}{\epsilon}} \left[ L^N + \frac{(1 + \tau^N)^{\frac{\epsilon}{\alpha}} A^S_d}{A^N_d} L^S \right]}{(1 + \tau^N)^\frac{\alpha}{\epsilon} \left( A^N_d \right)^{\frac{\alpha}{\epsilon}} + \left( A^N_c \right)^{\frac{\alpha}{\epsilon}}}, \\
L^N_d &= \frac{\left( A^N_c \right)^{\frac{\alpha}{\epsilon}} \left( L^N - \frac{(1 + \tau^N)^{\frac{\epsilon}{\alpha}} \tau^\alpha A^S_d}{A^N_d} L^S \right)}{(1 + \tau^N)^\frac{\alpha}{\epsilon} \left( A^N_c \right)^{\frac{\alpha}{\epsilon}} + \left( A^N_c \right)^{\frac{\alpha}{\epsilon}}}.
\end{align*}
\]

Non complete specialization in the North then imposes that $L^N_d > 0$, which in turn is equivalent to:

\[
\frac{L^N}{L^S} > \frac{\left( A^N_d \right)^{\frac{\alpha}{\epsilon}} (1 + \tau^N)^{\epsilon + \frac{\epsilon}{\alpha}} A^S_d}{A^N_d}.
\]

**Case 2: complete specialization in both countries**

Here, all labor in the South is allocated to the production of dirty input, and all labor in the North is allocated to the production of the clean input. This yields equations (46) and (45). Complete specialization in clean production in the North then requires $\frac{MPL^N}{MPL^d} \geq 1$, whereas complete specialization in dirty production in the South requires $\frac{MPL^S}{MPL^d} \leq 1$. Finally, profit maximization by the final good producer yields the equilibrium price ratio:

\[
\frac{p^c}{p^S} = \left( \frac{\bar{Y}_d^k}{\bar{Y}_d^k} \right)^{-\frac{1}{\epsilon}} \quad \text{for } k = N, S.
\]

\[\textnormal{Equation (60) and (58) yield the following expressions for the equilibrium consumption of inputs as:}\]

\[
\begin{align*}
\bar{Y}_d^N &= \frac{p^c Y^N_c + p^d Y^N_d}{p^c (1 + \tau^N)^\frac{\alpha}{\epsilon} \left( \frac{A^N_c}{A^N_d} \right)^{\frac{\alpha}{\epsilon}} (1 - \alpha) + p^d_d}, \\
\bar{Y}_d^S &= \frac{p^d Y^S_d}{p^c (1 + \tau^N)^\frac{\alpha}{\epsilon} \left( \frac{A^N_c}{A^N_d} \right)^{\frac{\alpha}{\epsilon}} (1 - \alpha) + p^d_d}, \\
\bar{Y}_S^N &= \frac{p^c Y^N_c + p^d Y^N_d}{p^c (1 + \tau^N)^\frac{\alpha}{\epsilon} \left( \frac{A^N_c}{A^N_d} \right)^{\frac{\alpha}{\epsilon}} (1 - \alpha) + p^d_d}, \\
\bar{Y}_S^S &= \frac{p^d Y^S_d}{p^c (1 + \tau^N)^\frac{\alpha}{\epsilon} \left( \frac{A^N_c}{A^N_d} \right)^{\frac{\alpha}{\epsilon}} (1 - \alpha) + p^d_d}.
\end{align*}
\]

now using this expression together with (59) and (16) leads to

\[
\left( 1 + \tau^N \right)^\frac{\alpha}{\epsilon} \left( \frac{A^N_c}{A^N_d} \right)^{(1 - \alpha)\epsilon} \left( p^d_d \right)^{\frac{\epsilon}{\alpha}} A^S_d L^S + \left( 1 + \tau^N \right)^\frac{\alpha}{\epsilon} \left( \frac{A^N_c}{A^N_d} \right)^{(1 - \alpha)\epsilon} \left( p^d_d \right)^{\frac{\epsilon}{\alpha}} A^N_d L^N = p^c \left( \frac{\alpha}{\epsilon} \right) A^S_d L^S \]

from which we can infer the equilibrium allocation of labor between the two sectors.
This, together with (58), yields the following expression for the consumption of input \( j \) in country \( k \):

\[
\tilde{Y}_d^{k} = \frac{p_c Y_c^k + p_d Y_d^k}{p_c^{1-\varepsilon} p_d^{\varepsilon} + p_d},
\]

and, together with (59), we obtain the equilibrium price levels:

\[
p_c = \frac{(A_S^d)^{1-\varepsilon}}{\left( (A_d^S)^{1-\varepsilon} + (A_c^N L^N)^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}}
\]

\[
p_d^S = \frac{(A_c^N L^N)^{1-\varepsilon}}{\left( (A_d^S)^{1-\varepsilon} + (A_c^N L^N)^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}}
\]

Substituting for these equilibrium input price in the condition \( \frac{MPL^N}{MPE_d^N} \geq 1 \), implies that:

\[
\frac{L^N}{L^S} \leq \frac{(A_d^S)^{\phi} \left( 1 + \tau^N \right)^{\varepsilon + \frac{\alpha}{1-\varepsilon}} A_d^S}{A_d^N} \]

and similarly substituting for the equilibrium input price in the condition \( \frac{MPL_d^S}{MPE^S} \leq 1 \), implies that:

\[
\frac{L^N}{L^S} \geq \frac{(A_d^S)^{\phi} A_c^S}{(A_c^S)^{\phi} A_c^N}. \]

Case 3: complete specialization in the North, non complete specialization in the South

The analysis is completely symmetric to case 1. We can derive the following expression for the equilibrium allocation of labor between the two sectors in the South:

\[
L_d^S = \frac{(A_d^S)^{\phi} \left( L^S + \frac{A_c^N}{A_c^S} L^N \right)}{(A_d^S)^{\phi} + (A_d^S)^{\phi}}
\]

\[
L_c^S = \frac{(A_d^S)^{\phi} \left( L^S - \left( \frac{A_c^S}{A_d^S} \right)^{\phi} \frac{A_c^N}{A_c^S} L^N \right)}{(A_d^S)^{\phi} + (A_d^S)^{\phi}}
\]

so that this case scenario is possible only if

\[
\frac{L^N}{L^S} \leq \left( \frac{A_d^S}{A_c^S} \right)^{\phi} \frac{A_c^S}{A_c^N}. \]

Taking stock
As long as the South has a comparative advantage in the dirty input production, the above conditions do not overlap, hence they are not only necessary but also sufficient. This establishes the lemma. ■

STEP 3

Lemma 6 The North cannot prevent a disaster under free trade for any initial environmental quality when $\alpha + \varphi > 0$ and $(1 + \gamma \eta_c)^{\alpha + \varphi} - 1 > \delta$

Proof. From Lemma 4, we know that the North must have a comparative advantage in clean input production when $\alpha + \varphi > 0$ and $(1 + \gamma \eta_c)^{\alpha + \varphi} - 1 > \delta$. Now we use Lemma 5, to describe the possible long-run scenarios and to show that in each of them the dirty input will grow at a rate higher than $\delta$.

A first possibility is to end up with complete specialization in the South but production of both inputs in the North. Indeed as long as the dirty input is the only one produced in the South, all imitation there occurs in the dirty sector, so $A_{ct}^S$ does not grow over time. However
\[
\frac{(A_{c}^N)^\alpha}{(A_{c}^N)^\varphi} (1 + \gamma c_N)^{\varphi + \frac{\alpha}{1 - \alpha}} A_{t}^S \geq \frac{(A_{d}^N)^\alpha}{(A_{d}^N)^\varphi} A_{t}^S \]
which grows exponentially, so the inequality $\frac{L_N}{L_S} \geq \frac{(A_{d}^N)^\alpha}{(A_{d}^N)^\varphi} (1 + \gamma c_N)^{\varphi + \frac{\alpha}{1 - \alpha}} A_{t}^S$ must be violated at some point. Hence in finite time the economy will display either complete specialization in both countries case or complete specialization in the North only.

A second possibility is to end up with complete specialization in both countries. Using (45) and (62), we can derive the equilibrium input production in the South, namely:

\[
Y_d^S = \left( \frac{A_c^N L_N}{(A_c^S L_S)^{\varepsilon + \frac{\alpha}{1 - \alpha}} + (A_c^N L_N)^{\varepsilon + \frac{\alpha}{1 - \alpha}}} \right)^{\frac{\alpha}{1 - \alpha}} A_d^S L^S
\]

Then, asymptotically we get

\[
Y_d^S \to (A_c^N L_N)^{\frac{\alpha}{\varepsilon(1 - \alpha) + \alpha}} (A_d^S L^S)^{\frac{\alpha}{\varepsilon(1 - \alpha) + \alpha}}
\]
so that $Y_d^S$ keeps increasing at rate $\frac{\alpha}{\varepsilon(1 - \alpha) + \alpha} - 1$. Given that $\alpha/(\varepsilon(1 - \alpha) + \alpha) > \varphi + \alpha$, this rate is higher than $\delta$, and therefore the above policy does not prevent a disaster.

A third possibility is to end up with no specialization in the South. We can then compute
the equilibrium dirty input production level as
\[ Y_d^S = \frac{(A^S_c)^{\alpha+\varphi} A_d^S (L^S + \frac{A^N}{A^S_c} L^N)}{((A^S_c)^{\varphi} + (A_d^S)^{\varphi})^{\frac{\alpha+\varphi}{\varphi}}} \]

Note that
\[ Y_d^S > \frac{(A^S_c)^{\alpha+\varphi-1} A_d^N A_c^N L^N}{((A^S_c)^{\varphi} + (A_d^S)^{\varphi})^{\frac{\alpha+\varphi}{\varphi}}} = O \left((A^S_c)^{\alpha+\varphi-1} A_c^N\right) \]

Now, given that \(A_c^S < A_c^N\), then \(Y_d^S\) must at least of the same order as \(A_c^N\) which grows faster than the regeneration rate of the environment. Thus again, a disaster cannot be avoided.

Finally, if the economy moves back and forth between these latter two cases, and given that in either case the production of dirty input is so high as to induce an environmental disaster, the policy cannot be successful either.

This establishes the Lemma. ■

STEP 4

Lemma 7 If
\[ \frac{A^S_{c, t-1}}{(1-\kappa_d)A^S_{d, t-1} + \kappa_d A_d^N_{t-1}} < \min \left( \frac{(1+\gamma_c) A^N_{c, t-1}}{A_d^N_{t-1} L^N}, \left(\frac{(1+\gamma_c) A^N_{c, t-1} L^N}{A_d^N_{t-1} L^N}\right)^{\frac{1}{\varphi}} \right) \]
then at time \(t\), there exists an equilibrium where the South has a comparative advantage in dirty technology and produces only in dirty technology. Moreover, \(\frac{A^S_{c, t}}{(1-\kappa_d)A^S_{d, t} + \kappa_d A_d^N_{t}} < \min \left( \frac{(1+\gamma_c) A^N_{c, t}}{A_d^N_{t}}, \left(\frac{(1+\gamma_c) A^N_{c, t} L^N}{A_d^N_{t} L^N}\right)^{\frac{1}{\varphi}} \right) \)

Proof. Assume that at time \(t\) the South produces the dirty input only. Then all imitation in the South will be in the dirty input, hence
\[ A^S_{d, t} = (1-\kappa_d) A^S_{d, t-1} + \kappa_d A_d^N_{t-1}. \]

This in turn implies that
\[ \frac{A^S_{c, t}}{A^S_{d, t}} < \min \left( \frac{A^N_{c, t}}{A_d^N_{t}}, \left(\frac{A^N_{c, t} L^N}{A^S_{d, t} L^N}\right)^{\frac{1}{\varphi}} \right), \]
so that it is indeed an equilibrium to have the South produce the dirty input only. Moreover, we have
\[ \frac{A^S_{c, t}}{(1-\kappa_d) A^S_{d, t} + \kappa_d A_d^N_{t}} < \min \left( \frac{(1+\gamma_c) A^N_{c, t}}{A_d^N_{t}}, \left(\frac{(1+\gamma_c) A^N_{c, t} L^N}{A_d^N_{t} L^N}\right)^{\frac{1}{\varphi}} \right) \].

Now assume that
\[ \frac{A^S_{c, 0}}{(1-\kappa_d) A^S_{d, 0} + \kappa_d A_d^N_{d, 0}} < \min \left( \frac{(1+\gamma_c) A^N_{c, 0}}{A_d^N_{d, 0}}, \left(\frac{(1+\gamma_c) A^N_{c, 0} L^N}{A_d^N_{d, 0} L^N}\right)^{\frac{1}{\varphi}} \right) \],
which is satisfied for $A_{50}^S$ sufficiently small. Then there will be an equilibrium where the South always has a comparative advantage in the production of the dirty input and where the South completely specializes in dirty input production. At some point \( \frac{(A_d^N)^c}{(A_d^N)^c} A_d^S \) will become sufficiently large that complete specialization must occur in equilibrium in the South. But then dirty input production is asymptotically equal to $Y_d^S \rightarrow (A_d^N L^N)^{\frac{\alpha}{(1-\alpha)+\alpha}} (A_d^S L^S)^{\frac{\epsilon(1-\alpha)}{(1-\alpha)+\alpha}}$, so that $Y_d^S$ will keep increasing at rate $(1 + \gamma \eta_c)^{\frac{\alpha}{(1-\alpha)+\alpha}} - 1$. Note that this rate is strictly greater than the rate $(1 + \gamma \eta_c)^{\alpha+\varphi} - 1$ under autarky. Thus there exists a non-empty range of $\delta$'s for which directing innovation towards the clean sector in the North only would prevent a disaster under autarky but no longer under free trade.