Search and Rest Unemployment

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Abstract

This paper develops a tractable version of the Lucas and Prescott (1974) search model. The economy consists of a continuum of labor markets, each of which produces a heterogeneous good which are imperfect substitutes. There is a constant returns to scale production technology in each labor market, but productivity is continually hit by idiosyncratic shocks, inducing the costly reallocation of workers across markets. In equilibrium, some workers search for new labor markets while others are rest unemployed, waiting for local labor market conditions to improve. We obtain closed-form expressions for key aggregate variables and use them to evaluate the model’s quantitative predictions for unemployment and wages. Both search and rest unemployment are important for understanding the behavior of wages at the industry level.

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1 Introduction

This paper develops a tractable version of Lucas and Prescott’s (1974) theory of unemployment. We distinguish between “search” and “rest” unemployment. Search unemployment is a costly reallocation activity in which workers look to improve their employment opportunity. Rest unemployment is a less costly activity where a worker waits for her current labor market conditions to improve. We obtain closed-form solutions for the search and rest unemployment rates as functions of important reduced-form parameters, which are in turn known functions of model parameters. The dynamics of industry wages are determined by the same set of reduced-form parameters, giving us a tight link between the incidence of search unemployment and the persistence of industry level wages. We exploit that link to evaluate our model.

Our paper extends Lucas and Prescott’s (1974) search model. The economy consists of a continuum of labor markets, each of which contains many workers and firms and produces a differentiated intermediate good with a constant returns to scale technology using labor only. Intermediate goods are imperfect substitutes in the production of a final consumption good, so industry demand curves slope down, although all workers and firms are price-takers. Labor productivity is continually hit by idiosyncratic shocks whose growth rate has a constant expected value and a constant variance per unit of time. That is, labor productivity is a geometric Brownian motion. Households have standard preferences and markets are complete. In any time period, households use their time endowment to engage in four mutually exclusive activities, from which they derive different amounts of leisure: work, search unemployment, rest unemployment, and inactivity, i.e. out of the labor force.

We assume that the reallocation of workers across labor markets requires a spell of search unemployment. A worker in a given labor market can either work, engage in rest unemployment, or leave the market. A rest unemployed worker is available to return to work in that labor market, and that labor market only, at no cost. If a worker leaves her market she can either be inactive or engage in search unemployment. We assume search unemployed workers find a job after a random, exponentially distributed amount of time. We consider both the case of directed search, as in Lucas and Prescott (1974), where a successful searcher moves to the market of her choice, and of random search, where the probability that a successful searcher moves to any particular market is proportional to the number of workers in that market. Finally, workers can costlessly move between search unemployment and inactivity.

To characterize the equilibrium, let $\omega$ denote the log of the wage that would prevail in a particular labor market if all workers in the market were employed, i.e. if there were no rest unemployment; we measure wages in utility-equivalent units. In the directed search model,
workers’ behavior is characterized by three thresholds, \( \bar{\omega} \leq \hat{\omega} < \bar{\omega} \). At intermediate wages, \( \omega \in (\bar{\omega}, \bar{\omega}) \), log wages are a regulated Brownian motion with drift and standard deviation that are simple functions of model parameters. Household optimization prevents log wages from rising above \( \bar{\omega} \) or falling below \( \bar{\omega} \). Workers who have successfully concluded their search arrive in the best labor markets, which keeps \( \omega \) from rising too high. Workers in depressed labor markets leave to become search unemployed, which keeps \( \omega \) falling too low. In markets with \( \omega \geq \hat{\omega} \) there is full employment and the log wage is \( \omega \). For \( \omega < \hat{\omega} \), log wages stay at \( \hat{\omega} \) and the rest unemployment rate in the market increases. Depending on parameter values, there may be no rest unemployment, \( \hat{\omega} = \omega \).

The random search model is characterized by only two thresholds \( \bar{\omega} < \hat{\omega} \). There is no upper threshold for wages, since a rapid increase in productivity can swamp the random arrival of workers into markets. Instead, the endogenous arrival rate of workers into labor markets at rate \( s \) puts downward pressure onto wages in all labor markets, so the drift in \( \omega \) is endogenous. As in the directed search model, only markets with \( \omega < \hat{\omega} \) have rest unemployment.

In the directed search model, the barriers \( \bar{\omega} \) and \( \hat{\omega} \) and the stochastic process for \( \omega \) imply an invariant distribution of \( \omega \) across workers. In the random search model, the barrier \( \bar{\omega} \) and the arrival rate of searchers \( s \) give the same implication. Aggregating across markets, we obtain closed-form solutions for the employment and unemployment rates and for the value of output and consumption. These are expressions are closely linked to the dynamics of the log full-employment wage \( \omega \). These findings do not depend on the details of how the thresholds are determined, offering a useful separation between optimization and aggregation.

Our closed-form solutions facilitate comparative statics and a quantitative evaluation of the model. We find a tight relationship between the search unemployment rate and the autocorrelation of wages at the labor market level. Using data for five-digit North American Industry Classification System (NAICS) industries, we show that annual average weekly earnings at the industry level are essentially a random walk. According to our model, this implies that wages rarely hit the barriers that regulate wages. But since a worker exits a labor market only when wages hit the lower barrier \( \bar{\omega} \), it follows that the incidence of search unemployment in our model is necessarily small.

We then calibrate the rest unemployment rate using evidence from Murphy and Topel (1987) and Loungani and Rogerson (1989) on the association between unemployment and switching industries—most unemployed workers subsequently return to their old industry. We find no conflict between a high rest unemployment rate and persistent wages with either random or directed search. Still, our conclusions that the search unemployment rate and wage fluctuations are persistent imply that search must be very costly. We conjecture that
some of those search costs actually stand in for labor market-specific human capital, a topic that we leave for future research.

There are three significant differences between our directed search model and Lucas and Prescott (1974). First, we introduce rest unemployment to the framework. Second, we make particular assumptions on the stochastic process for productivity which enable us to obtain closed-form solutions. This leads us to evaluate the model’s properties in novel ways; however, we believe our insights, e.g. on the link between search unemployment and the autocorrelation of wages and on the role of rest unemployment, carry over to alternative productivity processes. Third, in Lucas and Prescott (1974), all labor markets produce a homogeneous good but there are diminishing returns to scale in each labor market. In our model, each labor market produce a heterogeneous good and has constant returns to scale. We believe our approach is more attractive because the extent of diminishing returns is determined by the elasticity of substitution between goods, which is potentially more easily measurable than the degree of decreasing returns on variable inputs (Atkeson, Khan, and Ohanian, 1996). An online Appendix B.2 tightens the connections between these models by solving a market social planner’s problem and proving that the equilibrium is efficient. Finally, our characterization of the random search model is new.¹

Our concept of rest unemployment is closely related to the one Jovanovic (1987), from whom we borrow the term.² While in both his model and ours search and rest unemployment coexist, the aims of both papers and hence the setup of the models are different. Jovanovic (1987) focuses on the cyclical behavior of unemployment and productivity, and so allows for both idiosyncratic and aggregate productivity shocks. But to be able to analyze the model with aggregate shocks, Jovanovic (1987) assumes that at the end of each period, there is exactly one worker in each location.³


¹The closest random search model is the one in Alvarez and Veracierto (1999). That paper assumes that each searcher is equally likely to find any labor market, while we assume here that searchers are more likely to find labor markets with more workers, as in Burdett and Vishwanath (1988). We view this assumption as no less plausible and it is much more tractable in our current setup.

²An alternative is “wait unemployment,” but the literature that uses this term emphasizes the behavior of workers waiting for a job in a high wage primary labor market rather than accepting a readily available job in a low wage secondary labor market. We study study a related concept in our work on unionization in Alvarez and Shimer (2009). Our concept of rest unemployment corresponds closely to one notion of structural unemployment; see, for example Abel and Bernanke (2001, p. 95).

³This assumption implies that search unemployment is socially wasteful. Our model illustrates how search unemployment may play an important role in reallocating workers away from severely depressed labor markets, while rest unemployment may be an efficient use of workers’ time in marginal labor markets.
ployment is likely to be countercyclical, consistent with empirical evidence but not with many models of reallocation. The key difference between those papers and ours is the stochastic process for productivity. While we assume productivity follows a geometric Brownian motion, these earlier papers assume a two-state Markov process. This coarse parameterization makes their analysis of cyclical fluctuations tractable, but at the cost of making it harder to map the model into rich cross-sectional data, for example on wages. It also eliminates any distinction between random and directed search. With two types of markets, including one where agents are indifferent about exiting, the two assumptions are essentially the same. One of our main contributions is to show that when log productivity follows a Brownian motion, the model is still tractable and the mapping from model to cross-sectional data is direct. Of course, solving our version of the model with aggregate shocks is a much more daunting task. In this sense, we view the two approaches as complementary.

In Section 2, we describe the economic environment. We analyze a special case where workers can immediately move to the best labor market in Section 3. Without any search cost, there no rest unemployment, since either working in the best labor market or dropping out of the labor force dominates this activity. Instead, idiosyncratic productivity shocks lead to a continual reallocation of workers across labor markets.

Section 4 characterizes the directed search model. We describe the equilibrium as the solution to a system of two equations in two endogenous variables, \( \omega \) and \( \bar{\omega} \), and various model parameters. We prove that the equilibrium is unique and perform simple comparative statics. In particular, we find that there is rest unemployment only if the cost in terms of foregone leisure is low. We also provide closed form expressions for the employment, search unemployment, and rest unemployment rates, as well as aggregate output. Section 5 gives a parallel characterization of the random search model, now describing the equilibrium as two equations in \( \omega \) and \( s \). We again provide closed form expressions for employment, search unemployment, rest unemployment, and output. The arguments are similar to the directed search case and so we keep our presentation compact.

Section 6 evaluates our model quantitatively. We show the theoretical link between the search unemployment rate and the persistence of wages and we argue that rest unemployment is empirically relevant and consistent with very persistent wages, as we observe in U.S. data. Finally, Section 7 concludes.

2 Model

We consider a continuous time, infinite-horizon model. We focus for simplicity on an aggregate steady state and assume markets are complete.
2.1 Intermediate Goods

There is a continuum of intermediate goods indexed by $j \in [0, 1]$. Each good is produced in a separate labor market with a constant returns to scale technology that uses only labor. In a typical labor market $j$ at time $t$, there is a measure $l(j, t)$ workers. Of these, $e(j, t)$ are employed, each producing $x(j, t)$ units of good $j$, while the remainder are rest unemployed. $x(j, t)$ is an idiosyncratic shock that follows a geometric random walk,

$$d \log x(j, t) = \mu_x dt + \sigma_x dz(j, t),$$

where $\mu_x$ measures the drift of log productivity, $\sigma_x > 0$ measures the standard deviation, and $z(j, t)$ is a standard Wiener process, independent across labor markets. The price of good $j$, $p(j, t)$, and the wage in labor market $j$, $w(j, t)$, are determined competitively at each instant $t$ and are expressed in units of the final good.

To keep a well-behaved distribution of labor productivity, we assume that labor market $j$ shuts down according to a Poisson process with arrival rate $\delta$, independent across labor markets and independent of labor market $j$’s productivity. When this shock hits, all the workers are forced out of the market. A new labor market, also named $j$, enters with positive initial productivity $x_0$, keeping the total measure of labor markets constant. We assume a law of large numbers, so the share of labor markets experiencing any particular sequence of shocks is deterministic.

2.2 Final Goods

A competitive sector combines the intermediate goods into the final good using a constant returns to scale technology

$$Y(t) = \left( \int_0^1 y(j, t)^{\frac{1}{\theta}} dj \right)^{\frac{\theta}{\theta - 1}},$$

where $y(j, t)$ is the input of good $j$ at time $t$ and $\theta > 0$ is the elasticity of substitution across goods. We assume $\theta \neq 1$ throughout the paper and comment in Section 3 on the role of this assumption. The final goods sector takes the price of the intermediate goods $\{p(j, t)\}$ as given and chooses $y(j, t)$ to maximize profits. It follows that

$$y(j, t) = \frac{Y(t)}{p(j, t)^{\theta}}.$$
2.3 Households

There is a representative household consisting of a measure 1 of members. The large household structure allows for full risk sharing within each household, a standard device for studying complete markets allocations.

At each moment in time $t$, each member of the representative household engages in one of the following mutually exclusive activities:

- $L(t)$ household members are located in one of the labor markets.
  - $E(t)$ of these workers are employed at the prevailing wage and get leisure 0.
  - $U_r(t) = L(t) - E(t)$ of these workers are rest unemployed and get leisure $b_r$.
- $U_s(t)$ household members are search unemployed, looking for a new labor market and getting leisure $b_s$.
- The remaining household members are inactive, getting leisure $b_i > b_s$.

Household members may costlessly switch between employment and rest unemployment and between inactivity and searching; however, they cannot switch labor markets without going through a spell of search unemployment.

Workers exit their labor market for inactivity or search in three circumstances: first, they may do so endogenously at any time at no cost; second, they must do when their market shuts down, which happens at rate $\delta$; and third, they must do so when they are hit by an idiosyncratic shock, according to a Poisson process with arrival rate $q$, independent across individuals and independent of their labor market’s productivity. We introduce the idiosyncratic “quit” shock $q$ to account for separations that are unrelated to the state of the industry.

A worker in search unemployment finds a job in the labor market of her choice according to a Poisson process with arrival rate $\alpha$. In Section 5, we consider a variant of the model where search unemployed workers instead find a random labor market.

We can represent the household’s preferences via the utility function

$$\int_0^\infty e^{-\rho t} \left( u(C(t)) + b_i \left( 1 - E(t) - U_r(t) - U_s(t) \right) + b_r U_r(t) + b_s U_s(t) \right) dt,$$

where $\rho > 0$ is the discount rate, $u$ is increasing, differentiable, strictly concave, and satisfies the Inada conditions $u'(0) = \infty$ and $\lim_{C \to \infty} u'(C) = 0$, and $C(t)$ is the household’s consumption of the final good. The household finances its consumption using its labor income.
2.4 Equilibrium

We look for a competitive equilibrium of this economy. At each instant, each household chooses how much to consume and how to allocate its members between employment in each labor market, rest unemployment in each labor market, search unemployment, and inactivity, in order to maximize utility subject to technological constraints on reallocating members across labor markets, taking as given the stochastic process for wages in each labor market; each final goods producer maximizes profits by choosing inputs taking as given the price for all the intermediate goods; and each intermediate goods producer $j$ maximizes profits by choosing how many workers to hire taking as given the wage in its labor market and the price of its good. Moreover, the demand for labor from intermediate goods producers is equal to the supply from households in each labor market; the demand for intermediate goods from the final goods producers is equal to the supply from intermediate goods producers; and the demand for final goods from the households is equal to the supply from the final goods producers.

Standard arguments imply that for given initial conditions, there is at most one competitive equilibrium of this economy. We look for a stationary equilibrium where all aggregate quantities and the joint distribution of wages, productivity, output, employment, and rest unemployment across labor markets are constant.

3 Frictionless Model

To understand the mechanics of the model, we start with a version where nonworkers can instantaneously become workers; formally, this is equivalent to the limit of the model when $\alpha \to \infty$. In this limit, the household does not need to devote any workers to search unemployment. Moreover, assuming $b_i > b_r$, there is no rest unemployment, since resting is dominated by inactivity. Thus the household divides its time between employment and inactivity. Finally, with costless mobility all workers must earn a common wage $w(t)$.

The household therefore solves

$$\max_{E(t)} \int_0^\infty e^{-\rho t} (u(C(t)) + b_i (1 - E(t))) \, dt,$$

subject to the budget constraint $C(t) = w(t) E(t)$. Assuming an interior equilibrium, $E(t) \in (0, 1)$, the first order conditions imply that at each date $t$, $b_i = w(t) u'(C(t))$. Put differently,

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4The first welfare theorem implies that any equilibrium is Pareto optimal. Since all households have the same preferences and endowments, if there are multiple equilibria, household utility must be equal in each. But a convex combination of the equilibrium allocations would be feasible and Pareto superior, a contradiction.
let $\omega(t) \equiv \log w(t) + \log u'(C(t))$ denote the log wage, measured in units of marginal utility. This is pinned down by preferences alone, $\omega(t) = \log b_i$ for all $t$.

To determine the level of employment and consumption, we aggregate across markets. Consider labor market $j$ with productivity $x(j, t)$ and $l(j, t)$ workers at time $t$. Output is $y(j, t) = l(j, t)x(j, t)$ and so equation (3) implies the price of the good is $p(j, t) = (Y(t)/l(j, t)x(j, t))^{1/\theta}$. Since workers are paid their marginal revenue product, $p(j, t)x(j, t)$, the log wage measured in units of marginal utility is

$$
\omega(t) = \log Y(t) + (\theta - 1) \log x(j, t) - \log l(j, t) + \log u'(C(t)).
$$

(5)

We showed in the previous paragraph that $\omega(t)$ is pinned down by preferences, while consumption and output are aggregate variables and so are the same in different markets. Thus this equation determines how employment moves with productivity. When $\theta > 1$, more productive labor markets employ more workers, while if goods are poor substitutes, $\theta < 1$, an increase in labor productivity lowers employment so as to keep output relatively constant. If $\theta = 1$, employment is independent of productivity, an uninteresting case whose we omit its analysis from the rest of the paper.

To close the model, eliminate $l(j, t)$ from $y(j, t) = x(j, t)l(j, t)$ using equation (5). This gives $y(j, t) = Y(t)(x(j, t)u'(C(t))e^{-\omega(t)})^\theta$. Substitute this into equation (2) and simplify to obtain

$$
\frac{e^\omega(t)}{u'(C(t))} = \left( \int_0^1 x(j, t)^{\theta-1}dj \right)^{\frac{1}{\theta-1}}.
$$

With an invariant distribution for $x$, we can rewrite this equation by integrating across that distribution. Appendix A.1 finds an expression for the invariant distribution. Assuming

$$
\delta > (\theta - 1) \left( \mu_x + (\theta - 1) \frac{\sigma_x^2}{2} \right),
$$

(6)

we prove that

$$
\frac{e^\omega(t)}{u'(C(t))} = x_0 \left( \frac{\delta}{\delta - (\theta - 1) \left( \mu_x + (\theta - 1) \frac{\sigma_x^2}{2} \right)} \right)^{\frac{1}{\theta-1}}.
$$

(7)

This equation pins down aggregate consumption $C$ and hence aggregate output $Y$ in terms of model parameters. If condition (6) fails with $\theta > 1$, extremely productive firms would produce an enormous amount of easily-substitutable goods, driving consumption to infinity. If it fails with $\theta < 1$, extremely unproductive firms would require a huge amount of labor to produce any of the poorly-substitutable goods, driving consumption to zero.

Since $e^\omega(t) = b_i$, equation (7) determines the constant level of consumption $C = C(t)$. 

8
Finally, employment is also constant and determined from the budget constraint as \( E = C u'(C)e^{-\omega} \).

For future reference, we note two important properties of the frictionless economy, both of which carry over to the frictional economy. First, if \( \mu_x + \frac{1}{2}(\theta - 1)\sigma_x^2 = 0, \ w = x_0, \) independent of \( \delta \). The condition is equivalent to imposing that \( x^{\theta-1} \) is a martingale which, by equation (5), implies employment \( l \) is a martingale. This is a reasonable benchmark and ensures that aggregate quantities are well-behaved in the limit as \( \delta \) converges to zero. Second, an increase in the leisure value of inactivity \( b_i \) raises the marginal utility of consumption \( u'(C) \) by the same proportion, while employment \( E \) decreases in proportion to \( C \).

4 Directed Search Model

We now return to the model where it takes time to find a new labor market, \( \alpha < \infty \). We look for a steady state equilibrium where the household maintains constant consumption, obtains a constant income stream, and keeps a positive and constant fraction of its workers in each of the activities, employment, rest unemployment, search unemployment, and inactivity.

Equilibrium is a natural generalization of the frictionless model. Because it is costly to switch labor markets, different markets pay different wages at each point in time. Workers exit labor markets when wages fall too low, which puts a lower bound on wages. Searchers enter labor markets when wages rise too high, providing an upper bound. Between these bounds, there is no endogenous entry or exit of workers, and so wages fluctuate only due to exogenous quits and to productivity shocks.

Our approach to characterizing equilibrium mimics our approach in the previous section. Using the household optimization problem alone, we determine the stochastic process for the log wage measured in units of marginal utility. We then turn to aggregation to pin down output and employment.

4.1 Household Optimization

We assume parameter values are such that some household members work, some search, and some are inactive. The household values its members according to the expected present value of marginal utility that they generate either from leisure or from income; this lies in an interval \([v, \bar{v}]\). To find these bounds, first consider an individual who is permanently inactive. It is immediate from equation (4) that he contributes \( b_i/\rho \) to the household. Since the household
may freely shift workers into inactivity, this must be the lower bound on marginal utility,
\[ v = \frac{b_i}{\rho}. \tag{8} \]

The household can also freely shift workers into search unemployment, so \( v \) is also the expected present value of marginal utility for a searcher. But compared to an inactive worker, a searcher get low marginal utility today in return for the possibility of moving to a labor market where she generates high marginal utility. More precisely, a searcher gets flow utility \( b_s \) and finds the best labor market at rate \( \alpha \), giving a capital gain \( \bar{v} - v \). Since the present value of her utility is \( v \), this implies \( \rho v = b_s + \alpha (\bar{v} - v) \), pinning down \( \bar{v} \):

\[ \bar{v} = b_i \left( \frac{1}{\rho} + \kappa \right), \text{ where } \kappa \equiv \frac{b_i - b_s}{b_i \alpha} \tag{9} \]
is a measure of search costs, the percentage loss in current utility from searching rather than inactivity times the expected duration of search unemployment \( 1/\alpha \).

We turn next to the behavior of wages. As in the frictionless model, consider a typical labor market \( j \) at time \( t \) with productivity \( x(j, t) \) and \( l(j, t) \) workers. If all the workers were employed, labor demand would pin down the log wage measured in units of marginal utility,

\[ \omega(j, t) = \log Y + (\theta - 1) \log x(j, t) - \log l(j, t) \theta + \log u'(C). \tag{10} \]

We omit the derivation of this equation, which is unchanged from equation (5). But if \( \omega(j, t) < \log b_r \), the household get more utility from rest unemployment. In equilibrium, employment falls, raising the wage until it is equal to \( b_r \) and households are indifferent between the two activities. To stress this point, we call \( \omega(j, t) \) the log full-employment wage and note that the actual log wage is the maximum of this and \( \log b_r \).

We claim that in equilibrium, the expected present value of a household member in a labor market depends only on the current log full employment wage,

\[ v(\omega) = \mathbb{E} \left( \int_0^\infty e^{-(\rho + q + \delta) t} \left( \max\{b_r, e^{\omega(j, t)}\} + (q + \delta) v \right) dt \bigg| \omega(j, 0) = \omega \right), \tag{11} \]

where expectations are taken with respect to future values of the random variable \( \omega(j, t) \), whose behavior we discuss further below. The discount rate \( \rho + q + \delta \) accounts for impatience, for the possibility that the worker exits the market exogenously, and for the possibility that the labor market ends exogenously. The time-\( t \) payoﬀ is the prevailing wage; this holds whether the worker is employed or rest unemployed because when there is rest unemployment,
the worker is indifferent between the two states. In addition, if the worker exogenously leaves the market, which happens with hazard rate \( q + \delta \), the household gets a terminal value \( v \).

In equilibrium, the expected present value of a worker in a labor market must lie in the interval \([v, \bar{v}]\). If \( v(\omega) > \bar{v} \), searchers would enter the labor market, reducing \( \omega \) and with it \( v(\omega) \). If \( v(\omega) < \bar{v} \), workers would exit the labor market, raising \( \omega \) and \( v(\omega) \). Thus workers’ entry and exit decisions determine two thresholds for the log full employment wage, \( \underbar{\omega} < \bar{\omega} \), with the value function satisfying

\[
v(\omega) \in [v, \bar{v}] \text{ for all } \omega \\
v(\bar{\omega}) = \bar{v} \\
v(\underbar{\omega}) = v \text{ if } \omega > -\infty.
\]  

(12)

The last equation allows for the possibility that workers never choose to endogenously exit a labor market, as will be the case if the leisure from rest unemployment exceeds the leisure from inactivity, \( b_r \geq b_i \).

Finally we turn to the dynamics of the log full employment wage, a regulated Brownian motion on the interval \([\omega, \bar{\omega}]\). When \( \omega(j, t) \in (\omega, \bar{\omega}) \), only productivity shocks and the deterministic exit of workers change \( \omega \). From equation (10), this implies \( d\omega(j, t) = \mu dt + \sigma_\omega dz(j, t) \), where

\[
\mu \equiv \frac{\theta - 1}{\theta} \mu_x + \frac{q}{\theta}, \quad \sigma_\omega \equiv \frac{\theta - 1}{\theta} \sigma_x, \quad \text{and } \sigma \equiv |\sigma_\omega|,
\]  

(13)

i.e., in this range \( \omega(j, t) \) has drift \( \mu \) and instantaneous standard deviation \( \sigma \). When the thresholds \( \omega \) and \( \bar{\omega} \) are finite, they act as reflecting barriers, since productivity shocks that would move \( \omega \) outside the boundaries are offset by the entry and exit of workers. Expectations in equation (11) are taken with respect to the stochastic process for wages.

These equations uniquely determine the thresholds \( \omega \) and \( \bar{\omega} \):

**Proposition 1.** Equations (11) and (12) uniquely define \( \omega \) and \( \bar{\omega} \) as functions of model parameters. A proportional increase in \( b_i, b_r \), and \( b_s \) raises \( e^\omega \) and \( e^{\bar{\omega}} \) by the same proportion. Moreover, \( \underbar{\omega} < \log b_i < \bar{\omega} < \infty \), with \( \omega > -\infty \) if and only if \( b_r < b_i \).

This generalizes the frictionless equilibrium, where household optimization pins down \( \omega = \log b_i \). With frictions, the log full-employment wage can fluctuate within some boundaries, with an unchanged relationship between those boundaries are the preference for leisure.

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5Note that equation (11) assumes that a worker never endogenously chooses to exit a market. This is appropriate because a worker never strictly prefers to do so, but rather is always willing to stay if enough other workers exit.
There are two ways to prove uniqueness of these boundaries. The proof in Appendix A.2 relies on monotonicity of the value function in the boundaries, showing that an increase in either boundary raises the value function, but an increase in the upper (lower) bound affects \( v \) more when \( \omega \) is close to the upper(lower) bound. Alternatively, in online Appendix B.2 we develop an alternative proof relying on the solution to an “island planner’s problem.”

Notice that the state of an industry in our model is the one dimensional object \( \omega \), while in Lucas and Prescott (1974) the state is two dimensional.\(^6\) Lucas and Prescott (1974) must include productivity \( x \) and employment \( l \) as separate state variables because they consider a general class of processes for \( x \), in particular allowing for mean reversion. While \( \omega \) still determines current wages in their setup, it is not a Markov process, so there is no analog of equation (13) in their setup. The assumption that \( \log x \) is a Brownian motion with drift permits us to reduce the state variable to a single dimension. This simplification is common in the partial equilibrium literature on irreversible investment (Bentolila and Bertola, 1990; Abel and Eberly, 1996; Caballero and Engel, 1999) and enables us to provide a more complete analytical characterization of the equilibrium than could Lucas and Prescott (1974).

The Proposition establishes that \( \omega \) is finite when \( b_r < b_i \). Intuitively, if a market is hit by sufficiently adverse shocks, workers will leave since rest unemployment is costly and has low expected payoffs. In contrast, when \( b_r \geq b_i \), rest unemployment is costless and hence workers only leave labor markets when they shut down. Moreover, if \( b_r \leq b_i \), there is no rest unemployment in the best labor markets, \( \bar{\omega} > \log b_r \). The next proposition addresses whether there is rest unemployment in the worst labor markets, \( \omega \geq \log b_r \).

**Proposition 2.** There exists a \( \tilde{b}_r \) such that in an equilibrium, \( b_r \geq \epsilon \sim \tilde{b}_r \) if and only if \( b_r \geq \tilde{b}_r \), with \( \tilde{b}_r = B(\kappa, \rho + q + \delta, \mu, \sigma)b_i \) for some function \( B \), positive-valued and decreasing in \( \kappa \) with \( B(0, \rho + q + \delta, \mu, \sigma) = 1 \).

The proof is in Appendix A.3. This Proposition implies that there is rest unemployment if search costs \( \kappa \) are sufficiently high given any \( b_r > 0 \), or equivalently if the leisure value of resting \( b_r \) is sufficiently close to the leisure of inactivity \( b_i \) given any \( \kappa > 0 \). If a searcher finds a job sufficiently fast (so \( \kappa \) is small) or resting gives too little leisure (so \( b_r \) is small), there is no reason to wait for labor market conditions to improve, and so \( B \) is monotone in \( \kappa \).

\(^6\)\( \max\{b_r, e^\omega\} \) is analogous to \( R(x, l) \) in Lucas and Prescott’s (1974) notation. Their production technology implies that \( Y \) does not affect \( R \), while risk-neutrality ensures that \( u'(C) \) is constant. Lucas and Prescott (1974) also assume that \( R_x > 0 \)—see their equation (1)—which in our set-up is equivalent to \( \theta > 1 \).
4.2 Aggregation

Having pinned down the thresholds $\omega$ and $\bar{\omega}$, we now prove that consumption, employment, and unemployment are uniquely determined. The approach is again analogous to our analysis of the frictionless model.

**Proposition 3.** There exists a unique equilibrium. The steady state density of workers across log full employment wages is

$$f(\omega) = \frac{\sum_{i=1}^{2} |\eta_i + \theta| e^{\eta_i(\omega - \bar{\omega})} \eta_i}{\sum_{i=1}^{2} |\eta_i + \theta| e^{\eta_i(\omega - \bar{\omega})} - 1},$$

(14)

where $\eta_1 < 0 < \eta_2$ solve $q + \delta = -\mu \eta + \frac{\sigma^2}{2} \eta^2$.

The proof in Appendix A.4 also provides explicit equations for output and consumption (which are equal by market clearing) and the number of workers in labor markets $L$.

We turn next to the rest unemployment rate. Recall that $U_r$ is the fraction of household members who are rest unemployed. If $\log b_r \leq \omega$, this is zero. Otherwise, in a market with $\omega \in [\omega, \bar{\omega}]$, the rest unemployment rate is $1 - e^{\theta(\omega - \bar{\omega})}$. Integrating across such markets using equation (14) gives

$$U_r L = \int_{\omega}^{\bar{\omega}} (1 - e^{\theta(\omega - \bar{\omega})}) f(\omega) d\omega = \theta \frac{e^{\eta_2(\omega - \bar{\omega}) - 1}}{\eta_2} \frac{e^{\eta_1(\omega - \bar{\omega}) - 1}}{\eta_1} \sum_{i=1}^{2} |\eta_i + \theta| e^{\eta_i(\omega - \bar{\omega})} - 1.$$

(15)

The remaining household members who are in labor markets are employed, $E = L - U_r$.

Finally we pin down the search unemployment rate. Let $N_s$ be the number of workers among $L$ that leave their labor market per unit of time, either because conditions are sufficiently bad or because their labor market has exogenously shut down. Appendix A.6 takes limits of the discrete time, discrete state space model to show that this rate is given by

$$N_s = \frac{\theta \sigma^2}{2} f(\omega) L + (q + \delta) L.$$

(16)

The first term gives the fraction of workers who leave their labor market to keep $\omega$ above $\omega$. The second term is the fraction of workers who exogenously leave their market. In steady state, the fraction of workers who leave labor markets must balance the fraction of workers who arrive in labor markets. The latter is given by the fraction of workers engaged in search unemployment $U_s$, times the rate at which they arrive to the labor market $\alpha$, so $\alpha U_s = N_s$. Solve equation (16) using equation (14) to obtain an expression for the ratio of
search unemployment to workers in labor markets:

\[
\frac{U_s}{L} = \frac{1}{\alpha} \left( \frac{\theta \sigma^2}{2} \frac{\eta_2 - \eta_1}{\sum_{i=1}^{2} |\theta + \eta_i| \frac{e^{\eta_i (\omega - \hat{\omega})}}{\eta_i}} + q + \delta \right)
\]  

(17)

To have an interior equilibrium we require that \(U_s + U_r + E \leq 1\) so that the labor force is smaller than the total population.\(^7\)

We deliberately leave the expressions for unemployment as a function of the thresholds \(\omega\), \(\hat{\omega}\), and \(\bar{\omega}\) in order disentangle optimization—the choice of thresholds—from the mechanics of aggregation. This has two advantages. First, we find it useful to exploit this dichotomy in our numerical evaluation of the model in Section 6. Second, the expressions for rest and search unemployment as a function of the thresholds are identical in other variants of the model, including the original Lucas and Prescott (1974) model. For example, suppose the curvature of labor demand comes from diminishing returns at the island level, due to a fixed factor, rather than imperfect substitutability (see footnote 6). Then the analog of the elasticity of substitution is the reciprocal of the elasticity of revenue with respect to the fixed factor, while the expressions for unemployment are otherwise unchanged.

We close this section by noting some homogeneity properties of employment, rest unemployment, search unemployment, and consumption.

**Proposition 4.** Let \(b_r = \chi \bar{b}_r\), \(b_s = \chi \bar{b}_s\), \(b_i = \chi \bar{b}_i\) for fixed \(\bar{b}_r\), \(\bar{b}_s\), and \(\bar{b}_i\). The equilibrium value of the unemployment rate \(\frac{U_s + U_r}{U_s + U_r + E}\) and the share of rest unemployed \(\frac{U_r}{U_r + U_s}\) do not depend on \(\chi\), the level of productivity \(x_0\), or the utility function \(u\), while \(u'(Y)\) is proportional to \(\chi/x_0\).

**Proof.** By inspection, the unemployment rate and share of rest unemployed are functions of the difference in thresholds \(\omega - \hat{\omega}\) and \(\bar{\omega} - \hat{\omega}\) and the parameters \(\alpha, \delta, q, \theta, \mu\) (or \(\mu_x\)), and \(\sigma\) (or \(\sigma_x\)), either directly or indirectly through the roots \(\eta_i\). From Proposition 1, the thresholds depend on the same parameters and on the discount rate \(\rho\). This completes the first part of the proof.

Next, recall from Proposition 1 that \(e^{\omega}\) and \(e^{|\omega|}\) are proportional to \(\chi\). Then equation (55) implies \(u'(Y)\) inherits the same proportionality. On the other hand, Proposition 1 implies \(x_0\) does not affect any of the thresholds and so equation (55) implies \(u'(Y)\) is inversely proportional to \(x_0\).  

\(^7\)If this condition fails, all household members participate in the labor market. The equilibrium is equivalent to one with a higher leisure value of inactivity, the value of \(b_i\) such that \(U_s + U_r + E = 1\). In any case, Proposition 4 implies that for \(b_r\), \(b_s\), and \(b_i\) large enough, the equilibrium has \(U_s + U_r + E < 1\).
This proposition shows that the unemployment rate and composition of unemployment is determined by the relative advantage of different leisure activities, while output, and hence consumption and employment, depends on an absolute comparison of leisure versus market production. Indeed, the finding that \( u'(Y) \) is proportional to \( \chi/x_0 \) holds in the frictionless benchmark, where an interior solution for the employment rate requires \( b_i = u'(Y)w \), while the wage is proportional to \( x_0 \) (see equation 7). Whether higher productivity lowers or raises equilibrium employment depends on whether income or substitution effects dominate in labor supply. With \( u(Y) = \log Y \), an increase in productivity raises consumption proportionately without affecting employment or labor force participation.

4.3 The Limiting Economy

We close this section by discussing a useful limit of the model, when the exogenous shut-down rate of markets \( \delta \) is zero. We introduced the assumption that labor markets shut down for technical reasons, to ensure an invariant distribution of productivity and employment. Still, with the parameter restriction, \( \mu_x + \frac{1}{2}(\theta - 1)\sigma_x^2 = 0 \), discussed previously in Section 3, the economy is well behaved even when \( \delta \) limits to zero. It is clear from Proposition 1 that \( \bar{\omega} \) and \( \hat{\omega} \) converge nicely for any value of \( \mu_x \) as long as the discount rate \( \rho \) is positive. More problematic is whether aggregate employment, unemployment, and output converge. This section verifies that the same parameter restriction yields a well-behaved limit of the frictional economy.

When \( \mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2 \) and \( \delta \to 0 \), the roots \( \eta_1 < 0 < \eta_2 \) in Proposition 3 converge to \( \eta_1 = -\theta \) and \( \eta_2 = 2q/\theta\sigma^2 \). Substituting into equation (14), we find

\[
f(\omega) = \frac{\eta_2^{\eta_2(\omega-\bar{\omega})}}{e^{\eta_2(\bar{\omega}-\omega)} - 1}.
\]

If \( q = 0 \) as well, this simplifies further to \( f(\omega) = 1/(\bar{\omega} - \omega) \), i.e. \( f \) is uniform on its support, while for positive \( q \) the density is increasing in \( \omega \). We can also take limits of equations (54) and (55) in Appendix A.4 to prove that the number of workers in labor markets and output have well-behaved limits.

Turning next to unemployment, equations (15) and (17) imply that in the limit,

\[
\frac{U_r}{L} = \frac{\theta e^{\eta_2(\omega-\bar{\omega})} - 1}{\eta_2} + e^{-\theta(\bar{\omega}-\omega)} - 1 \quad \text{and} \quad \frac{U_s}{L} = \frac{q}{\alpha(1 - e^{\eta_2(\omega-\bar{\omega})})}.
\]
Each of these expressions simplifies further when there are no quits, \( q = 0 \) and so \( \eta_2 \to 0 \):\(^8\)

\[
\frac{U_r}{L} = \frac{\theta(\bar{\omega} - \omega) + e^{-\theta(\bar{\omega} - \omega)} - 1}{\theta(\bar{\omega} - \omega)} \quad \text{and} \quad \frac{U_s}{L} = \frac{\theta\sigma^2}{2\alpha(\bar{\omega} - \omega)}.
\]  

(19)

The last expression is particularly intuitive. With our parameter restrictions, \( \mu = -\frac{1}{2}\theta\sigma^2 \).

Then \(-\mu/(\bar{\omega} - \omega)\) reflects the average duration of an employment spell, since wages must fall from \( \bar{\omega} \) to \( \omega \) and drift down on average at rate \( \mu \), while \( 1/\alpha \) is the average duration of an unemployment spell.

## 5 Random Search Model

In this section we analyze an alternative technology for search. Instead of locating the best labor market, we assume that agents can only locate markets randomly. As in the directed search case, we obtain a separation between optimization and aggregation, with simple expressions for the reduced form expression for the unemployment rates. This shows that our approach is robust to the specification of the mobility technology. Additionally, we find this alternative specification interesting because wages are not regulated from above, and hence they can, in principle, have very different statistical behavior.

### 5.1 Setup

As much as possible, our setup parallels the one with directed search. We leave our notation unchanged and focus on the differences between the two models.

The search unemployed engage in one of two mutually exclusive activities. First, \( U_{s,r} \) search randomly, finding a labor market at rate \( \alpha \). The probability of contacting any particular market \( j \) is proportional to the number of workers in that market at time \( t \), \( l(j,t) \), as in Burdett and Vishwanath (1988). The assumption that workers are not more likely find a high wage industry is an extreme alternative to our directed search model. One interpretation is that a worker searching randomly contacts another worker currently in a market at rate \( \alpha \), and is equally likely to contact any worker, regardless of the state of her market. Second, \( U_{s,n} \) workers search for a new market, finding one at the same rate \( \alpha \). This ensures that there are some workers who can get a new market with productivity \( x_0 \) off the ground.

---

\(^8\)The order of convergence of \( \delta \) and \( q \) to zero does not affect these results.
5.2 Household Optimization

The value of permanent inactivity is given by $v$ as described in equation (8). In an equilibrium with inactivity and with search, the expected value of a worker concluding either search activity must be $\bar{v}$, defined in equation (9).

A market $j$ at time $t$ is characterized by the log full-employment wage $\omega(j, t)$, unchanged from equation (10), with the value of a worker in such a market still given by equation (11); however, random search affects the stochastic process for wages and thus the value function. As in the directed search model, $\omega$ is regulated from below by agents’ willingness to exit a market with bad prospects, but it is not regulated from above because of the absence of directed search. That is,

\[
\begin{align*}
v(\omega) &\geq v \text{ for all } \omega \\
v(\omega) &= v \text{ if } \omega > -\infty.
\end{align*}
\]

The arrival of random searchers causes gross labor force growth at some endogenous rate $s$, a key variable in what follows. This implies that $\omega$ follows a Brownian motion regulated below at $\underline{\omega}$ with $d\omega(j, t) = \mu dt + \sigma_{\omega}dz(j, t)$, where

\[
\mu \equiv \frac{\theta - 1}{\theta} \mu_x + \frac{q - s}{\theta}, \quad \sigma_{\omega} \equiv \frac{\theta - 1}{\theta} \sigma_x, \quad \text{and} \quad \sigma \equiv |\sigma_{\omega}|.
\]

Thus the arrival of random searchers puts downward pressure on wages, the opposite effect of exogenous quits. Indeed, it is not surprising that only the difference between these two rates affects the stochastic process for wages.

Turn next to the value of workers who have successfully completed a search spell,

\[
\int_{-\infty}^{\infty} v(\omega)f(\omega)d\omega = \bar{v}
\]

\[
v(\omega_0) = \bar{v}.
\]

The first equation states that the expected value of a completed random search spell must be $\bar{v}$, where $f(\omega)$ is the density of log full-employment wages across markets. The second equation states that the value of a completed search for a new market must be $\bar{v}$, which pins down $\omega_0$, the initial wage in new markets. We prove in Appendix A.7 that the density of log
full-employment wages satisfies
\[
f(\omega) = \begin{cases} 
\left(\eta_1 \eta_2 + \frac{2\bar{L}\sigma}{\bar{L}}\right) \left((\eta_2 + \theta) \exp(\omega - \omega_2) - (\eta_1 + \theta) \exp(\omega - \omega_1)\right) \\
\frac{\theta(\eta_2 - \eta_1)}{\eta_2 - \eta_1} \left(\frac{2\bar{L}}{\sigma^2} \left(\exp(\omega - \omega) - \exp(\omega - \omega_0)\right)\right)
\end{cases}
\]
if \( \omega \in [\omega, \omega_0] \)
\[
\frac{\theta(\eta_2 - \eta_1)}{\eta_2 - \eta_1} \left(\frac{2\bar{L}}{\sigma^2} \left(\exp(\omega - \omega_0) - \exp(\omega - \omega_0)\right)\right)
\]
if \( \omega > \omega_0 \),

where \( \eta_1 < \eta_2 < 0 \) solve \( q + \delta - s = -\mu \eta + \frac{\sigma^2}{2} \eta^2 \) and \( \bar{L}/L \) is the number of workers in a new labor market relative to the overall number of workers in labor markets. Note that there is a kink in the density at \( \omega_0 \), reflecting the entry of new markets; if \( \delta = 0 \), the density is smooth.

To close the model, we describe the ratio \( \bar{L}/L \) using the invariant distribution \( \tilde{f} \) for the process \( \{\omega(t), \log l(t)\} \) across markets. Formally, we can first describe \( \{\omega(t)\} \) as a one sided reflected diffusion and jump process on \( [\omega, \infty) \), and then use this process to describe \( \{\log l(t)\} \) on the real line. Let \( Z(t) \) be a standard Brownian motion, \( N(t) \) the counter associated with a homogeneous Poisson process with intensity \( \delta \), and \( B(t) \) an increasing singular process:
\[
\begin{align*}
\omega(0) &= \omega_0, \quad \log l(0) = \log \bar{L}, \\
d\omega(t) &= \mu dt + \sigma dZ(t) + (\omega_0 - \omega(t)) dN(t) + dB(t), \\
d\log l(t) &= (s - q) dt + (\log \bar{L} - \log l(t)) dN(t) - \theta dB(t),
\end{align*}
\]
(25)

where for all \( T > 0: \int_0^T I_{\{\omega(t) \geq \omega\}} dB(t) = 0 \). In this definition we start each industry at \( (\omega_0, \bar{L}) \). When the industry is destroyed, at rate \( \delta \) per unit of time, we replace it by a new one with the same initial conditions. Given this recurrence, this process has a unique invariant distribution for all \( \delta > 0 \). We denote this by \( \tilde{f}(\omega, l) \).

The distribution \( \tilde{f} \) implies a value for \( \bar{L}/L \). To see this notice that equation (25) implies that an increase in \( \log \bar{L} \) increases all the realizations of each path of \( \{\log l(t)\} \) by the same amount. Alternatively, equation (25) can be rewritten for the process \( \log(l(t)/\bar{L}) \), which makes no other reference to \( \bar{L} \). To denote the dependence of \( \tilde{f} \) on \( L \) we write \( \tilde{f}(\omega, l; L) \). We have that \( L \) is the measure of agents in markets, so that:
\[
L = \int_0^\infty \int_\omega^\infty f(\omega, l; L) d\omega dl = \bar{L} \int_0^\infty \int_\omega^\infty \tilde{f}(\omega, l; \bar{L}) d\omega dl.
\]
Then we can write a condition for the ratio $\bar{L}/L$ as

$$\bar{L}/L = \left( \int_0^\infty \int_0^\infty \tilde{f}(\omega, l; 1) dl \right)^{-1}.$$  

(26)

Although we do not explicitly solve for $\tilde{f}(\omega, l; 1)$, equation (25) implies that it depends only on $\omega$, $s$, and $\omega_0$.

Equations (20), (22), (23), and (26) determine $\omega$, $s$, $\omega_0$, and $\bar{L}/L$ in terms of model parameters. Although we do not have a general existence and uniqueness proof for the random search model, analogous to Proposition 1, later in this section we prove existence in the limit as $\delta \to 0$. We also establish uniqueness for a particular case, without exogenous quits or rest unemployment.

5.3 Aggregation

To close the model, first determine output $Y$ and the number of workers in labor markets $L$. We omit the results, which are unchanged from the directed search model in Appendix A.4. We focus instead on the unemployment rates. For the search unemployed, we have

$$U_{s,r}/L = s/\alpha \quad \text{and} \quad U_{s,n}/L = (\bar{L}/L)(\delta/\alpha).$$  

(27)

These equations balance inflows and outflows from each state. For example, random searchers flow into labor markets at rate $sL$ and find jobs at rate $\alpha$, so there must be $U_{s,r} = sL/\alpha$ such workers. For rest unemployment, we simply have

$$U_{r}/L = \int_{\hat{\omega}}^\infty (1 - e^{\theta(\omega-\hat{\omega})}) f(\omega) d\omega,$$  

(28)

where $\hat{\omega} \equiv \min\{\omega, \log b_r\}$.

5.4 The Limiting Economy

We focus again on the special case with $\mu_x = -\frac{1}{2}(\theta - 1)\sigma^2_x$, so we can take the limit as $\delta$ converges to zero. In this case, the first block of equations describing an equilibrium becomes a set of two equations in two unknowns, namely equations (20) and (22) determining $\bar{\omega}$ and $s$. This is because neither $\bar{\omega}$ nor $\bar{L}/L$ affect the density $f$ in this limit. Moreover, the roots $\eta_i$ satisfy

$$\eta_1 = -\theta, \quad \eta_2 = -\frac{2(s-q)\theta}{\theta \sigma^2},$$  

(29)
so that the density of workers across markets is exponential:

\[ f(\omega) = -\eta_2 e^{\eta_2 (\omega - \bar{\omega})}. \]  

(30)

Using this expression for \( f \), equation (28) reduces to

\[ \frac{U_r}{L} = 1 + \frac{\eta_1}{\eta_2 - \eta_1} e^{\eta_2 (\bar{\omega} - \omega)} + \frac{\eta_2}{\eta_1 - \eta_2} e^{\eta_1 (\bar{\omega} - \omega)}, \]  

(31)

while the search unemployment rate is still given by equation (27). It is straightforward to prove existence of equilibrium in this limiting economy and, under more restrictive conditions, uniqueness of equilibrium:

**Proposition 5.** There exists an equilibrium pair \((\bar{\omega}, s)\) for the random search model with \( \mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2 \) and \( \delta \to 0 \). Moreover, \( s > q + \frac{1}{2}\sigma^2 \). If \( q = b_r = 0 \), the equilibrium is unique. In that case, an increase in search costs \( \kappa \) reduces the equilibrium values of \( s \) and \( \bar{\omega} \), raising the search unemployment rate.

The proof is in Appendix A.8. We believe the uniqueness result should hold more generally.

### 6 Persistence of Wages

The Lucas and Prescott (1974) search model emphasizes that idiosyncratic productivity shocks cause wage dispersion across industries, although this is moderated by workers’ search for better job opportunities. Thus the effectiveness of search unemployment should determine the extent of wage dispersion in the economy. This motivates our empirical approach for evaluating the model: we examine whether model-generated data can approximate well the stochastic process for industry wages that we observe in the data. More precisely, we develop an auxiliary statistical model that describes the behavior of wages in the data and show that wages are extremely persistent. We then show that if all unemployment is due to search frictions, neither version of our model is capable of generating such persistent wages. Our full model with rest unemployment comes closer to matching the data. We use this evidence to help calibrate the structural parameters of our model.

#### 6.1 Auxiliary Statistical Model

We start by describing our auxiliary statistical model. To do this, we first must take a stand on the empirical counterpart of a labor market. According to our model, a labor market has two defining characteristics. First, the goods produced within a labor market are homogeneous
while the goods produced in different labor markets are heterogeneous, as captured by the
elasticity of substitution $\theta$. This suggests modeling a labor market as an industry. Second,
workers are free to move within a labor market but not between labor markets, perhaps
both because of some specificity of human capital or because of geographic mobility costs.
To the extent that human capital is occupation, not industry, specific (Kambourov and
Manovskii, 2007), this suggests that a labor market may be a cross between an occupation
and a geographic location. In the end, our definition of a labor market is governed by data
availability: we measure a labor market as a five-digit NAICS industry.

We then measure average weekly earnings in industry $j$ and year $t$, $\tilde{w}_{j,t}$, for $J = 312$ indus-
tries from 1990 to 2006 from the Current Employment Statistics (http://www.bls.gov/ces/),
all the industries with available data. Deflate this by average weekly earnings in the private
sector, $\tilde{w}_t$, so $w_{j,t} \equiv (\tilde{w}_{j,t}/\tilde{w}_t)$. Then estimate the autocorrelation of wages using a fixed
effects regression, as in Nickell (1981):

$$
\log w_{j,t+1} = \log w_j + \beta_w \log w_{j,t} + \sigma_w \epsilon_{j,t+1}, \quad (32)
$$

where $\epsilon_{j,t}$ has mean zero, standard deviation 1, and is independent across industries and
over time. $w_j$ represents an industry fixed effect, while $\sigma_w$ is the standard deviation of wage
innovations. We are primarily interested in the estimated value $\hat{\beta}_w$, which informs us about
the autocorrelation in wages.

We introduce the industry fixed effect $\log w_j$ in equation (32) to recognize that other
features of the labor market may induce permanent differences in wages across industries. In
Appendix C we develop a version of the model with heterogeneity in human capital, where
most workers are only able to work in a minority of labor markets, while a few workers can
work in all labor markets. Such scarce labor earns a higher average wage; however, we find
that the stochastic process for log wages across industries differs only by a constant. This
is consistent with our specification here, where we would estimate a high fixed effect for
industries where labor is scarce.

Still, log wages generated by our model do not follow an AR(1) as in equation (32), but
rather are a regulated Brownian Motion, or an even less linear process in the presence of
rest unemployment. We regard equation (32) as an auxiliary statistical model, useful for
summarizing properties of log wages in the data as well as in the model. Our use of this
model is in the spirit of indirect inference. Even if our economic model is misspecified, we can

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9Suppose that labor productivity in industry $j$ at time $t$ is $A(t)x(j,t)$. Fluctuations in aggregate pro-
ductivity $A(t)$ cause fluctuations in average earnings. With log utility, such fluctuations cause proportional
changes in wages but do not affect wages measured in units of marginal utility or the unemployment rate.
Deflating by $\tilde{w}_t$ is therefore a perfect control for aggregate fluctuations according to our theory.
Table 1: Estimates of equation (32) using CES data from 1990 to 2006 at different levels of aggregation. $J$ indicates the number of industries used in estimation, $\hat{\beta}_w$ is our estimate of autoregression in wages and $\hat{\sigma}_w$ is our estimate of the standard deviation of wage innovations.

<table>
<thead>
<tr>
<th>aggregation</th>
<th>$J$</th>
<th>$\hat{\beta}_w$</th>
<th>$\hat{\sigma}_w$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 digit</td>
<td>15</td>
<td>0.914</td>
<td>0.013</td>
</tr>
<tr>
<td>3 digit</td>
<td>75</td>
<td>0.913</td>
<td>0.023</td>
</tr>
<tr>
<td>4 digit</td>
<td>233</td>
<td>0.889</td>
<td>0.027</td>
</tr>
<tr>
<td>5 digit</td>
<td>312</td>
<td>0.905</td>
<td>0.033</td>
</tr>
<tr>
<td>6 digit</td>
<td>175</td>
<td>0.878</td>
<td>0.033</td>
</tr>
</tbody>
</table>

Nickell (1981) assumes that wages are generated for $J$ industries drawn from ergodic distribution implied by equation (32) and that each is observed for $T + 1$ periods. Under these ideal circumstances, the fixed effects regression estimate of $\beta_w$ satisfies

$$\hat{\beta}_w \rightarrow \beta_w + \left( \frac{2\beta_w}{1 - \beta^2_w} - \left[ \frac{1 + \beta_w}{T - 1} \left( 1 - \frac{1 - \beta^T_w}{T(1 - \beta_w)} \right) \right]^{-1} \right)^{-1}$$

when $J \rightarrow \infty$. For fixed $T$, this is an increasing function of $\beta_w$. If in fact $\beta_w = 1$, $\hat{\beta}_w \rightarrow \frac{T-2}{T+1}$. With our sample size of $T + 1 = 17$, this implies $\hat{\beta}_w \rightarrow 0.82$ as $J \rightarrow \infty$, substantially smaller than the estimates in Table 1. This suggests that empirically we are unlikely to be able reject a unit root in wages.

Table 1 also describes the behavior of the innovation in wages, $\hat{\sigma}_w$. Later we will find that this is also a useful target when calibrating our model. At the five-digit level, the standard deviation of wage changes is about three percent per year. This measure of volatility is smaller as we look at more aggregate data, presumably reflecting a less-than-perfect correlation in shocks. Although the results at the six-digit level are similar to those at the five-digit level, some caution is warranted due to the small number of observations at this level of aggregation, more than half of which are in manufacturing.

Blanchard and Katz (1992) also report a high persistence in relative wages in manufacturing across US states between 1952-1990. They fail to reject a unit root for 47 out of 51
states (including the District of Columbia). Pooling the 51 states, including fixed effects, they estimate an AR(4) with an implied half life of 11 years and an implied first order autocorrelation of 0.94.\footnote{Based upon the numbers reported in the third column of their Table 1.} Like our estimates in Table 1, these do not correct for finite sample biases.

6.2 Random Search Model without Rest Unemployment

We next examine the persistence of wages in our theoretical model. We assume that productivity drift satisfies \( \mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2 \). Recall that this parameter restriction implies that employment is a martingale in the frictionless model. With frictions, the expected change in employment per unit of time converges to zero as the time horizon increases. This parameter restriction allows us to focus on the limit without exogenous destruction of labor markets, \( \delta \to 0 \), since old labor markets and new ones have roughly the same number of workers. For now we also suppose that there is only search unemployment, as is the case if the leisure from rest unemployment is zero, \( b_r = 0 \).

Consider first the random search model in Section 5. Equation (27) implies that unemployed workers lose their job at rate \( \alpha U_{s,r}/L = s \). Then equation (21) implies that log wages satisfy

\[
d\omega(j, t) = -\left(\frac{1}{2}\theta \sigma^2 + \frac{\alpha U_{s,r}/L - q}{\theta}\right) dt + \sigma dz(j, t)
\]

on \([\omega, \infty)\). The estimated autocorrelation \( \hat{\beta}_w \) depends on two objects, \( \alpha U_{s,r}/L - q \) and \( \theta \sigma \). To prove this define \( \tilde{\omega}(j, t) = (\omega(j, t) - \omega) / \sigma \). The stochastic process for \( \tilde{\omega} \) is slightly simpler than the one for \( \omega \),

\[
d\tilde{\omega}(j, t) = -\left(\frac{1}{2}\theta \sigma + \frac{\alpha U_{s,r}/L - q}{\theta \sigma}\right) dt + dz(j, t)
\]

for \( \tilde{\omega}(j, t) \geq 0 \), but has the same estimated autoregressive coefficient \( \hat{\beta}_w \).

Fixing the other parameters, the measured autoregressive coefficient \( \hat{\beta}_w \) is a non-monotone function of \( \theta \sigma \) because the drift is a non-monotone function of this object. Indeed, the more negative is the drift, the more time that \( \tilde{\omega}(j, t) \) spends near zero, which reduces the measured autoregressive coefficient.\footnote{More precisely, the theoretical infinite sample correlation between \( \tilde{\omega}(j, t) \) and \( \tilde{\omega}(j, t + 1) \) decreases when the drift in \( \tilde{\omega}(j, t) \) becomes negative. As we discuss below, finite samples and time aggregation complicate our analysis, but the claim is supported by our numerical simulations of the model.} The autoregressive coefficient is maximized when the drift is
maximized, at $\theta \sigma = \sqrt{2(\alpha U_{s,r}/L - q)}$. This gives

$$d\tilde{\omega}(j, t) = -\sqrt{2(\alpha U_{s,r}/L - q)} dt + dz(j, t).$$

(34)

Thus to find a bound on the autoregressive coefficient, we simply need a value for $\alpha U_{s,r}/L - q$. $\alpha U_{s,r}/L$ is the rate that employed workers become unemployed: $\alpha U_{s,r}$ is the flow of workers from unemployment into employment and $L$ is the number of employed workers. This is easily measured. From 1990 to 2006, the unemployment rate in the United States averaged 5.5 percent. The mean duration of an in-progress unemployment spell was 0.31 years, which implies $\alpha = 3.2$ per year. Putting this together, employed workers become unemployed at rate $\alpha U_{s,r}/L \approx 0.186$ per year.

Now suppose there are no exogenous quits, $q = 0$. One can prove that the theoretical correlation between $\omega(j, t)$ and $\omega(j, t + 1)$ is 0.618 given the other parameter values. But in practice we have finite samples and the data we observe is time-averaged. Finite samples tend to reduce the measured autocorrelations—recall equation (33)—while time averaging raises them. To assess these offsetting effects, we thus turn to Monte Carlo simulations of a discrete time version of the model. We assume a period length is one day and measure the log of annual earnings for $J = 312$ industries and $T + 1 = 17$ years, the same as in the data. We then estimate equation (32) using the model-generated data. We repeat this 50 times to obtain an accurate estimate of $\hat{\beta}_w$. We find that on average $\hat{\beta}_w = 0.64$, with a standard deviation across samples of 0.03. This is significantly below the values we measure in U.S. data and show in Table 1.

Exogenous quits mitigate these results. Technically, this is because exogenous quits reduce the drift in equation (34). Economically, with more exogenous quits, the model needs to generate fewer endogenous separations. Since endogenous separations occur only when wages hit the lower bound $\omega$, a high rate endogenous separation rate implies that industry wages are frequently reflecting off the lower bound, reducing their measured persistence.

For example, set $q = 0.12$, so about two-thirds of unemployment spells are due to exogenous quits, independent of labor market conditions. Monte Carlo shows that the maximum attainable value of $\hat{\beta}_w$ increases to 0.76 but is still significantly short of a random walk. Now consider a sequence of economies with $q \rightarrow \alpha U_{s,r}/L$ and $\theta \sigma = \sqrt{2(\alpha U_{s,r}/L - q)} \rightarrow 0$. In the

---

12The empirical duration numbers were constructed by the Bureau of Labor Statistics from the Current Population Survey and may be obtained from http://www.bls.gov/cps/. Our choice of years is governed by the availability of industry wage data.

13We draw an initial value of $\omega$ from the ergodic distribution of $\omega$ across industries, say $g(\omega)$. This is not the same as the ergodic distribution of $\omega$ across workers, $f(\omega)$, because industries that experience negative productivity shocks shed workers but are no more likely to disappear. We find $g(\omega) = -(2\mu/\sigma^2)e^{(2\mu/\sigma^2)(\omega - \bar{\omega})}$ while $f$ is given in equation (30). Both distributions are exponential, but the mean of $g$ is lower.
limit, \( \tilde{\omega} \) follows a random walk and so the estimated value of \( \hat{\beta}_w \) converges to \( \frac{T-2}{T+1} = 0.82 \). While this is arguably close to the empirical value of 0.91, the model is not very interesting. Unemployed workers find jobs at rate \( \alpha \) and lose them at rate \( q \), both exogenous parameters. The variance in wages is infinitely large and wages are perfectly persistent and so unregulated by the entry and exit of workers. In other words, the equilibrating mechanism in Lucas and Prescott (1974) is shut down.

### 6.3 Directed Search Model without Rest Unemployment

Our analysis of the directed search model in Section 4 is similar. We still assume that productivity drift satisfies \( \mu_x = -\frac{1}{2}(\theta - 1)\sigma_x^2 \) and focus on the limit as \( \delta \to 0 \). We again shut down rest unemployment, \( b_r = 0 \).

In the directed search model, wages are a Brownian motion \( d\omega(j,t) = \mu dt + \sigma dz(j,t) \), regulated on \([\omega, \bar{\omega}]\). Using equation (13), our assumption on the relationship between the drift and standard deviation of productivity implies a similar relationship for wages,

\[
\mu = -\frac{1}{2}\theta\sigma^2 + \frac{q}{\theta}.
\]

In this case, the estimated autoregressive coefficient depends on \( \alpha U_s/L, q \), and \( \theta\sigma \). To prove this, again let \( \tilde{\omega}(j,t) = (\omega(j,t) - \bar{\omega})/\sigma \). This has the same autoregressive coefficient \( \tilde{\beta}_w \) as \( \omega \), but satisfies a simpler law of motion,

\[
d\tilde{\omega}(j,t) = \left(-\frac{1}{2}\theta\sigma + \frac{q}{\theta\sigma}\right) dt + dz,
\] for \( \tilde{\omega}(j,t) \in [0, (\bar{\omega} - \omega)/\sigma] \). Invert equation (18) to express this interval as

\[
\tilde{\omega}(j,t) \in \left[0, -\frac{\theta\sigma}{2q} \log \left(1 - \frac{qL}{\alpha U_s}\right)\right].
\]

In the random search case, the autoregressive coefficient depended only on the difference \( \alpha U_s/L - q \), while here the two objects enter separately. Still, the spirit of our previous analysis carries over to this case. In particular, for fixed \( \alpha U_s/L \) and \( q \), the autoregressive coefficient is a nonmonotonic function of \( \theta\sigma \). When \( \theta\sigma \) is close to zero, \( \tilde{\omega} \) lies in a small interval and so is nearly serially uncorrelated. When \( \theta\sigma \) is large, \( \tilde{\omega} \) has a strong negative drift. As in the random search model, this again eliminates the serial correlation. The estimated autoregressive coefficient is therefore maximized at an intermediate value of \( \theta\sigma \), although unlike in the random search model, we do not have an explicit expression for the maximizing value, but rather proceed using Monte Carlo.
Again fix $\alpha U_s/L \approx 0.186$ per year and initially assume $q = 0$; note that in this limit, the upper bound on $\tilde{\omega}$ converges to $\frac{\theta\sigma}{2\alpha U_s/L}$. Our Monte Carlo simulations indicate a maximum value of $\hat{\beta}_w \approx 0.56$ when $\theta\sigma = 1.3$. Introducing exogenous quits again mitigates but does not eliminate these issues; the economic intuition is unchanged from the random search model. With $q = 0.12$, we find that the maximum value of $\hat{\beta}_w$ increases to $0.71$, attained when $\theta\sigma \approx 1.2$. Both of these autoregressive coefficients are somewhat smaller than in the random search model. The limit as $q \to \alpha U_s/L$ is actually unchanged from the random search model—an appropriate choice of $\theta\sigma$ implies wages a random walk and so the estimated value of $\hat{\beta}_w$ converges to $\frac{T-2}{T+1} \approx 0.82$.\footnote{For fixed $q$, let $\theta\sigma = \sqrt{2q}$, so $\tilde{\omega}(j, t)$ has no drift. In the ergodic distribution, $\tilde{\omega}(j, t)$ is uniformly distributed on $[0, -\sqrt{-2q\log(1 - qL/\alpha U_s)}]$. Ball and Roma (1998) prove that the autocorrelation of wages depends only on the width of this interval. In particular, in the limit as $q \to \alpha U_s/L$, the upper bound on the interval converges to infinity and wages follow a random walk.} Once again, we do find this limit particularly interesting.

### 6.4 Rest Unemployment

If some unemployment is accounted for by workers waiting for labor market conditions to improve, our model generates more persistent fluctuations in wages. This happens for two reasons. First, the arguments behind equations (34) and (35) suggest that if the search unemployment rate is lower, the maximum possible autocorrelation in the log full-employment wage $\omega$ is higher. Second, we measure the actual wage, not the full-employment wage, in the data. Rest unemployment imposes a lower bound on wages, which further increases their persistence.

To evaluate the quantitative importance of these effects, we calibrate the model. The first critical issue is the empirical counterpart of search and rest unemployment. We refer to the evidence on “stayers” and “switchers” developed by Murphy and Topel (1987) from the March Current Population Survey and by Loungani and Rogerson (1989) from the Panel Study of Income Dynamics (PSID). These papers classify unemployed workers as stayers if they remain in the same two-digit industry after an unemployment spell and as switchers otherwise. Despite differences in their data sources and methodology, both papers find that workers switch industries after about one quarter of all unemployment spells and otherwise return their old industry. Moreover, Loungani and Rogerson (1989) report that switchers account for about a third of all weeks of unemployment, i.e. they stay unemployed slightly longer than stayers.

To map this into our model, we assume that a labor market is the theoretical counterpart of an industry in Murphy and Topel (1987) and Loungani and Rogerson (1989). Then according to our model, all stayers must have experienced a spell of rest unemployment,
Table 2: Results from random and directed search models with 1.3% search unemployment, 4.2% rest unemployment, and $\alpha = 3.2$. Each row shows a different value of the exogenous quit rate $q$ (first column) and the value of $\theta \sigma$ (second column) that maximizes the autoregressive coefficient $\hat{\beta}_w$ (third column). The fourth column shows the standard deviation of the residual from estimating equation (33), which is proportional to $\sigma$. The numbers in parenthesis in the third and fourth columns are the standard deviation of the estimates of $\hat{\beta}_w$ and $\hat{\sigma}_w$ across runs of the model. The fifth column shows the value of $\theta$ such that $\hat{\sigma}_w \approx 0.033$. The last two columns impose $\rho = 0.05$ and show the implied structural parameters $b_r/b_i$ and $\kappa$.

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\theta \sigma$</th>
<th>$\hat{\beta}_w$ (s.d.)</th>
<th>$\hat{\sigma}_w/\sigma$ (s.d.)</th>
<th>$\theta$</th>
<th>$b_r/b_i$</th>
<th>$\kappa$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.29</td>
<td>0.82 (0.01)</td>
<td>0.49 (0.01)</td>
<td>4.3</td>
<td>0.96</td>
<td>1.8</td>
</tr>
<tr>
<td>0.02</td>
<td>0.23</td>
<td>0.84 (0.01)</td>
<td>0.51 (0.01)</td>
<td>3.6</td>
<td>0.96</td>
<td>3.4</td>
</tr>
<tr>
<td>0.03</td>
<td>0.11</td>
<td>0.85 (0.01)</td>
<td>0.50 (0.01)</td>
<td>1.7</td>
<td>0.97</td>
<td>2.4</td>
</tr>
<tr>
<td>0.04</td>
<td>0.06</td>
<td>0.87 (0.01)</td>
<td>0.53 (0.02)</td>
<td>0.96</td>
<td>0.99</td>
<td>52</td>
</tr>
<tr>
<td>0</td>
<td>0.64</td>
<td>0.79 (0.01)</td>
<td>0.46 (0.01)</td>
<td>8.9</td>
<td>0.96</td>
<td>2.9</td>
</tr>
<tr>
<td>0.02</td>
<td>0.52</td>
<td>0.82 (0.01)</td>
<td>0.48 (0.01)</td>
<td>7.6</td>
<td>0.96</td>
<td>3.2</td>
</tr>
<tr>
<td>0.03</td>
<td>0.49</td>
<td>0.84 (0.01)</td>
<td>0.46 (0.01)</td>
<td>6.8</td>
<td>0.97</td>
<td>4.7</td>
</tr>
<tr>
<td>0.04</td>
<td>0.44</td>
<td>0.85 (0.02)</td>
<td>0.37 (0.02)</td>
<td>4.9</td>
<td>0.99</td>
<td>13</td>
</tr>
</tbody>
</table>

while switchers’ unemployment spell ended with search but may have also included some rest unemployment. Thus the share of rest unemployment in the overall unemployment rate should be slightly larger than Loungani and Rogerson’s (1989) two-third share of stayers. This motivates our targets for the search unemployment rate, $U_s/(U_s + L) = 0.013$, and for the rest unemployment rate $U_r/(U_r + L) = 0.042$, totalling the same 5.5% unemployment rate as before. We keep the arrival rate of job offers to searchers fixed at $\alpha = 3.2$.

We again simulate finite panels using our model and estimate the autoregressive coefficient $\hat{\beta}_w$ using equation (33). As in the model without rest unemployment, we find that $\hat{\beta}_w$ is a nonmonotonic function of $\theta \sigma$. Table 2 reports the value of $\theta \sigma$ that maximizes the average measured autoregressive coefficient $\hat{\beta}_w$ for different values of the exogenous quit rate $q$ and for both random and directed search. Note that the structure of our model imposes $q \leq \alpha U_s/L \approx 0.042$.

Two patterns emerge from this table, both consistent with the models without rest unemployment: higher values of $q$ imply more persistence in wages; and a given value of $q$, the directed search model generates slightly less persistent wages than the random search model. But rest unemployment significantly raises the measured persistence of wages. Recall that with only search unemployment, we found that the autoregressive coefficient was bounded above by $\hat{\beta}_w = \frac{T-2}{T+1} = 0.82$ and we only obtained that bound in the limit when all unemployment was due to exogenous quits. With rest unemployment, the random search model exceeds that bound for almost any value of $q$, while the directed search model exceeds it if
Although the empirical counterpart of this object is around 0.9, rest unemployment helps close the gap between model and data.

### 6.5 Structural Parameters

We now return to our structural model and look at how preference parameters affect the measured persistence of wages. We consider both the random and directed search models. We assume throughout that the search unemployment rate is $U_s/(U_s + L) = 0.013$ and the rest unemployment rate is $U_r/(U_r + L) = 0.042$. The search unemployed find jobs at rate $\alpha = 3.2$. For a given value of the exogenous quit rate $q$, we set $\theta \sigma$ so as to maximize the autoregressive coefficient in wages, i.e. at the values in Table 2.

To determine $\theta$ and $\sigma$ separately, we use evidence on the estimated standard error of the residual in equation (33), given in the fourth column of Table 2. According to our model, this is proportional to the standard deviation of wages $\sigma$, while the data in Table 1 indicate that $\hat{\sigma}_w \approx 0.033$ at the five-digit level. The fifth column shows the implied value of the elasticity of substitution $\theta$, which varies from slightly less than 1 to almost 9, depending on the choice of $q$ and whether search is random or directed. Evidence in Broda and Weinstein (2006) gives us a sense that the middle of this range may be reasonable. They estimate the elasticity of substitution between goods at the five-digit SITC level using international trade data, and report a median elasticity of 2.8 (see their Table IV).

We again start with the random search model. Equation (27) implies that search unemployed workers arrive in labor markets at rate $s = 0.042$. Then equations (29) and (31) determine $\hat{\omega} - \omega$, 0.08 with $q = 0$ and 0.31 with $q = 0.04$. Next, building on our analysis of the value function in Appendix A.8, we use equations (8) and (20) to pin down the relative leisure values of rest unemployment and inactivity,

$$\frac{b_r}{b_i} = \frac{\zeta_2 - 1}{\zeta_2 - 1 + e^{-\zeta_2 (\hat{\omega} - \omega)}},$$

where $\zeta_2$ is the larger root of equation (61). With $q = 0$, we find $b_r/b_i = 0.96$, while $q = 0.04$ requires a higher relative value of rest unemployment, $b_r/b_i = 0.99$. More generally, in order to generate rest unemployment, $b_r/b_i$ must exceed $1 - 1/\zeta_2$, which is 0.91 without exogenous quits and 0.87 with $q = 0.04$. This suggests that, while the rest unemployed must pay some

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15This elasticity is in line with the one used in much of the literature that quantitatively evaluates the Lucas and Prescott (1974) model. Recall that the analog of $\theta$ in a model with diminishing returns at the labor market level due to a fixed factor is the reciprocal of the elasticity of revenue with respect to the fixed factor. If the fixed factor is capital, then a capital share of $\frac{1}{3}$ is empirically reasonable. Alvarez and Veracierto (1999) set the elasticity of fixed factor to 0.36, Alvarez and Veracierto (2001) set it to 0.23, and Kambourov and Manovskii (2007) set it to 0.32, in line with values of $\theta$ between 2.8 and 4.3.
cost to remain in contact with their labor market, the cost is small. Rest unemployment and inactivity may look quite similar to an outsider who observes individuals’ time use, even though the rest unemployed are much more likely to return to work.

Next we consider the search cost \( \kappa = (\bar{v} - \underline{v})/b_i \). We find this by computing the expected value of finding a new labor market from equation (22). With \( q = 0 \), we obtain \( \kappa = 1.8 \), while \( q = 0.04 \) requires a much higher value, \( \kappa = 52.16 \). In words, the expected cost of moving to a new market must be equal to 1.8 years of inactivity in the model without exogenous quits, and much higher with a high exogenous quit rate. A strong autocorrelation in wages requires that most markets are far from the lower threshold \( \omega \), but this implies that the wage in an average market is much higher than the wage in the least productive market. In order for workers to be willing to endure such a market, the cost of moving must be large. This is essentially the contrapositive of Hornstein, Krusell, and Violante’s (2006) finding that when search costs are small, search models cannot generate much wage dispersion. Introducing other mobility costs, such as market-specific human capital, may alleviate this issue.

The determination of the structural parameters in the directed search model is similar. We first use the expression for the search unemployment rate in equation (18) to pin down \( \bar{\omega} - \omega \), 0.55 with \( q = 0 \) and 1.47 with \( q = 0.04 \). We then use the equation for the rest unemployment rate to find \( \hat{\bar{\omega}} - \omega \), 0.08 without quits and 0.45 with the high quit rate. Finally, equation (12) gives us the remaining parameters. Without quits, we get \( b_r/b_i = 0.96 \) and \( \kappa = 2.9 \). With \( q = 0.04 \), both numbers are larger, \( b_r/b_i = 0.99 \) and \( \kappa = 13 \). These numbers are comparable to the ones in the random search model, and the intuition is similar.

We have focused here on the relative leisure values from search unemployment, rest unemployment, and inactivity. To do this, we did not need to take a stand on the utility function. For the model to be consistent with balanced growth, we would additionally require \( u(\cdot) = \log(\cdot) \). In this case, data on labor force participation rates would pin down the absolute level of the three leisure parameters.

Finally, it is worth stressing that these high values of search costs \( \kappa \) are a consequence of persistent wages and not due to rest unemployment per se. Suppose we shut down rest unemployment but let \( q \to \alpha U_s/L \) so that wages are nearly a random walk, regardless of whether search is random or directed. In either case, the incentive to search for a job is large and so search costs must be large in equilibrium as well, reaching infinity in the limit when wages a random walk. For example, with a 5.5 percent search unemployment rate and no rest unemployment, \( \alpha U_s/L = 0.186 \). Set \( q = 0.18 \) to generate \( \hat{\beta}_w = 0.85 \) and \( \hat{\sigma}_w = 0.70 \) in the random search model. These reduced form parameters imply \( \theta = 2.4 \), \( b_r/b_i = 0.94 \), and

\[ ^{16} \text{An increase in } \theta \text{ raises } b_r/b_i \text{ and reduces } \kappa. \text{ Indeed, part of the reason that the implied search costs are so high when } q = 0.04 \text{ is because } \theta \text{ is so small. With } q = 0.04 \text{ and } \theta = 4.3, \text{ we obtain } b_r/b_i = 0.972 \text{ and } \kappa = 2.25; \text{ however, the model then generates too little wage volatility.} \]
$\kappa = 2.6$, comparable to values in the full model with rest unemployment.

7 Concluding Remarks

This paper characterizes the equilibrium of the Lucas-Prescott search model with directed and random search. Our characterization features a separation between optimization and aggregation which keeps our analysis tractable. We use this separation to argue that we can distinguish economies between economies with different amounts of search and rest unemployment, arguing that rest unemployment helps to explain why wages are so persistent yet unemployment incidence is high.

Although our’s is the first paper to separate optimization from aggregation in the context of the Lucas and Prescott (1974) search model, others have fruitfully used the same idea in different contexts. Caballero and Engel (1993) evaluate the magnitude and effect of labor adjustment costs by looking at the statistical behavior of employment at the firm level. Doms and Dunne (1998) perform a similar exercise for capital adjustment costs and irreversibilities. Cole and Rogerson (1999) examine whether the Mortensen and Pissarides (1994) model is consistent with the behavior of job creation and destruction at business cycle frequencies by using a reduced-form version of the original model. Hall (1995) similarly studies the propagation of one-time shocks in an elaboration of this model through a statistical representation of the labor market as a 19-state Markov process.

One important question is the empirical counterpart of rest unemployment. We used evidence on whether workers switch industries after a spell of unemployment to pin down the rest unemployment rate, but a more direct link may be desirable. Workers on temporary layoff are an obvious example of rest unemployment, since they are waiting for conditions in the industry to improve. Interestingly, Katz and Meyer (1990) and Starr-McCluer (1993) find that the hazard rates of workers on temporary layoff are more strongly decreasing as a function of unemployment duration, compared to other unemployed workers. While the current version of the model does not have an explicit characterization unemployment duration, our extension in Alvarez and Shimer (2009) does. We develop a version of this model where unions can keep wages above the market-clearing level and allocate jobs based on seniority. We show that the hazard rate of reentering unemployed is decreasing in duration for workers who are rationed out of jobs. Moreover, we think that in that version of the model, the identification of the rest unemployed as workers on temporary layoff is quite descriptive. While the determination of the thresholds $\varnothing$ and $\bar{\varnothing}$ is more complicated in that framework, the same separation between optimization and aggregation applies and the link between search unemployment rates and wages is unchanged.
A Appendix

A.1 Density of Productivity $x$

Let $f_x(\tilde{x})$ denote the steady state density of log productivity, $\tilde{x} \equiv \log x$, across labor markets. This solves a Kolmogorov forward equation:

$$
\delta f_x(\tilde{x}) = -\mu_x f'_x(\tilde{x}) + \frac{\sigma^2_x}{2} f''_x(\tilde{x})
$$

at all $\tilde{x} \neq \tilde{x}_0 \equiv \log x_0$. The solution to this equation takes the form

$$
f_x(\tilde{x}) = \begin{cases} 
D_1^1 e^{\tilde{\eta}_1 \tilde{x}} + D_2^1 e^{\tilde{\eta}_2 \tilde{x}} & \text{if } \tilde{x} < \log x_0 \\
D_1^2 e^{\tilde{\eta}_1 \tilde{x}} + D_2^2 e^{\tilde{\eta}_2 \tilde{x}} & \text{if } \tilde{x} > \log x_0,
\end{cases}
$$

where $\tilde{\eta}_1 < 0 < \tilde{\eta}_2$ are the two real roots of the characteristic equation

$$
\delta = -\mu_x \tilde{\eta} + \frac{\sigma^2_x}{2} \tilde{\eta}^2. \tag{36}
$$

For this to be a well-defined density, integrating to 1 on $(-\infty, \infty)$, we require that $D_1^1 = D_2^2 = 0$. To pin down the remaining constants, we use two more conditions: the density is continuous at $\tilde{x} = \log x_0$; and it integrates to 1. Imposing these boundary conditions delivers

$$
f_x(\tilde{x}) = \begin{cases} 
\frac{\tilde{\eta}_1 \tilde{\eta}_2}{\tilde{\eta}_1 - \tilde{\eta}_2} e^{\tilde{\eta}_1 (\tilde{x} - \log x_0)} & \text{if } \tilde{x} < \log x_0 \\
\frac{\tilde{\eta}_1 \tilde{\eta}_2}{\tilde{\eta}_1 - \tilde{\eta}_2} e^{\tilde{\eta}_2 (\tilde{x} - \log x_0)} & \text{if } \tilde{x} > \log x_0,
\end{cases} \tag{37}
$$

With this notation, we obtain

$$
\left(\int_0^1 x(j, t)_{\theta-1} dj\right)^{\frac{1}{\theta-1}} = \left(\int_{-\infty}^{\infty} e^{(\theta-1)\tilde{x}} f_x(\tilde{x}) d\tilde{x}\right)^{\frac{1}{\theta-1}}. \tag{38}
$$

The interior integral converges if $\tilde{\eta}_1 + \theta - 1 < 0 < \tilde{\eta}_2 + \theta - 1$. The definition of $\tilde{\eta}_i$ in equation (36) implies these inequalities are equivalent to condition (6). With this restriction, we can then simplify equation (38) to obtain equation (7).

A.2 Proof of Proposition 1

Throughout this proof we express the value of a worker $v$ as a function not only on the current log wage $\omega$ but also of the lower and upper bound on wages $\omega$ and $\bar{\omega}$, say $v(\omega; \underline{\omega}, \bar{\omega})$. We start by proving a preliminary result.
**Lemma 1.** \( v \) is continuous and nondecreasing in \( \omega, \underline{\omega}, \) and \( \bar{\omega}. \) It is strictly increasing in each argument if \( \omega \in (\underline{\omega}, \bar{\omega}) \) and \( \bar{\omega} > \log b_r. \)

**Proof.** Define

\[
\Pi(\omega'; \omega; \underline{\omega}, \bar{\omega}) \equiv E\left( \int_0^\infty e^{-(\rho+q+\delta)t} I_{\omega'}(\omega(j, t))dt \middle| \omega(j, 0) = \omega \right),
\]

where \( I_{\omega'}(\omega(j, t)) \) is an indicator function, equal to 1 if \( \omega(j, t) < \omega' \) and equal to zero otherwise. This *discounted occupancy function* evaluates to zero at \( \omega' \leq \underline{\omega} \) and to \( \frac{1}{\rho+q+\delta} \) at \( \omega' \geq \bar{\omega}. \)

We use \( \Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) \) for the density of \( \omega' \) or the *discounted local time function*, where the subscript denotes the partial derivative with respect to the first argument. Then switching the order of integration in equation (11), which is permissible since for \( -\infty \leq \omega \leq \bar{\omega} < \infty \) and \( \rho + q + \delta > 0, \) the function max\{\( b_r, e^{\omega'} \)\} \((q + \delta) v \) is integrable, we get

\[
v(\omega; \underline{\omega}, \bar{\omega}) = \int_\underline{\omega}^{\bar{\omega}} \left( \max\{b_r, e^{\omega'}\} + (q + \delta) v \right) \Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) d\omega'. \tag{39}\]

The value of being in a market with current log full-employment wage \( \omega \) is equal to the expected value of future \( \omega' \) weighted by the appropriate discounted local time function. Stokey (2009) proves in Proposition 10.4 that for all \( \omega \in [\underline{\omega}, \bar{\omega}], \)

\[
\Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) = \begin{cases} 
\left( \frac{\zeta_2 e^{\zeta_1 \omega} + \zeta_2 \bar{\omega} - \zeta_1 e^{\zeta_1 \omega} + \zeta_2 \bar{\omega}}{\rho + q + \delta} \right) & \text{if } \omega \leq \omega' < \omega \\
\left( \frac{\zeta_2 e^{\zeta_1 \omega} + \zeta_2 \omega - \zeta_1 e^{\zeta_1 \omega} + \zeta_2 \omega}{\rho + q + \delta} \right) & \text{if } \omega \leq \omega' \leq \bar{\omega},
\end{cases}
\]

where \( \zeta_1 < 0 < \zeta_2 \) are the two roots of the characteristic equation

\[
\rho + q + \delta = \mu \zeta + \frac{\sigma^2}{2} \zeta^2. \tag{41}\]

For \( \omega < \underline{\omega}, \Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) = \Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) \) and for \( \omega > \bar{\omega}, \Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) = \Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}). \)

That \( v \) is continuous follows immediately from equations (39) and (40). In particular, the latter equation defines \( \Pi_{\omega'} \) as a continuous function.

We next prove that the distribution \( \Pi(\cdot; \omega; \underline{\omega}, \bar{\omega}) \) is increasing in each of \( \omega, \underline{\omega}, \) and \( \bar{\omega} \) in the sense of first order stochastic dominance. This follows from differentiating equation (40) with respect to each variable and using simple algebra. One can verify that an increase in \( \underline{\omega} \) strictly increases \( \Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) \) for all \( \omega' \in (\underline{\omega}, \bar{\omega}). \) This therefore strictly reduces \( \Pi(\omega'; \omega; \underline{\omega}, \bar{\omega}) \) for \( \omega' \in (\underline{\omega}, \bar{\omega}). \) Similarly, an increase in \( \bar{\omega} \) strictly reduces \( \Pi_{\omega'}(\omega'; \omega; \underline{\omega}, \bar{\omega}) \) for
all \( \omega' \in (\omega, \bar{\omega}) \), which also strictly reduces \( \Pi(\omega'; \omega, \bar{\omega}) \) for \( \omega' \in (\omega, \bar{\omega}) \). Finally, an increase in \( \omega \) when \( \omega \in (\omega, \bar{\omega}) \) reduces \( \Pi_{\omega'}(\omega'; \omega, \bar{\omega}) \) for \( \omega' \in (\omega, \omega) \) and raises it for \( \omega' \in (\omega, \bar{\omega}) \). Once again, this implies a stochastic dominating shift in \( \Pi \).

Since the return function \( \max\{b_r, e^{\omega'}\} + (q + \delta)\bar{v} \) is nondecreasing in \( \omega' \), weak monotonicity of \( v \) in each argument follows immediately from equation (39). In addition, the return function is strictly increasing when \( \omega' > \log b_r \), and so we obtain strict monotonicity when the support of the integral includes some \( \omega' > \log b_r \), i.e. when \( \bar{\omega} > \log b_r \).

Our approach to solving for the \( v \) may be unfamiliar to some readers, and so it is worth stressing that equations (12) and (39) imply some familiar conditions:

\[
(\rho + q + \delta)v(\omega; \omega, \bar{\omega}) = \max\{b_r, e^{\omega'}\} + (q + \delta)v + \mu v(\omega; \omega, \bar{\omega}) + \frac{\sigma^2}{2} v_{\omega, \omega}(\omega; \omega, \bar{\omega}),
\]

and

\[
v_{\omega}(\bar{\omega}; \omega, \bar{\omega}) = v_{\omega}(\omega; \omega, \bar{\omega}) = 0,
\]

where subscripts denote partial derivatives with respect to the first argument.\(^{17}\) Together with the “value-matching” conditions \( v(\bar{\omega}; \omega, \bar{\omega}) = \bar{v} \) and \( v(\omega; \omega, \bar{\omega}) = \bar{v} \) in equation (12), this is an equivalent representation of the labor force participant’s value function.

We now turn to the proof of Proposition 1. We start by proving the result when \( b_r < b_i \) and defer \( b_r \geq b_i \) until the end.

\(^{17}\)The first condition, the Hamilton-Jacobi-Bellman equation, can be verified directly by differentiating equation (39) using the definition of \( \Pi_{\omega'} \) in equation (40). The interested reader can consult the online Appendix B.1 for the details of the algebra. The second pair of conditions, “smooth-pasting,” follow from equation (39) because equation (40) implies \( \frac{\partial \Pi_{\omega'}(\omega'; \omega, \bar{\omega})}{\partial \omega} = 0 \) when \( \omega = \bar{\omega} \) or \( \omega = \omega \).
First, define \( \bar{\omega}^* \) to solve \( v(\bar{\omega}^*; \omega^*, \bar{\omega}^*) = \bar{v} \). When \( \omega \) is regulated at the point \( \bar{\omega}^* \), it is trivial to solve equation (11) to obtain

\[
\frac{e^{\bar{\omega}^*} + (q + \delta)v}{\rho + q + \delta} = \bar{v}.
\]

This point is depicted along the 45° line in Figure 1. Lemma 1 ensures \( v \) is continuous and strictly increasing in its first three arguments. Moreover, for any \( \omega < \bar{\omega}^* \), we can make \( v(\omega; \omega, \bar{\omega}) \) unboundedly large by increasing \( \bar{\omega} \), while we can make it smaller than \( \bar{v} \) by setting \( \bar{\omega} = \omega^* \). Then by the intermediate value theorem, for any \( \omega < \bar{\omega}^* \), there exists \( \bar{\Omega}(\omega) > \bar{\omega}^* \) solving \( v(\bar{\Omega}(\omega); \omega, \bar{\Omega}(\omega)) \equiv \bar{v} \). Continuity of \( v \) ensures \( \bar{\Omega} \) is continuous while monotonicity of \( v \) ensures it is decreasing. Thus \( \bar{\Omega}(\omega) < \omega^* \). Figure 1 illustrates this function.

Similarly, define \( \omega^* \) to solve \( v(\omega^*; \omega^*, \omega^*) = v \). Again solve equation (11) to obtain

\[
\frac{e^{\omega^*} + (q + \delta)v}{\rho + q + \delta} = v.
\]

Since \( v < \bar{v} \), \( \omega^* < \bar{\omega}^* \), while equation (8) implies \( \omega^* = \log b_i \). For any \( \omega > \omega^* \), we can make \( v(\omega; \omega, \bar{\omega}) \) approach \( \frac{\rho b_i + (q + \delta)b_i}{\rho(p + q + \delta)} < v \) by making \( \omega \) arbitrarily small, while we can make it bigger than \( \bar{v} \) by setting \( \omega = \omega^* \). Then by the intermediate value theorem, for any \( \omega > \omega^* \), there exists a \( \bar{\Omega}(\omega) < \omega^* \) solving \( v(\bar{\Omega}(\omega); \omega, \bar{\Omega}(\omega)) \equiv v \). Continuity of \( v \) ensures \( \bar{\Omega} \) is continuous while monotonicity of \( v \) ensures it is decreasing. Thus \( \bar{\Omega}(\omega) < \omega^* \) for any \( \omega > \omega^* \).

An equilibrium is simply a fixed point \( \bar{\omega} \) of the composition of the functions \( \bar{\Omega} \circ \Omega \). The preceding argument implies that this composition maps \([\omega^*, \omega^*] \) into itself and is continuous, and hence has a fixed point.

To prove the uniqueness of the fixed point when \( b_i < b_i \), we prove that the composition of the two functions has a slope less than 1, i.e. \( \Omega'(\bar{\Omega}(\omega))\bar{\Omega}'(\omega) < 1 \). To start, simple transformations of equation (40) imply that the cross partial derivatives of the discounted occupancy function satisfy

\[
\Pi_{\omega, \bar{\omega}}(\omega'; \omega; \omega, \bar{\omega}) = \frac{\zeta_1 \zeta_2 e^{(\zeta_1 + \zeta_2)\omega} (e^{-\zeta_1(\omega' - \omega)} - e^{-\zeta_2(\omega' - \omega)}) (e^{\zeta_1 \omega + \zeta_2 \bar{\omega}} - e^{\zeta_1 \omega + \zeta_2 \bar{\omega}})}{(e^{\zeta_1 \omega + \zeta_2 \omega} - e^{\zeta_1 \bar{\omega} + \zeta_2 \bar{\omega}})^2 (\rho + q + \delta)} < 0
\]

and

\[
\Pi_{\omega, \bar{\omega}}(\omega'; \omega; \omega, \bar{\omega}) = \frac{-\zeta_1 \zeta_2 e^{(\zeta_1 + \zeta_2)\omega} (e^{\zeta_2(\omega - \omega') - e^{\zeta_1(\bar{\omega} - \omega)}}) (e^{\zeta_1 \omega + \zeta_2 \bar{\omega}} - e^{\zeta_1 \bar{\omega} + \zeta_2 \bar{\omega}})}{(e^{\zeta_1 \omega + \zeta_2 \omega} - e^{\zeta_1 \bar{\omega} + \zeta_2 \bar{\omega}})^2 (\rho + q + \delta)} > 0,
\]

where the inequalities use the fact that all the terms in parenthesis are positive. Then use
integration-by-parts on equation (39) to write

\[ v(\omega; \bar{\omega}, \bar{\omega}) = \frac{s(\bar{\omega})}{\rho + q + \delta} - \int_{\omega}^{\bar{\omega}} s'(\omega') \Pi(\omega'; \omega; \bar{\omega}, \bar{\omega}) d\omega', \]

where the period return function \( s(\omega') \) is nondecreasing and strictly increasing for \( \omega' > \log b_r \), and \( \Pi \) is the discounted occupancy function. Taking the cross partial derivatives of this expression gives

\[ v(\omega; \bar{\omega}, \bar{\omega}) > 0 > v_{\omega, \bar{\omega}}(\omega; \bar{\omega}, \bar{\omega}). \]

In particular,

\[ v_{\omega}(\bar{\omega}; \bar{\omega}, \bar{\omega}) > v_{\omega}(\omega; \bar{\omega}, \bar{\omega}) \text{ and } v_{\omega}(\omega; \omega, \bar{\omega}) > v_{\omega}(\bar{\omega}; \bar{\omega}, \bar{\omega}). \]

Now since \( v_{\omega}(\omega; \bar{\omega}, \bar{\omega}) > 0 \) from Lemma 1, these inequalities imply

\[ \frac{v_{\omega}(\bar{\omega}; \bar{\omega}, \bar{\omega})}{v_{\omega}(\bar{\omega}; \bar{\omega}, \bar{\omega}) + v_{\omega}(\omega; \bar{\omega}, \bar{\omega})} < 1. \]

In particular, this is true when evaluated at any point \( \{\omega, \bar{\omega}\} \) where \( \bar{\omega} = \bar{\Omega}(\omega) \) and \( \omega = \Omega(\omega) \). Implicit differentiation of the definitions of these functions shows that the first term in the above inequality is \(-\bar{\Omega}'(\omega)\) and the second term is \(-\Omega'(\omega)\), which proves \( \bar{\Omega}'(\Omega(\omega))\Omega'(\omega) < 1 \).

Next we prove proportionality of the thresholds \( e^{\bar{\omega}} \) and \( e^{\omega} \) to the leisure values \( b_r, b_i, \) and \( b_s \). From equations (8) and (9), \( v \) and \( \bar{v} \) are homogeneous of degree one in the three leisure values. The function \( \max\{b_r, e^{\omega}\} + (q + \delta)v \) is also homogeneous of degree 1 in the leisure values and \( e^{\omega} \). By inspection of equation (40), \( \Pi_{\omega} \) is unaffected by an equal absolute increase in each of its arguments. Then the integral in equation (39) is homogeneous of degree one in the \( b \)'s and \( e^{\bar{\omega}} \) and \( e^{\omega} \). The result follows from equation (12).

Finally we consider \( b_r \geq b_i \), so the period return function \( s(\omega) \geq b_i + (q + \delta)v \) for all \( \omega \). This implies \( v(\omega; -\infty, \bar{\omega}) \geq v \) for all \( \omega \) and \( \bar{\omega} \). Then an equilibrium is defined by \( v(\bar{\omega}; -\infty, \bar{\omega}) = \bar{v} \).

As discussed above, the solution of this equation is \( \bar{\omega}^{**} \in (\bar{\omega}^*, \infty) \).

### A.3 Proof of Proposition 2

First, set \( b_r = 0 \). By Proposition 1, there exists a unique equilibrium characterized by thresholds \( \bar{\omega}_0 \) and \( \bar{\omega}_0 \). We now prove that \( \bar{b}_r \equiv e^{\omega_0} \). To see why, observe that for all \( b_r \leq \bar{b}_r \), the equations characterizing equilibrium are unchanged from the case of \( b_r = 0 \) because \( \log b_r \leq \omega_0 \), and hence the equilibrium is unchanged. Conversely, for all \( b_r > \bar{b}_r \), the equations characterizing equilibrium necessarily are changed, and so the equilibrium must have \( \log b_r > \omega_{b_r} \).

Next we prove that \( \bar{b}_r / b_i = B(\kappa, \rho + q + \delta, \mu, \sigma) \). Again with \( b_r = 0 \), combine equations (12) and (39), noting the discounted local time function \( \Pi_{\omega} \) integrates to \( \frac{1}{\rho + q + \delta} \), and use the
definitions of $\varphi$ and $\bar{v}$ in equations (8) and (9):

$$
\frac{b_i}{\rho + q + \delta} = \int_{\bar{\omega}_0}^{\omega_0} e^{\omega} \Pi_\omega(\omega; \bar{\omega}_0; \omega_0, \bar{\omega}_0) d\omega \quad \text{and}
$$

$$
\frac{1}{\rho + q + \delta + \kappa} = \int_{\bar{\omega}_0}^{\omega_0} e^{\omega} \Pi_\omega(\omega; \bar{\omega}_0; \omega_0, \bar{\omega}_0) d\omega.
$$

Since $\Pi_\omega$ is homogeneous of degree zero in the exponentials of its arguments (see equation 40), this implies $e^{\omega}$ and $e^{\bar{\omega}_0}$ are homogeneous of degree 1 in $b_i$. Moreover, $\zeta_i$ depends on $\rho + q + \delta$, $\mu$, and $\sigma$ by equation (41) and so the density $\Pi_\omega$ in equation (40) depends on these same parameters. It follows that the solution to these equations can depend only on these parameters and the parameters on the left hand side of the above equations. In particular, this proves

$$
e^{\omega_0} = b_i B(\kappa, \rho + q + \delta, \mu, \sigma).
$$

Since $\bar{b}_r = e^{\omega_0}$, that establishes the dependence of $\bar{b}_r$ on this limited set of parameters.

Obviously $B$ is positive-valued. By Proposition 1, $\omega_0 < \log b_i$ and so $B < 1$. We finally prove it is decreasing in $\kappa$. Since $\kappa$ affects $\varphi$ and $\bar{\omega}$ only through $\bar{v}$, to establish that $B$ is decreasing in $\kappa$ it suffices to show that the $\varphi$ and $\bar{\omega}$ that solve equations (12) and (39) is decreasing in $\bar{v}$. This follows because $\bar{\Omega}(\omega)$ is increasing in $\bar{v}$ and $\Omega(\bar{\omega})$ is unaffected, where these functions are defined in the proof of Proposition 1. A decrease in $\bar{v}$ then reduces the composition $\bar{\Omega} \circ \Omega$. Since the slope of this function is less than 1, it reduces the location of the fixed point $\bar{\omega}$ and hence raises $\omega = \Omega(\bar{\omega})$.

**A.4 Proof of Proposition 3**

We start by characterizing the stationary distribution of workers across log full-employment wages $\omega$, $f$, defined on $[\underline{\omega}, \bar{\omega}]$. Since $f$ is a density,

$$
\int_{\underline{\omega}}^{\bar{\omega}} f(\omega) d\omega = 1.
$$

(44)

By taking the limit of a discrete time, discrete state-space analog of our model, we prove in Appendix A.5 that this density has to satisfy three conditions, equations (45)–(47) below. First, in the interior of its support, it must solve a Kolmogorov forward equation,

$$(q + \delta)f(\omega) = -\mu f'(\omega) + \frac{\sigma^2}{2} f''(\omega) \quad \text{for all } \omega \in (\underline{\omega}, \bar{\omega}).$$

(45)
This captures the requirement that inflows and outflows balance at each point in the support of the density. Workers exit markets either because of quits or shutdowns at rate \( q + \delta \), while otherwise \( \omega \) is a Brownian motion with drift \( \mu \) and standard deviation \( \sigma \). Workers whose \( \omega \) changes leave this point in the density for higher or lower values of \( \omega \), while the density picks up mass from points above and below when they are hit by appropriate shocks. In a short period of time, this relates the density of \( f \) at nearby points, i.e. it relates the level of \( f \) and its derivatives.

Second, at the lower bound \( \omega \),

\[
\frac{\sigma^2}{2} f'(\omega) - \left( \mu + \frac{\theta \sigma^2}{2} \right) f(\omega) = 0. \tag{46}
\]

The elasticity of substitution \( \theta \) appears in this equation because it determines how many workers must exit from depressed markets required to regulate \( \omega \) above \( \omega \). The exogenous separation rate \( q + \delta \) does not appear in this equation because the ratio of endogenous to exogenous exits is infinite in a short time interval for a market at the lower bound. Since by definition there are no markets with smaller \( \omega \), \( f(\omega) \) is not fed from below, which explains the difference between equations (45) and (46). The second order differential equation (45) and the boundary conditions equations (44) and (46) yield equation (14) using standard calculations.

We now turn to output and consumption (which are equal by the final goods market clearing condition) and employment. Appendix A.5 also establishes that at the upper bound \( \bar{\omega} \),

\[
\frac{\sigma^2}{2} f'(\bar{\omega}) - \left( \mu + \frac{\theta \sigma^2}{2} \right) f(\bar{\omega}) = \delta L_0, \tag{47}
\]

where \( L_0 \) is the (endogenous) average number of workers in a new labor market. The logic for the left hand side of this equation parallels the logic behind equation (46). There is an extra inflow at \( \bar{\omega} \) coming from newly-formed markets, which absorb \( \delta L_0 \) workers per unit of time; dividing by \( L \) expresses this inflow as a percentage of the workers located in labor markets.

Next, equation (5) implies that the number of workers in a new labor market is

\[
L_0 = Y u'(Y)^\frac{\theta}{\theta - 1} \sigma_{\theta} e^{-\theta \bar{\omega}}, \tag{48}
\]

which ensure a log full-employment wage \( \bar{\omega} \).

Our last condition relates intermediate and final goods output. It is convenient to first define the productivity of a location \( x \) consistent with \( l \) workers present in the location, a log full-employment wage \( \omega \), and aggregate output and consumption \( Y \). From equation (5), this
solves
\[ x = \xi(l, \omega, Y) \equiv \left( \frac{lt^\omega}{Y u'(Y) \theta} \right)^{\frac{1}{\theta-1}}. \tag{49} \]

Use this to compute output in a market with \( l \) workers and log full-employment wage \( \omega \), recognizing that there may be rest unemployment:
\[ Q(l, \xi(l, \omega, Y)) = Y^{\frac{1}{\theta-1}} \left( \frac{e^{\omega l}}{u'(Y)} \right)^{\frac{\theta}{\theta-1}} \min\{1, e^\omega/b\}^\theta. \tag{50} \]

Using this notation, we can write equation (2) as
\[ Y = \left( \int_0^1 Q(l(j,t), \xi(l(j,t), \omega(j,t), Y) \right)^{\frac{\theta-1}{\theta}} dj) \right)^{\frac{\theta}{\theta-1}} \tag{51} \]
\[ = \left( \int_0^1 Q(L, \xi(L, \omega(j,t), Y) \right)^{\frac{\theta-1}{\theta}} \frac{l(j,t)}{L} dj) \right)^{\frac{\theta}{\theta-1}} \]

Note that these equations have to hold for all \( t \) in steady state, so the choice of \( t \) is arbitrary.

The second equation follows because \( Q(\cdot, \xi(\cdot, \omega, Y))^{\frac{\theta-1}{\theta}} \) is linear (equation 50). To solve this, we change the variable of integration from the name of the market \( j \) to its log full-employment wage \( \omega \) and number of workers \( l \). Let \( \tilde{f}(\omega, l) \) be the ergodic density of the joint distribution of workers in markets \((\omega, l)\). The joint distribution of \( (\omega, l) \) for an individual worker is a strongly convergent Markov process whenever markets shut down at a positive rate, \( \delta > 0 \), which ensures that \( \tilde{f} \) is unique. Without characterizing the distribution explicitly, we have
\[ Y = \int_0^{\omega} \int_0^\infty Q(L, \xi(L, \omega, Y) \right)^{\frac{\theta-1}{\theta}} \frac{l}{L} \tilde{f}(\omega, l) dl d\omega. \]

Since \( f(\omega) = \int_0^{\infty} \frac{l}{L} \tilde{f}(\omega, l) dl \), we can solve the inner integral to obtain
\[ Y = \left( \int_0^{\omega} Q(L, \xi(L, \omega, Y) \right)^{\frac{\theta-1}{\theta}} f(\omega) d\omega \right)^{\frac{\theta}{\theta-1}}. \tag{52} \]

This depends on the known density \( f \) rather than the more complicated density \( \tilde{f} \).

Finally, we solve equations (47), (48), and (52) for \( L_0, L, \) and \( Y \). First eliminate \( L_0 \) between equations (47) and (48) and evaluate \( f(\bar{\omega}) \) using equation (14) to get
\[ L = \frac{-2\delta Y u'(Y) \theta x_0^{\theta-1} e^{-\theta \bar{\omega}}}{\sigma^2(\theta + \eta_1)(\theta + \eta_2) \left( e^{\theta (\bar{\omega} - \omega)} - e^{\eta_1 (\bar{\omega} - \omega)} \right) \sum_{i=1}^2 |\theta + \eta_i| e^{\eta_i (\bar{\omega} - \omega)} - 1} \frac{\eta_i}{\eta_i}. \tag{53} \]
Substitute equation (50) into equation (52) to get

\[
Y = \frac{L}{u'(Y)} \int_{\mathbb{R}} e^\omega \min\{1, e^{\omega/b_r}\}^{\theta-1} f(\omega) d\omega
\]

\[
= \frac{L e^\omega}{u'(Y)} \sum_{i=1}^2 |\theta + \eta_i| e^{-\omega}\frac{e^{\eta_i(\omega-\bar{\omega})} - e^{-\theta(\omega-\bar{\omega})}}{\theta + \eta_i} \sum_{i=1}^2 |\theta + \eta_i| e^{\eta_i(\omega-\bar{\omega}) - 1} \eta_i,
\]

where we solve the integral using the expression for \( f \) in equation (14) and define \( \bar{\omega} \equiv \max\{\omega, \log b_r\} \). Eliminating \( L \) between these equations and solving for \( u'(Y) \) gives an implicit equation for output:

\[
u'(Y)^{1-\theta} = \frac{-2\delta x_0^{\theta-1} e^{-(\theta \omega - \bar{\omega})}}{\sigma^2(\theta + \eta_1)(\theta + \eta_2)(e^{\eta_2(\omega-\bar{\omega})} - e^{\eta_1(\omega-\bar{\omega})})} \times \sum_{i=1}^2 |\theta + \eta_i| \left( \frac{e^{\eta_i(\omega-\bar{\omega})} - e^{-\theta(\omega-\bar{\omega})}}{\theta + \eta_i} + e^{\eta_i(\omega-\bar{\omega})} \frac{e^{(\omega-\bar{\omega})} - e^{-\eta_i(\omega-\bar{\omega})}}{1 + \eta_i} \right).
\]

Finally, equation (54) determines the number of workers in labor markets, \( L \).

A.5 Derivation of the Density \( f \)

We use a discrete time, discrete state space model to obtain the Kolmogorov forward equations and boundary conditions for the density \( f \). Divide \([\omega, \bar{\omega}]\) into \( n \) intervals of length \( \Delta \omega = (\bar{\omega} - \omega)/n \). Let the time period be \( \Delta t = (\Delta \omega/\sigma)^2 \) and assume that when \( \omega < \bar{\omega} \), it decreases with probability \( \frac{1}{2}(1 + \Delta p) \) where \( \Delta p = \mu \Delta \omega/\sigma^2 \); when \( \omega > \bar{\omega} \), it increases with probability \( \frac{1}{2}(1 - \Delta p) \); and otherwise \( \omega \) stays constant. Note that for \( \omega < \omega(t) < \bar{\omega} \), the expected value of \( \omega(t + \Delta t) - \omega(t) \) is \( \mu \Delta t \) and the second moment is \( \sigma^2 \Delta t \). As \( n \) goes to infinity, this converges to a regulated Brownian motion with drift \( \mu \) and standard deviation \( \sigma \).

Now let \( f_n(\omega, t) \) denote the fraction of workers in markets with log full employment wage \( \omega \) at time \( t \) for fixed \( n \). With a slight abuse of notation, let \( f_n(\omega) \) be the stationary distribution. We are interested in characterizing the density \( f(\omega) = \lim_{n \to \infty} \frac{f_n(\omega)}{\Delta \omega} \). For \( \omega \in [\omega + \Delta \omega, \bar{\omega} - \Delta \omega] \), the dynamics of \( \omega \) imply

\[
f_n(\omega, t + \Delta t) = (1 - (q + \delta) \Delta t) \left( \frac{1}{2}(1 + \Delta p) f_n(\omega - \Delta \omega, t) + \frac{1}{2}(1 - \Delta p) f_n(\omega + \Delta \omega, t) \right).
\]

In any period of length \( \Delta t \), a fraction \((q+\delta)\Delta t\) of workers leave due to market shut downs and idiosyncratic quits. Thus the workers in markets with \( \omega \) at \( t + \Delta t \) are a fraction \( 1 - (q + \delta) \Delta t \) of those who were in markets at \( \omega - \Delta \omega \) at \( t \) and had a positive shock, plus the same fraction of those who were in markets at \( \omega + \Delta \omega \) at \( t \) and had a negative shock. Now impose
stationarity on $f_n$. Take a second order approximation to $f_n(\omega + \Delta \omega)$ and $f_n(\omega - \Delta \omega)$ around $\omega$, substituting $\Delta t$ and $\Delta p$ by the expressions above:

$$f_n(\omega) = \left(1 - (q + \delta)\frac{\Delta \omega^2}{\sigma^2}\right) \left(f_n(\omega) - \mu\frac{\Delta \omega^2}{\sigma^2} f'_n(\omega) + \frac{\Delta \omega^2}{2} f''_n(\omega)\right)$$

$$\Rightarrow (q + \delta) f_n(\omega) = \left(1 - (q + \delta)\frac{\Delta \omega^2}{\sigma^2}\right) \left(-\mu f'_n(\omega) + \frac{\sigma^2}{2} f''_n(\omega)\right)$$

Taking the limit as $n$ converges to infinity, $\frac{f_n'(\omega)}{\Delta \omega} \to f(\omega)$ solving equation (45).

Now consider the behavior of $f_n$ at the lower threshold $\omega$. A similar logic implies

$$f_n(\omega, t + \Delta t) = (1 - (q + \delta)\Delta t)^\frac{1}{2}(1 - \Delta p)\left(f_n(\omega + \Delta \omega, t) + f_n(\omega, t)(1 - \Delta \bar{l})\right).$$

The workers at $\omega$ at $t+\Delta t$ either were at $\omega + \Delta \omega$ or at $\omega$ at $t$; in both cases, they had a negative shock. Moreover, in the latter case, a fraction $\Delta \bar{l} \equiv \theta \Delta \omega$ of the workers exited the market to keep $\omega$ above $\omega$. Again impose stationarity but now take a first order approximation to $f_n(\omega + \Delta \omega)$ at $\omega$; the higher order terms will drop out later in any case. Replacing $\Delta t$, $\Delta p$, and $\Delta \bar{l}$ with the expressions described above gives

$$f_n(\omega) = \left(1 - (q + \delta)\frac{\Delta \omega^2}{\sigma^2}\right) \left(1 - \frac{\mu \Delta \omega}{\sigma^2}\right) \left(f_n(\omega) \left(1 - \frac{\theta \Delta \omega}{2}\right) + \frac{\Delta \omega}{2} f'_n(\omega)\right)$$

Again eliminating terms in $f_n(\omega)$ and taking the limit as $n \to \infty$, we obtain $\frac{f_n(\omega)}{\Delta \omega} \to f(\omega)$ solving equation (46).

Now consider the behavior of $f_n$ at the upper threshold $\bar{\omega}$:

$$f_n(\bar{\omega}, t + \Delta t) = (1 - (q + \delta)\Delta t)^\frac{1}{2}(1 + \Delta p)\left(f_n(\bar{\omega} - \Delta \omega, t) + f_n(\bar{\omega}, t)(1 + \Delta \bar{l})\right) + \delta \Delta t L_0 / L.$$
A.6 Exit Rates from Labor Markets

A worker exits her labor market if the log full-employment wage is \( \omega \) and the market is hit by an adverse shock, if the labor market closes, or if she quits. In the discrete time, discrete state space model, the first event hits a fraction \( \frac{1}{2} \Delta \tilde{l}(1 - \Delta p) \) of the workers who survive in a labor market with \( \omega = \bar{\omega} \):

\[
N_s \Delta t \equiv (1 - (q + \delta) \Delta t) \frac{1}{2} (1 - \Delta p) \Delta \tilde{l} f_n(\omega)L + (q + \delta) \Delta t L
\]

Reexpress \( \Delta \omega \), \( \Delta \tilde{l} \), and \( \Delta p \) in terms of \( \Delta t \), take the limit as \( n \to \infty \), and use \( \frac{f_n(\omega)}{\Delta \omega} \to f(\omega) \), to get equation (16).

A.7 Density \( f \): Random Search

Since \( f \) is a density,

\[
\int_{\omega}^{\infty} f(\omega) d\omega = 1.
\] (57)

By taking the limit of a discrete time, discrete state-space analog of our model, we find that this density has to satisfy three additional conditions. First, at \( \omega > \omega \), it must solve a Kolmogorov forward equation, unchanged from equation (45) except for the effect of random search:

\[
(q + \delta - s) f(\omega) = -\mu f'(\omega) + \frac{\sigma^2}{2} f''(\omega) \text{ for all } \omega > \omega.
\] (58)

Second, at the lower bound \( \omega \), we have the same condition as in the directed search case, equation (46). Finally, in contrast with the directed search case, there is a kink in \( f \) at the point where new markets are created:

\[
f'_-(\bar{\omega}) - f'_+(\bar{\omega}) = \frac{2 \delta \bar{L}}{\sigma^2 L}.
\] (59)

where \( \bar{L} \) is the average number of workers in a new labor market. The derivation of this condition, which reflects the addition of directed searchers into new markets, is standard and hence omitted. Solving these equations and emphasizing the dependence of \( f \) on the boundary \( \omega \) and the arrival rate of new workers \( s \), we obtain equation (24).

A.8 Proof of Proposition 5

Express the value of a worker \( v \) as a function not only on the current log wage \( \omega \) but also of the lower bound on wages \( \omega \) and the arrival rate of search unemployed workers \( s \), \( v(\omega; \omega, s) \). We start by proving a preliminary result.
Lemma 2. $v$ is continuous and nondecreasing in $\omega$ and continuous and nonincreasing in $s$. It is strictly monotone in each argument if $\omega > \omega'$. 

Proof. Taking limits of equation (40), we obtain

$$
\Pi_{\omega'}(\omega'; \omega, \omega, s) = \begin{cases} 
\frac{\zeta_2 e^{\zeta_1 (\omega-\omega)} (\zeta_2 e^{\zeta_2 (\omega-\omega)} - \zeta_1 e^{\zeta_1 (\omega-\omega)})}{(\rho + q + \delta)(\zeta_2 - \zeta_1)} & \text{if } \omega' \leq \omega \\
\frac{\zeta_2 e^{\zeta_2 (\omega-\omega)} (\zeta_2 e^{\zeta_1 (\omega-\omega)} - \zeta_1 e^{\zeta_2 (\omega-\omega)})}{(\rho + q + \delta)(\zeta_2 - \zeta_1)} & \text{if } \omega \leq \omega' \leq \infty,
\end{cases}
$$

(60)

where $\zeta_1 < 0 < \zeta_2$ are the two roots of the characteristic equation

$$
\rho + q + \delta = \mu \zeta + \frac{\sigma^2}{2} \zeta^2.
$$

(61)

Of course, $\mu$ is a decreasing function of $s$ from equation (21). The remainder of the proof follows the structure of the proof of Lemma 1 as is omitted. ■

Now rewrite equations (20) and (22) to define two implicit functions of $s$:

$$
v(\omega_1(s); \omega_1(s), s) = \bar{v},
$$

(62)

$$
\int_{\omega_2(s)}^{\infty} v(\omega; \omega_2(s), s) f(\omega; \omega_2(s), s) d\omega = \bar{v},
$$

(63)

where our notation also emphasizes the dependence of the cross-sectional density of $\omega$ on $\omega$ and $s$. An equilibrium is a solution $s$ to

$$
\omega_1(s) = \omega_2(s).
$$

(64)

Lemma 2 and equation (30) imply that these are continuous and increasing functions of $s$. We then argue that

$$
\omega_2 \left( q + \frac{1}{2} \theta \sigma^2 \right) = -\infty < \omega_1 \left( q + \frac{1}{2} \theta \sigma^2 \right) \quad \text{and} \quad \lim_{s \to \infty} \omega_2(s) > \lim_{s \to \infty} \omega_1(s).
$$

(65)

If $s \leq q + \frac{1}{2} \theta \sigma^2$, the integral on the left hand side of equation (63) does not converge for all $\omega_2(s)$, while as $s \to q + \frac{1}{2} \theta \sigma^2$, $\omega_2 \to -\infty$. On the other hand, $\omega_1$ is finite for any $s$. At the other end of the range, as $s \to \infty$, the density $f$ puts almost all its weight at $\omega$. This implies that the left hand side of equation (63) converges to $v(\omega_2(\infty); \omega_2(\infty), \infty)$. Since $\bar{v} > \bar{v}$, we must have $\omega_2(\infty) > \omega_1(\infty)$. The existence of $\omega$ solving equation (64) then follows from the intermediate value theorem.
For the case where \( b_r = 0 \), so there is no rest unemployment, and \( q = 0 \), equations (62) and (63) simplify further. In this case, the left hand sides of both equations are proportional to \( e^\omega \), hence taking the ratio gives

\[
\frac{\tilde{v}}{v} \equiv 1 + \frac{\rho}{\kappa} = \frac{\eta_2(s)(1 + \zeta_1(s) + \eta_2(s))}{(1 + \eta_2(s))(\zeta_1(s) + \eta_2(s))},
\]

where the notation emphasizes that the roots \( \eta_i \) for \( f \) and \( \zeta_i \) given by equation (61) are functions of \( \mu \), which depends on \( s \) as in equation (21). In a Mathematica file available on request, we prove that the right hand side of equation (66) converges to \( \infty \) as \( s \) goes to its lower bound, and it converges to 1 as \( s \) goes to \( \infty \). Moreover, a lengthy algebraic argument shows that this ratio is decreasing in \( s \) everywhere. Thus, since \( \tilde{v}/v \geq 1 \), there is a unique solution \( s^* \), and hence a unique equilibrium. It is immediate that the equilibrium value of \( s^* \) is decreasing in the search cost \( \kappa \), and since \( \omega_1 \) is increasing, the value of \( \omega \) decreases too.

References


B Additional Appendixes not for Publication

B.1 Derivation Hamilton-Jacobi-Bellman

This appendix proves that if \( v(\omega) \) is given by:

\[
v(\omega) = \int_{\omega}^{\infty} R(\omega') \Pi_{\omega'}(\omega'; \omega) d\omega'
\]  

(67)

for an arbitrary continuous function \( R(\cdot) \) and where the local time function \( \Pi_{\omega'}(\cdot) \) is given as in Stokey (2009) Proposition 10.4:

\[
\Pi_{\omega'}(\omega'; \omega) = \begin{cases} 
\frac{(\zeta_2 e^{\zeta_1 \omega + \zeta_2 \omega} - \zeta_1 e^{\zeta_1 \omega + \zeta_2 \omega})(\zeta_2 e^{\zeta_1 \omega' - \zeta_1 \omega}) - \zeta_1 e^{\zeta_1 \omega' - \zeta_1 \omega}}{(\rho + q + \delta)(\zeta_2 - \zeta_1)(e^{\zeta_1 \omega + \zeta_2 \omega} - e^{\zeta_1 \omega + \zeta_2 \omega})} & \text{if } \omega \leq \omega' < \omega \\
\frac{(\zeta_2 e^{\zeta_1 \omega + \zeta_2 \omega} - \zeta_1 e^{\zeta_1 \omega + \zeta_2 \omega})(\zeta_2 e^{\zeta_1 \omega - \zeta_1 \omega}) - \zeta_1 e^{\zeta_1 \omega - \zeta_1 \omega}}{(\rho + q + \delta)(\zeta_2 - \zeta_1)(e^{\zeta_1 \omega + \zeta_2 \omega} - e^{\zeta_1 \omega + \zeta_2 \omega})} & \text{if } \omega \leq \omega' \leq \bar{\omega},
\end{cases}
\]  

(68)

where \( \zeta_1 < 0 < \zeta_2 \) are the two roots of the characteristic equation \( \rho + q + \delta = \mu \zeta + \frac{\sigma^2}{2} \zeta^2 \), then

\[
(\rho + q + \delta)v(\omega) = R(\omega) + \mu v'(\omega) + \frac{\sigma^2}{2} v''(\omega).
\]

**Proof.** Differentiating \( v \) with respect to \( \omega \) we get

\[
v'(\omega) = \int_{\omega}^{\infty} R(\omega') \Pi_{\omega'}(\omega'; \omega) d\omega'
\]

\[
v''(\omega) = \int_{\omega}^{\infty} R(\omega') \Pi_{\omega'}(\omega'; \omega) d\omega' + R(\omega) \left( \lim_{\omega' \uparrow \omega} \Pi_{\omega'}(\omega'; \omega) - \lim_{\omega' \downarrow \omega} \Pi_{\omega'}(\omega'; \omega) \right)
\]

where we use that \( \Pi_{\omega'} \) is continuous but \( \Pi_{\omega'\omega} \) has a jump at \( \omega' = \omega \). Then

\[
(\rho + q + \delta)v(\omega) - \mu v'(\omega) - \frac{\sigma^2}{2} v''(\omega)
\]

\[
= \int_{\omega}^{\infty} R(\omega) \left( (\rho + q + \delta)\Pi_{\omega'}(\omega'; \omega) - \mu \Pi_{\omega'}(\omega'; \omega) - \frac{\sigma^2}{2} \Pi_{\omega'\omega}(\omega'; \omega) \right) d\omega'
\]

\[
- \frac{\sigma^2}{2} R(\omega) \left( \lim_{\omega' \uparrow \omega} \Pi_{\omega'}(\omega'; \omega) - \lim_{\omega' \downarrow \omega} \Pi_{\omega'}(\omega'; \omega) \right).
\]

Using the functional form of \( \Pi_{\omega'} \) we have, for \( \omega' < \omega \):

\[
\Pi_{\omega'}(\omega'; \omega) = e^{\zeta_1 \omega' \tilde{h}_1(\omega')} - e^{\zeta_2 \omega' \tilde{h}_2(\omega')}
\]
Then use the expression for the roots:
\[ ζ_i e^{ζ_i ω} (ζ_i e^{ζ_i (ω-ω')} - ζ_i e^{ζ_i (ω-ω')}) \]

\[ \frac{ζ_2 e^{ζ_2 ω} (ζ_2 e^{ζ_2 (ω-ω')} - ζ_1 e^{ζ_1 (ω-ω')})}{(ρ + q + δ)(ζ_2 - ζ_1)(e^{ζ_2 ω} + e^{ζ_2 ω'})} \]

Thus for all \( ω' < ω \):
\[
(ρ + q + δ)Π_ω(ω'; ω) - μΠ_ω(ω'; ω) - \frac{σ^2}{2} Π_ω(ω'; ω) =
\]
\[
[(ρ + q + δ) - ζ_1 μ](e^{ζ_1 ω}) - [(ρ + q + δ) - ζ_2 μ](e^{ζ_2 ω}) = 0
\]

where the last equality follow from the definition of the roots \( ζ_i \). Hence
\[
\int_{ω}^{ω'} R(ω') \left( (ρ + q + δ)Π_ω(ω'; ω) - μΠ_ω(ω'; ω) - \frac{σ^2}{2} Π_ω(ω'; ω) \right) dω' = 0.
\]

Using a symmetric calculation for \( ω' > ω \) we have:
\[
\int_{ω'}^{ω} R(ω') \left( (ρ + q + δ)Π_ω(ω'; ω) - μΠ_ω(ω'; ω) - \frac{σ^2}{2} Π_ω(ω'; ω) \right) dω' = 0.
\]

Next, differentiating \( Π_ω(ω'; ω) \) when \( ω' < ω \) and when \( ω' > ω \) and let \( ω' → ω \) from below and from above, tedious—but straightforward—algebra, gives:
\[
\lim_{ω' → ω} Π_ω(ω'; ω) - \lim_{ω' → ω} Π_ω(ω'; ω) = - \frac{ζ_1 ζ_2}{ρ + q + δ}.
\]

Then use the expression for the roots: \( ζ_1 ζ_2 = -(ρ + q + δ)/(σ^2/2) \). Putting this together proves the result. \( □ \)

### B.2 Market Social Planner’s Problem

In this section we introduce a dynamic programming problem whose solution gives the equilibrium value for the thresholds \( ω, ω' \). This problem has the interpretation of a fictitious social planner located in a given market who maximizes net consumer surplus by deciding how many of the agents currently located in the market work and how many rest and whether to adjust the number of workers in the market. The equivalence of the solution of this problem with the equilibrium value of the labor market participant has the following implications. First, it
establishes that our market decentralization is rich enough to attain an efficient equilibrium, despite the presence of search frictions. Second, it gives an alternative argument to establish the uniqueness of the equilibrium values for the thresholds $\omega$ and $\bar{\omega}$. Third, it connects our results with the decision theoretic literature analyzing investment and labor demand model with costly reversibility.

The market planner maximizes the net surplus from the production of the final good in a market with current log productivity $\tilde{x}$ and $l$ workers, taking as given aggregate consumption $C$ and aggregate output $Y$. The choices for this planner are to increase the number of workers located in this market (hire), paying $\bar{v}$ to the households for each or them, or to decrease the number of workers located at the market number (fire), receiving a payment $v$ for each. Increases and decreases are non-negative, and the prices associated with them have the dimension of an asset value, as opposed to a rental. We let $M(\tilde{x}, l)$ be the value function of this planner, hence:

$$M(\tilde{x}, l) = \max_{l_h(t), l_f(t)} \mathbb{E}\left( \int_0^\infty e^{-(\rho+\delta)t} \left( (S(\tilde{x}(t), l(t)) + vl(t)) dt - \bar{v}dl_h(t) + vdl_f(t) \right) \bigg| \tilde{x}(0) = \tilde{x}, l(0) = l \right)$$

subject to $dl(t) = -ql(t)dt + dl_h(t) - dl_f(t)$ and $d\tilde{x} = \mu_x dt + \sigma_x dz$. \tag{69}

The $l_h(t)$ and $l_f(t)$ are increasing processes describing the cumulative amount of “hiring” and “firing” and hence $dl_h(t)$ and $dl_f(t)$ intuitively have the interpretation of hiring and firing during period $t$. The term $ql(t)dt$ represent the exogenous quits that happens in a period of length $dt$. The planner discounts at rate $\rho + \delta$, accounting both for the discount rate of households and for the rate at which her labor market disappears.

The function $S(\tilde{x}, l)$ denotes the return function of the market social planner per unit of time and is given by

$$S(\tilde{x}, l) = \max_{E \in [0,l]} u'(C) \int_0^{EAe^\tilde{x}} \left( \frac{Y}{y} \right)^{\frac{1}{\theta}} dy + b_r(l - E) + \delta lv.$$ 

The first term is the consumer’s surplus associated with the particular good, obtained by the output produced by $E$ workers with log productivity $\tilde{x}$. The second term is value of the workers that the planner chooses to send back to the household, receiving $v$ for each. The third term is the value of the “sale” of all the workers if the market shuts down. Setting $q = \delta = b_r = 0$ our problem is formally equivalent to Bentolila and Bertola’s (1990) model of a firm deciding employment subject to a hiring and firing cost and to Abel and Eberly’s (1996) model of optimal investment subject to costly irreversibility, i.e. a different buying and selling price for capital.
Using the envelope theorem, we find that the marginal value of an additional worker is:

\[
S_t(\bar{x}, l) = \max \left\{ u'(C) \left( \frac{Y(Ae^{\bar{x}})^{\theta-1}}{l} \right)^{\frac{1}{\theta}}, b_r \right\} + \delta v
\]

\[
\equiv s \left( \frac{(\theta - 1)(\bar{x} + \log A) + \log Y - \log l}{\theta} + \log u'(C) \right)
\]

where the function \(s(\cdot)\) is given by \(s(\omega) = \max\{e^\omega, b_r\} + \delta v\) and is identical to the expression for the per-period value of a labor market participant in our equilibrium, except that \(\delta v\) is in place of \((q + \delta)\). This is critical to the equivalence between the two problems.

To prove this equivalence, we write the market social planner’s Hamilton-Jacobi-Bellman equation. For each \(\bar{x}\), there are two thresholds, \(l(\bar{x})\) and \(\bar{l}(\bar{x})\) defining the range of inaction. The value function \(M(\cdot)\) and thresholds functions \(\{l(\cdot), \bar{l}(\cdot)\}\) solve the Hamilton-Jacobi-Bellman equation if the following two conditions are met:

1. For all \(\bar{x}\), and \(l \in (l(\bar{x}), \bar{l}(\bar{x}))\) employment decays exponentially with the quits at rate \(q\) and hence the value function \(M\) solves

\[
(\rho + \delta)M(\bar{x}, l) = S(\bar{x}, l) - qM_l(\bar{x}, l) + \mu_x M_{\bar{x}}(\bar{x}, l) + \frac{\sigma^2}{2} M_{\bar{x}\bar{x}}(\bar{x}, l).
\]

2. For all \((\bar{x}, l)\) outside the interior of the range of inaction,

\[
(\rho + \delta)M(\bar{x}, l) + qM_l(\bar{x}, l) - \mu_x M_x(\bar{x}, l) - \frac{\sigma^2}{2} M_{xx}(\bar{x}, l) \leq S(\bar{x}, l),
\]

\[
v = M_l(\bar{x}, l) \forall l \geq \bar{l}(\bar{x}), \text{ and } \bar{v} = M_l(\bar{x}, l) \forall l \leq l(\bar{x})
\]

Equation (73) is also referred to as smooth pasting. Since \(M(\bar{x}, \cdot)\) is linear outside the range of inaction, a twice-continuously differentiable solution implies super-contact, or that for all \(\bar{x}\):

\[
0 = M_{ll}(\bar{x}, \bar{l}(\bar{x})) = M_{ll}(\bar{x}, l(\bar{x})).
\]

According to Verification Theorem 4.1, Section VIII in Fleming and Soner (1993), a twice-continuously differentiable function \(M(\bar{x}, l)\) satisfying equations (71), (73), and (74) solves the market social planner’s problem.

If \(M\) is sufficiently smooth, finding the optimal thresholds functions \(\{l(\cdot), \bar{l}(\cdot)\}\) can be stated as a boundary problem in terms of the function \(M_l(\bar{x}, l)\) and its derivatives. To see this,
differentiate both sides of equation (71) with respect to \( l \) and replace \( S_l \) using equation (70):

\[
(\rho + \delta + q)M_l(\bar{x}, l) = s \left( \frac{(\theta - 1)(\bar{x} + \log A) + \log Y - \log l}{\theta} + \log u'(C) \right) \\
- qlM_{ll}(\bar{x}, l) + \mu_x M_{l\bar{x}}(\bar{x}, l) + \frac{\sigma^2_x}{2} M_{x\bar{x}}(\bar{x}, l).
\]

(75)

If the required partial derivatives exist, any solution to the market social planner’s problem must solve equations (73)–(75). Moreover, there is a clear relationship between the value function \( v(\omega) \) in the decentralized problem and the marginal value of a worker \( M_l \) in the market social planner’s problem:

**Lemma 3.** Assume that \( \theta \neq 1 \) and that the functions \( M_l(\cdot) \) and \( v(\cdot) \) satisfy

\[
M_l(\bar{x}, l) = v(\omega), \text{ where } \omega = \frac{\log Y + (\theta - 1)(\log A + \bar{x}) - \log l}{\theta} + \log u'(C)
\]

(76)

and that thresholds functions \( \{\underline{l}(\cdot), \bar{l}(\cdot)\} \) and the thresholds levels \( \{\underline{\omega}, \bar{\omega}\} \) satisfy

\[
\log \underline{l}(\bar{x}) = \log Y + (\theta - 1)(\bar{x} + \log A) - \theta(\underline{\omega} - \log u'(C))
\]

(77)

\[
\log \bar{l}(\bar{x}) = \log Y + (\theta - 1)(\bar{x} + \log A) - \theta(\bar{\omega} - \log u'(C)).
\]

(78)

Then, \( M_l(\cdot) \) and \( \{\underline{l}(\cdot), \bar{l}(\cdot)\} \) solve equations (73)–(75) for all \( \bar{x} \) and \( l \in [\underline{l}(\bar{x}), \bar{l}(\bar{x})] \) if and only if \( v(\cdot) \) and \( \{\underline{\omega}, \bar{\omega}\} \) solve equations (12).

**Proof.** Differentiate equation (76) with respect to \( \bar{x} \) and \( l \) to get

\[
M_{\bar{x}}(\bar{x}, l) = v'(\omega) \frac{\theta - 1}{\theta}, \quad M_{\bar{x}\bar{x}}(\bar{x}, l) = v''(\omega) \left( \frac{1}{\theta} \right)^2 \quad \text{and} \quad M_{ll}(\bar{x}, l) = -v'(\omega) \frac{1}{\theta}.
\]

Recall that a solution of equation (12) is equivalent to a solution to equations (42), (43), and \( v(\bar{\omega}) = \bar{v} \) and \( v(\underline{\omega}) = \underline{v} \). The equivalence between equation (12) and equations (73)–(75) is immediate, recalling the definitions of \( \mu \) and \( \sigma \).

This lemma has important implications. First, it establishes, not surprisingly, that the equilibrium allocation is Pareto Optimal. Second, since the market social planner’s problem is a maximization problem, the solution is easy to characterize. For instance, since the problem is convex, it has at most one solution and hence the equilibrium value of a labor market participant is uniquely defined, for given \( u'(C) \) and \( Y \). The fact that \( v \) is increasing is then equivalent to the concavity of \( S(\bar{x}, \cdot) \). Finally, notice that Proposition 1 establishes
existence and uniqueness of the solution to equation (12) only under mild conditions on \( s(\cdot) \), i.e. that it was weakly increasing and bounded below. Proposition 1 can be used to extend the uniqueness and existence results of the literature of costly irreversible investment to a wider class of production functions. Currently the literature uses that the production function is of the form \( x^{a_x} l^{a_l} \) for some constants \( a_x \) and \( a_l \), with \( 0 < a_l < 1 \), as in Abel and Eberly (1996). Proposition 1 shows that the only assumption required is that the production function be concave in \( l \), and that the marginal productivity of the factor \( l \) can be written as a function of the ratio of the quantity of the input \( l \) to (a power of) the productivity shock \( x \).

C Heterogeneous Industries

In this section we extend the model to include heterogeneity in households human capital. We specify the heterogeneity so that in equilibrium industries can be divided into different classes, with the ones attracting the high human capital households paying higher wages. In particular, the stochastic process for the wages of a higher human capital industry class is a scaled up version of the one for an industry class with lower human capital. Otherwise industry classes are identical, in particular they have the same process for \( \{\omega\} \) for the full employment log wage (measured in utils) and hence the same values of thresholds \( \omega, \bar{\omega} \), rest and search unemployment rates, and job destruction rates. This version of the model justifies our fixed effect treatment—removing industry means in logs of the US industry wage data in Section 6.1.

We turn to the description of the model. Households are indexed by one of the \( K \) human capital types, denoted by \( h_k \) satisfying \( 0 < h_1 < h_k < h_{k+1} \), for \( k = 1, 2, ..., K \), and \( h_K = 1 \). For notational convenience we use \( h_0 = 0 \). Recall that industries are indexed by \( j \) which belong to \([0, 1]\). The economic meaning of type \( h_k \) human capital, is that such household can work in any industry labelled \( j \in (0, h_k) \). We let \( H_k \) denote the CDF of households’ human capital types, so that there are \( H_k \) households with human capital \( h_j \leq h_k \), and there are \( \Delta H_k \equiv H_k - H_{k-1} \) household with human capital type \( h_k \) for \( k = 1, ..., K \).

We are looking for an equilibrium where households of type \( h_k \) work on industries of type \( j \in (h_{k-1}, h_k) \). We let \( \Delta h_k \equiv h_k - h_{k-1} \) the industries that will hire workers of type \( k \). In this equilibrium we can talk of both household and industries of type \( k \). For workers to sort themselves across industries in this way, it requires that wages are increasing in industry type. To obtain this pattern for wages we will assume that the ratio of households to industries is lower for higher labelled industry classes:

\[
\frac{\Delta H_{k+1}}{\Delta h_{k+1}} < \frac{\Delta H_k}{\Delta h_{k-1}}, \tag{79}
\]
for \( k = 1, \ldots, K - 1 \).

We use \( L_k \) for the fraction of households of type \( k \) that are located in industries of that class, and \( L_{0,k} \) for the fraction located in newly created industries within the \( k \) class. Thus \( L_k \Delta H_k \) is the number of households of human capital \( k \) located in industries, i.e. working or in rest unemployment.

Households with different human capital will have different consumption, and hence different marginal utility. Letting \( C_k \) be the consumption per household for those with human capital \( k \) we have that the log full-employment wage for household of type \( k \) follows:

\[
\omega_k(t) \equiv \frac{\log Y + (\theta - 1) \log(Ax(t)) - \log l_k(t)}{\theta} + \log u'(C_k) \tag{80}
\]

where \( l_k(t) \) is the number of workers of type \( k \) in per industry of class \( k \). We will characterize an equilibrium where the process for \( \omega_k \) are identical across industries.

**Proposition 6.** Let \((L^*, \bar{\omega}^*, \bar{\omega}^*)\) be the equilibrium values for the model without heterogeneity and utility function is log, \( u(C) = \log(C) \). Furthermore, assume that \( \theta > 1 \) and that the distribution of human capital satisfies condition equation (79). Then, there is an equilibrium for the model with heterogeneity with \((L_k, \bar{\omega}_k, \bar{\omega}_k) = (L^*, \bar{\omega}^*, \bar{\omega}^*)\) for all \( k \).

**Proof.** For the processes \( \{\omega_k(t)\} \) to be identical across industries, the difference in the log of the marginal utilities must be compensated by a difference in the level of the employment per industry \( l_k \) so that two industries in different classes \( k \) and \( k' \) created at the same time, and with the same history of shocks \( x' \)’s up to time \( t \) will have

\[
\log(l_k(t)) - \log(l_{k'}(t)) = \theta(\log u'(C_k) - \log u'(C_{k'})) ,
\]

This means that there must be a number \( \beta > 0 \) such that the number of workers located in \( k \) industries must satisfy

\[
\Delta H_k L_k / \Delta h_k (u'(C_k))^\theta \equiv \beta
\]

for all \( k = 1, \ldots, K \).

The boundary behaviour of the distribution \( f \) evaluated at the upper bound gives:

\[
\frac{\sigma^2}{2} f'(\bar{\omega}) - \left( \mu + \frac{\theta \sigma^2}{2} \right) f(\bar{\omega}) = \delta \frac{L_{0,k}}{L_k}, \tag{82}
\]

which gives the same solution \( \frac{L_{0,k}}{L_k} \) for all \( k \). The requirement that at the upper bound log
full-employment wages must be equal to $\bar{\omega}$ gives

$$L_k \frac{L_{0,k}}{L_k} \Delta H_k = \Delta h_k \int_0^\infty Y u'(C_k)\theta (Ax_0)^{\theta-1}e^{-\theta \bar{\omega}} \, dF(x_0)$$

$$= \Delta h_k Y u'(C_k)\theta (AX_0)^{\theta-1}e^{-\bar{\omega}}, \quad (83)$$

Combining equation (82) and equation (83) we have:

$$\beta = Y \left( \frac{\delta}{2} f'(\bar{\omega}) - \left( \mu + \frac{\theta \sigma^2}{2} \right) f(\bar{\omega}) \right) (AX_0)^{\theta-1}e^{-\bar{\omega}} \quad (84)$$

which gives an expression for $Y$ and $\beta$.

Total output in the economy will be consumed by all households, so:

$$\sum_{k=1}^K C_k \Delta H_k = Y \quad (85)$$

In each industry class $k$ we can solve for the productivity consistent with $(l, \omega, Y, C_k)$ as:

$$x = \xi(l, \omega, Y, C_k) \equiv \frac{1}{A} \left( \frac{l e^{\bar{\omega}}}{Y u'(C_k)^\theta} \right)^{\frac{\theta}{\theta-1}} \quad (86)$$

Then using the production function, output in a market in such an industry class, with $l$ workers and log full-employment wage $\omega$ is

$$Q(l, \xi(l, \omega, Y, C_k)) = Y^{\frac{1}{\theta-1}} \left( \frac{e^{\omega}l}{u'(C_k)^\theta} \right)^{\frac{\theta}{\theta-1}} \min\{1, e^{\omega}/b_r\}^\theta \quad (87)$$

Using this notation, we can write equation (51) as

$$Y = \left( \sum_{k=1}^K \int_{h_{k-1}}^{h_k} Q(l(j,t), \xi(l(j,t), \omega(j,t), Y, C_k)) \frac{\Delta H_k L_k}{\Delta h_k} \, dj \right)^{\frac{\theta}{\theta-1}}$$

$$= \left( \sum_{k=1}^K \int_{h_{k-1}}^{h_k} Q \left( \frac{\Delta H_k L_k}{\Delta h_k}, \xi \left( \frac{\Delta H_k L_k}{\Delta h_k}, \omega(j,t), Y, C_k \right) \right) \frac{\Delta H_k L_k}{\Delta h_k} \, dj \right)^{\frac{\theta}{\theta-1}} \quad (88)$$

The second equation follows because $Q(\cdot, \xi(\cdot, \omega, Y, C_k))^{\frac{\theta}{\theta-1}}$ is linear (equation 50). To solve this, we change the variable of integration from the name of the market $j$ to its log full-employment wage $\omega$ and number of workers $l$. The quantity $\frac{\Delta H_k L_k}{\Delta h_k}$ is the average number of households per industry in industry class $k$. Let $\tilde{f}(\omega, l)$ be the density of the joint invariant
distribution of workers in markets \((\omega, l)\), as discussed in Appendix A.4. Notice that under our hypothesis this distribution is the same for all \(k\). Then

\[
Y = \left( \sum_{k=1}^{K} \Delta h_k \int_{\omega}^{\omega} \int_{0}^{\infty} Q \left( \frac{\Delta H_k L_k}{\Delta h_k}, \xi \left( \frac{\Delta H_k L_k}{\Delta h_k}, \omega, Y, C_k \right) \right) \frac{\theta - 1}{\theta} \frac{l}{\Delta H_k L_k} \tilde{f}(\omega, l) \, dl \, d\omega \right)^{\frac{\theta}{\theta - 1}}.
\]

Since \(f(\omega) = \int_{0}^{\infty} \frac{1}{L} \tilde{f}(\omega, l) \, dl\), where \(L = \int_{0}^{\infty} \int_{\omega}^{\omega} \tilde{f}(\omega, l) \, dld\omega\), we can solve the inner integral to obtain

\[
Y = \left( \sum_{k=1}^{K} \Delta h_k \int_{\omega}^{\omega} Q \left( \frac{\Delta H_k L_k}{\Delta h_k}, \xi \left( \frac{\Delta H_k L_k}{\Delta h_k}, \omega, Y, C_k \right) \right) \frac{\theta - 1}{\theta} \frac{l}{\Delta H_k L_k} \tilde{f}(\omega)\right)^{\frac{\theta}{\theta - 1}},
\]

without characterizing the joint density \(\tilde{f}\). Using equation (87) and simplifying,

\[
Y = \left( \sum_{k=1}^{K} \frac{\Delta H_k L_k}{u'(C_k)} \right) \int_{\omega}^{\omega} e^{\omega \min\{1, e^{\omega/b_r}\}^{\theta-1} f(\omega) \, d\omega}.
\] (88)

Finally, we describe the budget constraint for each household type in steady state. This constraint can be written as \(C_k = W_k L_k (1 - U_{r,k} L_k)\), where \(W_k\) is the average wage of type \(k\). Using that the rest unemployment rate will be the same across type \(k\) and that the labor demand in equation (80) has slope \(-1/\theta\), we can write the budget constraint as:

\[
C_k = W_k L_k (1 - \frac{U_{r,k}}{L_k}) = \gamma \left[ \frac{L_k \Delta H_k}{\Delta h_k} \right]^{-1/\theta} L_k
\] (89)

for \(k = 1, \ldots, K\) and for some number \(\gamma > 0\) to be determined.

Now we have a system of \(2K + 3\) equilibrium objects are \(\{C_k, L_k, Y, \beta, \gamma\}\) in \(2K + 3\) equations: equation (85) equation (84), equation (88), \(K\) versions of equation (81), and \(K\)
versions of equation (89). We can write this system as:

\[ C_k = \gamma \left[ \frac{L_k \Delta H_k}{\Delta h_k} \right]^{-1/\theta} L_k, \quad k = 1, \ldots, K \]  
(90)

\[ \beta = \frac{\Delta H_k L_k}{u'(C_k)^{\theta} \Delta h_k}, \quad k = 1, \ldots, K \]  
(91)

\[ \beta = Y \phi_2, \]  
(92)

\[ Y = \sum_{k=1}^{K} \Delta H_k C_k, \]  
(93)

\[ Y = \phi_1 \sum_{k=1}^{K} \Delta H_k L_k \frac{1}{u'(C_k)}, \]  
(94)

where

\[ \phi_2 \equiv \left( \frac{\delta(A X_0)^{\theta} e^{-\theta \bar{\omega}}}{\frac{a^2}{2} f'(\bar{\omega}) - (\mu + \frac{\theta a^2}{2}) f(\bar{\omega})} \right) \quad \text{and} \quad \phi_1 \equiv \int_{\bar{\omega}}^{\omega} e^{\omega} \min\{1, e^{\omega} / b_r\}^{\theta-1} f(\omega) d\omega \]

If utility is log, we replace \( u'(C_k) = 1/C_k \), in equation (94), and using equation (93) we obtain

\[ \sum_{k=1}^{K} \Delta H_k C_k = \left( \sum_{k=1}^{K} L_k (\Delta H_k C_k) \right) \phi_1. \]

Guess that \( L_k = L \) for all \( k \) so that this equation becomes \( L_k = L = 1/\phi_1 \), which is the same equation than in the log case without heterogeneity.

Combining equation (91), equation (92), equation (93) and \( L_k = 1/\phi_1 \) to obtain:

\[ C_k = \left( \frac{\phi_1 \phi_2 \Delta h_k}{\Delta H_k} \right)^{1/\theta} \left( \sum_{j=1}^{K} \Delta H_j C_j \right)^{1/\theta} \quad \text{for } k = 1, \ldots, K. \]

Defining \( c_k \equiv \log C_k \) we can write this system as:

\[ c_k = (T)(c)_k \equiv \frac{1}{\theta} \varphi_k + \frac{1}{\theta} \log \left( \sum_{j=1}^{K} \Delta H_j e^{e_j} \right) \]  
(95)

where

\[ \varphi_k \equiv \log (\phi_1 \phi_2) + \log \left( \frac{\Delta h_k}{\Delta H_k} \right) \quad \text{for } k = 1, \ldots, K. \]

We regard the left hand side of equation (95) as defining an operator \( T : \)
mapping $R^K$ into itself. We notice that

\[ T_i(c) \leq T(c) \leq T_h(c) \]  \hspace{1cm} (96)

for all $c \in R^K$ where

\[ (T_h)(c)_k = \frac{1}{\theta} \bar{q}_k + \frac{1}{\theta} \log \left( \sum_{j=1}^{K} \Delta H_j e^{c_j} \right) \]

where $\bar{q} \equiv \max_{j=1,...,K} \{ q_j \}$ and where $T_i$ is defined analogous with the smallest $q_j$ instead. It is immediate to see that $c_k = \frac{1}{\theta-1} \bar{q}$ is the unique fixed point of $T_h$. Analogously, we have that $c_k = \frac{1}{\theta-1} \bar{q}$ is the unique fixed point of $T_i$. Now we can restrict the domain of $T$ to the following $K$ dimensional compact set: $\mathcal{K} = \left[ \frac{1}{\theta-1} \bar{q}, \frac{1}{\theta-1} \bar{q} \right]^K$. Using equation (96) it is easy to show that $T : \mathcal{K} \to \mathcal{K}$. Finally, we verify that $T$ satisfy the Blackwell’s sufficient conditions for a contraction. That $T$ is monotone is immediate. That $T$ discount follows from the following inequality for $a > 0$:

\[ (T)(c + a)_k = \frac{1}{\theta} q_k + \frac{1}{\theta} \log \left( \sum_{j=1}^{K} \Delta H_j e^{c_j+a} \right) = \frac{1}{\theta} a + (T)(c) \]

where we use that $\sum_{j=1}^{K} \Delta H_j = 1$. Hence, $T$ is a contraction of modulus $1/\theta$ and hence it has a unique fixed point on $\mathcal{K}$.

Using the computed values of $C_k$ and equation (93) we find $Y$. Using the value of $Y$ and $L$ in equation equation (91) we compute $\beta$. From equation (90) and equation (91) we obtain: $\gamma^\theta = \beta$.

Finally, we must check that a type $k$ household prefer to work on industry $k$ to other industries $j = 1, ..., k - 1$. Since in the constructed equilibrium $L_k$ and $U_{s,k}$ are the same for all $k$ we just need to verify that $C_k > C_{k-1}$ for all $k = 2, ..., K$. This follows immediately from equation (90) and the assumption equation (79). \hfill \Box