Economics 202A Midterm Exam

October 22, 2013

Instructions: You have 1 hour 20 minutes for this exam. Take the first few minutes to look it over, and pace yourself. Don't panic; the problems are easier than they look. (Really.) If you get stuck, move on to something else and come back to the difficult bit later if you have time. Each question is worth 50 points.

1. *A Ramsey model with human capital.* Assume a constant-returns production function of the form $Y = F(K, H, L)$ where H denotes the stock of human capital. The labor force grows according to $\dot{L} = nL$, and letting lower case letters denote ratios to L as usual, we shall write $y = f(k, h) \equiv F(K/L, H/L, 1)$. Technology does not change over time.

(a) A fraction s of saving goes into accumulating K and a fraction $1 - s$ goes into accumulating H. If both types of capital depreciate at the same rate δ, then:

$$\begin{aligned} \dot{K} &= s\,[F(K, H, L) - C] - \delta K, \\ \dot{H} &= (1 - s)\,[F(K, H, L) - C] - \delta H. \end{aligned}$$

Show how to write these two equations in terms of k, h, per capita consumption $c = C/L$, and the "intensive" production function $f(k, h)$.

(b) We would like to endogenize the share s above (in principle, it need not be constant over time without further assumptions). To that end, set up the optimal control problem of maximizing

$$\int_0^\infty u[c(t)]e^{-(\theta-n)t}dt$$

(where $\theta > n$), subject to your equations from part (a) for $\dot{k}$ and $\dot{h}$, as well as initial conditions for physical and human capital. What are the control variables and what are the state variables?

(c) Write down the Hamiltonian for the optimal control problem, letting λ_k and λ_h denote the costate variables for the two accumulation constraints. What are the first-order conditions for the controls, and what do you learn from them? Explain your answer intuitively. (To economize on writing effort, denote $\partial F/\partial K = \partial f/\partial k = f_k$, and likewise for f_h. Also, recall I pointed out in class that everything I said about the Maximum Principle applies to *vectors* of control and state variables.)

(d) Now derive the equations of motion for the costate variables. What do these tell you about the optimal relationship between k and h?

(e) Write the optimal ratio of h to k for the production function $Y = K^{\alpha}H^{\beta}L^{1-\alpha-\beta}$. Is it constant over time or not? Find the optimal values of s and $1-s$ for this Cobb-Douglas production function.

(f) Using the consumption Euler equation and your answer to (e), show that the model has a balanced growth path and calculate the steady state values $\bar{k}$ and $\bar{h}$. How do they depend on θ and δ, and why?

(g) Using the aggregate relation $\dot{k}+\dot{h} = f(k,h) - c - (n+\delta)(k+h)$, find the golden-rule steady-state levels of k and h. (I want only the first-order conditions, not algebraic solutions for k and h at the golden rule.)

(h) Is the balanced growth path dynamically efficient? How can you tell?

2. *Government non-interest-bearing debt in an overlapping generations model.* Consider an overlapping generation model like the Diamond economy except that (i) population size is constant and (ii) instead of output being produced, it comes as a pure endowment y to the young, where y is fixed over time and the old get absolutely nothing in the way of income. A representative generation born on date t maximizes $u\left(c_t^y\right) + \beta u\left(c_{t+1}^o\right)$.

(a) We will assume throughout that the output y that the young receive is *perishable*: there is no way to save any of it for the future. Under this assumption, what are the consumption levels of the young and the old in the absence of any governmental intervention?

(b) Assume instead that the government prints pieces of paper that people believe others will accept in return for output. I will call each piece of paper a "dollar." Let p_t denote the price of one dollar in terms of output on date t. Let M_t denote the number of dollars that a representative young person will wish to hold, and explain why his/her lifetime consumption levels will now be given by:

$$\begin{aligned} c_t^y &= y - p_t M_t, \\ c_{t+1}^o &= p_{t+1} M_t. \end{aligned}$$

Provided the price of money p is always positive, and money enters the economy initially as a gift to the existing old, does the invention of dollars make everyone better off?

(c) By eliminating M_t above, derive the *intertemporal budget constraint* of a young person born on date t. How is the term p_{t+1}/p_t in that constraint related to the real rate of interest that appears in the Diamond model?

(d) Show the first-order Euler condition from maximizing $u\left(c_t^y\right)+\beta u\left(c_{t+1}^o\right)$ subject to the last constraint. Explain the intuition briefly.

(e) Suppose the total supply of dollars is fixed at m. In equilibrium (with one representative person per generation), $M_t = m$ and $c_t^y + c_t^o = (y - p_t m) + p_t m = y$. Use this to explain why in equilibrium, the economy's path is governed by the difference equation in p_t:

$$u'\left(y - p_t m\right) = \beta\left(p_{t+1}/p_t\right) u'\left(p_{t+1} m\right).$$

(f) Argue graphically that there is a constant (steady-state) price of dollars, $\bar{p}$, that satisfies the preceding equation and therefore is an equilibrium. Calculate this $\bar{p}$ explicitly for the case that $u(c) = \ln(c)$.

(g) Now assume that the government issues new dollars over time by going to the output market and buying resources g from the young. Because the young are accepting these dollars in payment, then in every period t, the government increases the supply of dollars according to its budget constraint:

$$p_t\left(M_t - M_{t-1}\right) = g.$$

We can no longer assume that the price of dollars, p_t, will remain constant over time, but let us conjecture that the *real* amount of dollars (dollars measured in terms of output) that the young wish to purchase will be constant over time in equilibrium, say, at $p_t M_t = \bar{\mu}$ for all t. Let us assume also that p_t will fall in equilibrium at a constant rate over time, such that $p_{t+1}/p_t = 1/(1+\bar{\pi})$ – because there are more dollars competing to buy the same amount of goods. Explain why, in this equilibrium, the Euler-equation-based equilibrium condition of part (e) becomes

$$u'\left(y - \bar{\mu}\right) = \beta\left(\frac{1}{1+\bar{\pi}}\right) u'\left(\frac{\bar{\mu}}{1+\bar{\pi}}\right),$$

while the government budget constraint becomes

$$\frac{\bar{\pi}}{1+\bar{\pi}}\bar{\mu} = g.$$

(h) Show that in this conjectured equilibrium, $c_t^y + c_t^o + g = y$ for all t (so that it is indeed an equilibrium).

(i) For the case $u(c) = \ln c$, use the two equations from part (g) to calculate the equilibrium values of $\bar{\mu}$ and $\frac{\bar{\pi}}{1+\bar{\pi}}$. (The number $\frac{\bar{\pi}}{1+\bar{\pi}} < 1$ is interpreted as the inflation tax rate on real dollar holdings $\bar{\mu}$, and that tax finances the government's spending g.) Given that the level of the date t money supply is M_t, what equation determines the price of money p_t?

(j) Is there a maximum ratio $g/y < 1$ that the government can squeeze out of the private sector by printing dollars?