University of California
Department of Economics

Economics 202A
Fall 2014
P. Gourinchas/D. Romer

Problem Set 7
Due in Lecture Thursday Oct. 23

1 Romer 8.1

2 Romer 8.13

3 Interest rates and Saving. Based on Summers (1981)

Consider a household with a finite lifetime T, working until age $N < T$ and solving the following continuous-time problem under certainty:

$$\max_{c_t} \int_0^T e^{-\rho t} u(c_t) dt$$

s.t.

$$\int_0^T c_t e^{-rt} dt = \int_0^N w_t e^{-rt} dt$$

where w_t denotes the labor income of the household, ρ is the discount factor and r is the constant real interest rate. $u(.)$ takes the constant relative risk aversion form:

$$u(c) = c^{1-\theta}/(1-\theta), \quad \theta > 0$$

At any instant, all workers receive the same wage w_t which grows at a constant rate g. We assume that $r \neq g$ and that $r(1-\theta) \neq \rho$.

1. Using the dynamic optimization tools you studied in class, write the conditions that characterize the solution to the household's maximization problem. In particular, show that consumption satisfies:
$$c_t = c_0 e^{t(r-\rho)/\theta}$$

2. Show that the consumption level c_0 satisfies

$$c_0 = \Phi(r) w_0$$

where

$$\Phi(r) = \frac{e^{(g-r)N} - 1}{e^{((r-\rho)/\theta - r)T} - 1} \frac{r - (r-\rho)/\theta}{r - g}$$

3. Sketch qualitatively the time path of household asset holdings a_t (recall that asset holdings follow: $\dot{a}_t = ra_t + w_t - c_t$) when $r > \rho$). Discuss how your answer may or may not depend on the growth rate g in relation to the interest rate r.

4. Now suppose that at any given point in time, the economy is populated with people born from different cohorts but solving a problem similar to the one above. People from different cohorts differ in the initial wage w_0 that they receive when they are born, but are otherwise identical. Denote l_t the size of the cohort born at time t and suppose that l_t grows at a constant rate n.

   (a) show that the working age population $L_t = \int_{t-N}^{t} l_s ds$ also grows at rate n.

   (b) Show that aggregate consumption C_t satisfies:

$$C_t = (w_t L_t)\Phi(r)\frac{1 - e^{-\left(g+n-\frac{r-\rho}{\theta}\right)T}}{g + n - \frac{r-\rho}{\theta}}\frac{n}{1 - e^{-nN}}$$

5. In the steady state of this economy, aggregate savings S_t must be such that aggregate capital K_t grows at rate $n + g$. It follows that

$$(n + g)K_t = w_t L_t + rK_t - C_t = S_t$$

   Deduce an expression for the ratio of saving to labor income $S/(wL)$ as a function of $C/(wL)$.

6. Assume that $n = 1.5\%$, $g = 2\%$, $T = 60$, $N = 40$ and $\rho = 3\%$. Using the expression above, plot the ratio of saving to labor income as r varies from 4% to 8% for the following values of θ: 0.5, 1, and 2. What do you conclude about the elasticity of aggregate savings to the interest rate in this model?

4 Time-varying discount rate (Uzawa (1968))

Consider the canonical problem of a household choosing its path of consumption to maximize lifetime utility, with one change: the household's discount rate is not constant. Specifically, the household maximizes:

$$\int_0^T e^{-\Delta_t} u(c_t)dt$$

where u satisfies the usual properties and $\Delta_t = \int_{\tau=0}^{t} \rho_s ds$. The household has initial wealth a_0 that evolves according to:

$$\dot{a}_t = ra_t + y_t - c_t$$

where y_t is labor income and r is a constant interest rate, both of which are taken as given by the household.

1. The household can borrow and lend freely, but it cannot die in debt. Write down the constraint that this imposes on a_T.

2. Set up the present value Hamiltonian

3. Find the conditions for optimality

4. Use your results to find an expression for $\dot{c}_t/c_t$ and interpret.

5. **[This will not be graded]** Suppose now that the discount rate ρ_t is a function of the instantaneous utility achieved by the household: $\rho_t = \rho(u(c_t))$ where the discount function $\rho(.)$ satisfies $\rho'(.) > 0$, $\rho''(.) > 0$ and $\rho(u) - u\rho'(u) > 0$. You can think of the previous case as what happens when the household does not realize that the discount rate varies with its consumption decisions. Now the household realizes what is going on. In theory, you could apply the Maximum Principle directly to this problem, as you did above. However, things get quickly messy (four differential equations, two co-state variables...). Uzawa observed that there is a way to simplify the problem by changing variables from 'calendar time' t to 'psychological' time Δ.

   (a) Using the law of motion of Δ_t, show that the problem is equivalent to:

$$\max_{c_\Delta} \int_0^{\Delta_T} e^{-\Delta} \frac{u(c_\Delta)}{\rho(u(c_\Delta))} d\Delta$$

s.t.

$$\frac{da_\Delta}{d\Delta} = \frac{ra_\Delta + y_\Delta - c_\Delta}{\rho(u(c_\Delta))}$$

   (b) Apply the Maximum Principle to this problem and find the conditions for optimality

   (c) Express these conditions in terms of 'calendar time' (i.e. c_t instead of c_Δ etc...) and interpret.