The Deflation Bias and Committing to being Irresponsible

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Abstract

I model deflation, at zero nominal interest rate, in a microfounded general equilibrium model. I show that deflation can be analyzed as a credibility problem if the government has only one policy instrument, i.e. increasing money supply by open market operations in short-term bonds, and cannot commit to future policies. I propose several policies to solve the credibility problem. They involve printing money or issuing nominal debt and either 1) cutting taxes, 2) buying real assets such as stocks, or 3) purchasing foreign exchange. The government credibly "commits to being irresponsible" by using these policy instruments. It commits to higher money supply in the future so that the private sector expects inflation instead of deflation. This is optimal since it curbs deflation and increases output by lowering the real rate of return.

Key words: Deflation, liquidity traps, zero bound on nominal interest rates

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Can the government lose control over the price level so that no matter how much money it prints, it has no effect on inflation or output? Ever since Keynes' General Theory this question has been hotly debated. Keynes answered yes, Friedman and the monetarists said no. Keynes argued that at low nominal interest rates increasing money supply has no effect. This is what he referred to as the liquidity trap. The zero short-term nominal interest rate in Japan today, together with the lowest short-term interest rate in the US in 45 years, make this old question urgent again. The Bank of Japan (BOJ) has nearly doubled the monetary base over the past 5 years, yet the economy still suffers deflation, and growth is stagnant or negative. Was Keynes right? Is increasing money supply ineffective when the interest rate is zero? In this paper I revisit this question using a microfounded intertemporal general equilibrium model and assuming rational expectations. I find support for both views under different assumptions about policy expectations. Expectations about future policy are crucial, because they determine long-term interest rates. Even if short-term interest rates are binding, increasing money supply by open market operations in certain assets can stimulate demand by changing expectations about future short-term interest rates, thus reducing long-term interest rates.

The paper has three key results. The first is that monetary and fiscal policy are irrelevant in a liquidity trap *if* expectations about future money supply are independent of past policy decisions, and certain restrictions on fiscal policy apply. I show this in a standard New Keynesian general equilibrium model widely used in the literature. The key message is not that monetary and fiscal policy are irrelevant. Rather, the point is that monetary and fiscal policy have their largest impact in a liquidity trap through expectations. This indicates that the old fashion IS-LM model is a blind alley. That model assumes that expectations are exogenous. In contrast, expectations are at the heart of the model in this paper.

I assume that expectations are rational. The government maximizes social welfare and I analyze two different equilibria. First I assume that the government is able to commit to future policy. This is what I call the commitment equilibrium. Then I assume that the government is unable to commit to any future policy apart from paying back the nominal value of its debt. This is what I call the Markov equilibrium. The commitment equilibrium in this paper is almost identical to the one analyzed by Eggertsson and Woodford (2003) in a similar model. They find that if the zero bound is binding due to temporary shocks, the optimal commitment is to commit to low future interest rates, modest inflation and output boom once the exogenous shocks subside. This reduces the real rate of return in a liquidity trap and increases demand. The main contribution of this paper is the analysis of the Markov equilibrium, i.e. the case when the government is unable to commit to future policy.

The second key result of the paper is that in a Markov equilibrium, deflation can be modelled as a credibility problem if the government has only one policy instrument, i.e. open market operations in government bonds. This theory of deflation, derived from the analysis of a Markov equilibrium, is in sharp contrast to conventional wisdom about deflation in Japan today (or, for that matter, US during the Great Depression). The conventional wisdom blames deflation on policy mistakes by the central bank or bad policy rules (see e.g. Friedman and Schwartz (1963), Krugman (1998), Buiter (2003), Bernanke (2000) and Benabib et al (2002)).[1] Deflation in this paper, however, is not attributed to an inept central bank or bad policy rules. It is a direct consequence of the central bank's *policy constraints* and inability to commit to the optimal policy when faced with large negative demand shocks. This result, however, does not to absolve the government of responsibility for deflation. Rather, it identifies the possible policy constraints that result in inefficient deflation in equilibrium (without resorting to an irrational policy maker). The result indicates two sources of deflation of equal importance. The first is the inability of the government to commit. The second is that open market operations in short-term government bonds is the only policy instrument. The result does not give the government a free pass on deflation because the government can clearly use more policy instruments to fight it (even if acquiring more credibility may be harder in practice). The central question of the paper, therefore, is how the government can use additional policy instruments to fight deflation even if it cannot commit to future policy.

The third key result of the paper is that in a Markov equilibrium the government can eliminate deflation by deficit spending. Deficit spending eliminates deflation for the following reason: If the government cuts taxes and increases nominal debt, and taxation is costly, inflation expectations increase (i.e. the private sector expects higher money supply in the future). Inflation expectation increase because higher nominal debt gives the government an incentive to inflate to reduce the real value of the debt. To eliminate deflation the government simply cuts taxes until the private sector expects inflation instead of deflation. At zero nominal interest rates higher inflation expectations reduce the real rate of return, and thereby raise aggregate demand and the price level. The central assumption behind this result is that there is some cost of taxation which makes this policy credible.[2]

Deficit spending has exactly the same effect as the government following Friedman's famous suggestion to "drop money from helicopters" to increase inflation. At zero nominal interest rates money and bonds are perfect substitutes. They are one and the same: A government issued piece of paper that carries no interest but has nominal value. It does not matter, therefore, if the government drops money from helicopters or issues government bonds. Friedman's proposal thus increases the price level through the same mechanism

[1] There is a large literture that discusses optimal monetray policy rules when the zero bound is binding. Contributions include Summers (1991), Fuhrer and Madigan (1997), Woodford and Rotemberg (1997), Wolman (1998), Reifschneider and Williams (1999) and references there in. Since monetary policy rules arguably become credible over time these contributions can be viewed as illustration of how to *avoid* a liquidity trap rather than a prescription of how to escape them which is the focus here.

[2] The Fiscal Theory of the Price Level (FTPL) popularized by Leeper (1992), Sims (1994) and Woodford (1994,1996) also stresses that fiscal policy can influence the price level. What separates this analysis from the FTPL (and the seminal contribution of Sargent and Wallace (1982)) is that in my setting fiscal policy only affects the price level because it changes the *inflation incentive* of the government. In contrast, according to the FTPL fiscal policy affects the price level because it is *assumed* that the monetary authority commits to a (possibly suboptimal) interest rate rule and fiscal policy is modelled as a (possibly suboptimal) exogenous path of real government surpluses. Under these assumptions innovations in real government surpluses can influence the price level, since the prices may have to move for the government budget constraint to be satisfied. In my setting, however, the government budget constraint is a *constraint* on the policy choices of the government.

as deficit spending This result, however, is not a vindication of the quantity theory of money. Dropping money from helicopters does *not* increase prices in a Markov equilibrium because it increases the *current* money supply. It creates inflation by increasing government debt which is defined as the sum of money and bonds. In a Markov equilibrium *it is government debt that determines the price level in a liquidity trap* because it determines expectations about *future* money supply.

The key mechanism that increases inflation expectation in this paper is government nominal debt. The government, however, can increase its debt in several ways. Cutting taxes or dropping money from helicopters are only two examples. The government can also increase its debt by printing money (or issuing nominal bonds) and buy real assets, such as stocks, or foreign exchange. In a Markov equilibrium these operations increase prices and output *because they change the inflation incentive of the government by increasing government debt* (money+bonds) (this is discussed in better detail in Eggertsson (2003b)). Hence, when the short-term nominal interest rate is zero, open market operations in real assets and/or foreign exchange increase prices through the same mechanism as deficit spending in a Markov equilibrium. This channel of monetary policy does not rely on the portfolio effect of buying real assets or foreign exchange. This paper thus compliments Meltzer's (1999) and McCallum (1999) arguments for foreign exchange interventions that rely on the portfolio channel.[3]

Deflationary pressures in this paper are due to temporary exogenous real shocks that shift aggregate demand.[4] The paper, therefore, does not address the origin of the deflationary shocks during the Great Depression in the US or in Japan today. These deflationary shocks are most likely due to a host of factors, including the stock market crash and banking problems. I take these deflationary pressures as given and ask: How can the government eliminate deflation by monetary and fiscal policy even if the zero bound is binding and it cannot commit to future policy? There is no doubt that there are several other policy challenges for a government that faces large negative shocks, and various structural problems, as in Japan.[5] Stabilizing the price level (and reducing real rates) by choosing the optimal mix of monetary and fiscal policy, however, is an obvious starting point and does not preclude other policy measures and/or structural reforms.

I study this model, and some extensions, in a companion paper with explicit reference to the current situation in Japan and some historical episodes (the Great Depression in particular). The contribution of the current paper is mostly methodological, so that even if I will on some occasions refer to the current experience in Japan (as a way of motivating the assumptions used) a more detailed policy study, with explicit reference to the rich institutional features of different countries at different times, is beyond the scope of this paper.

[3] The argument in the paper is also complimentary to Svensson's (2000) "foolproof"' way of escaping the liquidity trap by foreign exchange intervention. I show explicitly how foreign exchange rate intervention increase inflation expectation even if the government cannot commit to future policy and maximizes social welfare.

[4] In contrast to Benabib et al (2002) where deflation is due to selffulfilling deflationary spirals.

[5] See for example Caballero et al (2003) that argue that banking problems are at the heart of the Japaneese recession.

1 The Model

Here I outline a simple sticky prices general equilibrium model and define the set of feasible equilibrium allocations. This prepares the grounds for the next section, which considers whether "quantitative easing" – a policy currently in effect at the Bank of Japan – and/or deficit spending have any effect on the feasible set of equilibrium allocations.

1.1 The private sector

1.1.1 Households

I assume there is a representative household that maximizes expected utility over the infinite horizon:

$$E_t \sum_{T=t}^{\infty} \beta^T U_T = E_t \left\{ \sum_{T=t}^{\infty} \beta^T [u(C_T, \frac{M_T}{P_T}, \xi_T) + g(G_T, \xi_T) - \int_0^1 v(h_T(i), \xi_T) di] \right\} \tag{1}$$

where C_t is a Dixit-Stiglitz aggregate of consumption of each of a continuum of differentiated goods,

$$C_t \equiv [\int_0^1 c_t(i)^{\frac{\theta}{\theta-1}}]^{\frac{\theta-1}{\theta}}$$

with elasticity of substituting equal to $\theta > 1$, G_t is is a Dixit-Stiglitz aggregate of government consumption, ξ_t is a vector of exogenous shocks, M_t is end-of-period money balances, P_t is the Dixit-Stiglitz price index,

$$P_t \equiv [\int_0^1 p_t(i)^{1-\theta}]^{\frac{1}{1-\theta}}$$

and $h_t(i)$ is quantity supplied of labor of type i. $u(.)$ is concave and strictly increasing in C_t for any possible value of ξ. The utility of holding real money balances is increasing in $\frac{M_t}{P_t}$ for any possible value of ξ up to a satiation point at some finite level of real money balances as in Friedman (1969).[6] $g(.)$ is the utility of government consumption and is concave and strictly increasing in G_t for any possible value of ξ. $v(.)$ is the disutility of supplying labor of type i and is increasing and convex in $h_t(i)$ for any possible value of ξ. E_t denotes mathematical expectation conditional on information available in period t. ξ_t is a vector of r exogenous shocks. The vector of shocks ξ_t follows a stochastic process as described below.

A1 (i) $pr(\xi_{t+j}|\xi_t) = pr(\xi_{t+j}|\xi_t, \xi_{t-1}, ...)$ for $j \geq 1$ where $pr(.)$ is the conditional probability density function of ξ_{t+j}. (ii) All uncertainly is resolved before a finite date K that can be arbitrarily high.

Assumption A1 (i) is the Markov property. This assumption is not very restrictive since the vector ξ_t can be augmented by lagged values of a particular shock. Assumption A1 (ii) is added for tractability. Since K can be arbitrarily high it is not very restrictive.

[6] The idea is that real money balances enter the utility because they facilitate transactions. At some finite level of real money balances, e.g. when the representative household holds enough cash to pay for all consumption purchases in that period, holding more real money balances will not facilitate transaction any further and thereby add nothing to utility. This is at the "satiation" point of real money balances. We assume that there is no storage cost of holding money so increasing money holding can never reduce utility directly through $u(.)$. A satiation level in real money balances is also implied by several cash-in-advance models such as Lucas and Stokey (1987) or Woodford (1998).

For simplicity I assume complete financial markets and no limit on borrowing against future income. As a consequence, a household faces an intertemporal budget constraint of the form:

$$E_t \sum_{T=t}^{\infty} Q_{t,T}[P_T C_T + \frac{i_T - i^m}{1+i_T} M_T] \leq W_t + E_t \sum_{T=t}^{\infty} Q_{t,T}[\int_0^1 Z_T(i)di + \int_0^1 n_T(j)h_T(j)dj - P_T T_T] \quad (2)$$

looking forward from any period t. Here $Q_{t,T}$ is the stochastic discount factor that financial markets use to value random nominal income at date T in monetary units at date t; i_t is the riskless nominal interest rate on one-period obligations purchased in period t, i^m is the nominal interest rate paid on money balances held at the end of period t, W_t is the beginning of period nominal wealth at time t (note that its composition is determined at time $t-1$ so that it is equal to the sum of monetary holdings from period $t-1$ and the (possibly stochastic) return on non-monetary assets), $Z_t(i)$ is the time t nominal profit of firm i, $n_t(i)$ is the nominal wage rate for labor of type i, T_t is net real tax collections by the government. The problem of the household is: at every time t the household takes W_t and $\{Q_{t,T}, n_T(i), P_T, T_T, Z_T(i), \xi_T; T \geq t\}$ as exogenously given and maximizes (1) subject to (2) by choice of $\{M_T, h_T(i), C_T; T \geq t\}$.

1.1.2 Firms

The production function of the representative firm that produces good i is:

$$y_t(i) = f(h_t(i), \xi_t) \quad (3)$$

where f is an increasing concave function for any ξ and ξ is again the vector of shocks defined above (that may include productivity shocks). I abstract from capital dynamics. As in Rotemberg (1983), firms face a cost of price changes given by the function $d(\frac{p_t(i)}{p_{t-1}(i)})$.[7] Price variations have a welfare cost that is separate from the cost of expected inflation due to real money balances in utility. I show that the key results of the paper do not depend on this cost being particularly large, indeed they hold even if the cost of price changes is arbitrarily small. The Dixit-Stiglitz preferences of the household imply a demand function for the product of firm i given by

$$y_t(i) = Y_t(\frac{p_t(i)}{P_t})^{-\theta}$$

The firm maximizes

$$E_t \sum_{T=t}^{\infty} Q_{t,T} Z_T(i) \quad (4)$$

where

$$Q_{t,T} = \beta^{T-t} \frac{u_c(C_T, \frac{M_T}{P_T}, \xi_T)}{u_c(C_t, \frac{M_t}{P_t}, \xi_t)} \frac{P_t}{P_T} \quad (5)$$

I can write firms period profits as:

$$Z_t(i) = (1+s)Y_t P_t^{\theta} p_t(i)^{1-\theta} - n_t(i) f^{-1}(Y_t P_t^{\theta} p_t^{-\theta}) - P_t d(\frac{p_t(i)}{p_{t-1}(i)}) \quad (6)$$

[7] I assume that $d'(\Pi) > 0$ if $\Pi > 1$ and $d'(\Pi) < 0$ if $\Pi < 1$. Thus both inflation and deflation are costly. $d(1) = 0$ so that the optimal inflation rate is zero (consistent with the interepretation that this represent a cost of changing prices). Finally, $d'(1) = 0$ so that in the neighborhood of the zero inflation the cost of price changes is of second order.

where s is an exogenously given production subsidy that I introduce for computational convenience (for reasons described later sections).[8] The problem of the firm is: at every time t the firm takes $\{n_T(i), Q_{t,T}, P_T, Y_T, C_T, \frac{M_T}{P_T}, \xi_T; T \geq t\}$ as exogenously given and maximizes (4) by choice of $\{p_T(i); T \geq t\}$.

1.1.3 Private Sector Equilibrium Conditions: AS, IS and LM Equations

In this subsection I show the necessary conditions for equilibrium that stem from the maximization problems of the private sector. These conditions must hold for *any* government policy. The first order conditions of the household maximization imply an Euler equation of the form:

$$\frac{1}{1+i_t} = E_t\{\frac{\beta u_c(C_{t+1}, m_{t+1}\Pi_{t+1}^{-1}, \xi_{t+1})}{u_c(C_t, m_t\Pi_t^{-1}, \xi_t)}\Pi_{t+1}^{-1}\} \tag{7}$$

where $\Pi_t \equiv \frac{P_t}{P_{t-1}}$, $m_t \equiv \frac{M_t}{P_{t-1}}$ and i_t is the nominal interest rate on a one period riskless bond. As I discuss below the central banks policy instrument is $M_{t.}$ Since P_{t-1} is determined in the previous period I can define $m_t \equiv \frac{M_t}{P_{t-1}}$ as the instrument of monetary policy and this notation will be convenient in coming sections. The equation above is often referred to as the IS equation. Optimal money holding implies:

$$\frac{u_m(C_t, m_t\Pi_t^{-1}, \xi_t)\Pi_t^{-1}}{u_c(C_t, \xi_t)} = \frac{i_t - i^m}{1+i_t} \tag{8}$$

This equation defines money demand or what is often referred as the "LM" equation. Utility is weakly increasing in real money balances. Utility does not increase further at some finite level of real money balances. The left hand side of (8) is therefore weakly positive. Thus there is bound on the short-term nominal interest rate given by:

$$i_t \geq i^m \tag{9}$$

In most economic discussions it is assumed that the interest paid on the monetary base is zero so that (9) becomes $\dot{i}_t \geq 0$. The intuition for this bound is simple. There is no storage cost of holding money in the model and money can be held as an asset. It follows that i_t cannot be a negative number. No one would lend 100 dollars if he or she would get less than 100 dollars in return.

The optimal consumption plan of the representative household must also satisfy the transversality condition[9]

$$\lim_{T\to\infty} \beta^T E_t(Q_{t,T}\frac{W_T}{P_t}) = 0 \tag{10}$$

to ensure that the household exhausts its intertemporal budget constraint. I assume that workers are wage takers so that households optimal choice of labor supplied of type j satisfies

$$n_t(j) = \frac{P_t v_h(h_t(j); \xi_t)}{u_c(C_t, m_t\Pi_t^{-1}, \xi_t)} \tag{11}$$

[8] I introduce it so that I can calibrate an inflationary bias that is independent of the other structural parameters, and this allows me to define a steady state at the fully efficient equilibrium allocation. I abstract from any tax costs that the financing of this subsidy may create.

[9] For a detailed discussion of how this transversality condition is derived see Woodford (2003).

I restrict my attention to a symmetric equilibria where all firms charge the same price and produce the same level of output so that

$$p_t(i) = p_t(j) = P_t; \quad y_t(i) = y_t(j) = Y_t; \quad n_t(i) = n_t(j) = n_t; \quad h_t(i) = h_t(j) = h_t \quad \text{for} \quad \forall\ j, i \tag{12}$$

Given the wage demanded by households I can derive the aggregate supply function from the first order conditions of the representative firm, assuming competitive labor market so that each firm takes its wage as given. I obtain the equilibrium condition often referred to as the AS or the "New Keynesian" Phillips curve:

$$\begin{aligned} &\theta Y_t[\frac{\theta-1}{\theta}(1+s)u_c(C_t, m_t\Pi_t^{-1}, \xi_t) - \tilde{v}_y(Y_t, \xi_t)] + u_c(C_t, m_t\Pi_t^{-1}, \xi_t)\Pi_t d'(\Pi_t) \\ &- E_t\beta u_c(C_{t+1}, m_{t+1}\Pi_{t+1}^{-1}, \xi_{t+1})\Pi_{t+1}d'(\Pi_{t+1}) = 0 \end{aligned} \tag{13}$$

where for notational simplicity I have defined the function:

$$\tilde{v}(y_t(i), \xi_t) \equiv v(f^{-1}(y_t(i)), \xi_t) \tag{14}$$

1.2 The Government

There is an output cost of taxation (e.g. due to tax collection costs as in Barro (1979)) captured by the function $s(T_t)$.[10] For every dollar collected in taxes $s(T_t)$ units of output are waisted without contributing anything to utility. Government real spending is then given by:

$$F_t = G_t + s(T_t) \tag{15}$$

I could also define cost of taxation as one that would result from distortionary taxes on income or consumption. The specification used here, however, focuses the analysis on the channel of fiscal policy that I am interested in. This is because for a constant F_t the level of taxes has no effect on the private sector equilibrium conditions (see equations above) but only affect the equilibrium by reducing the utility of the households (because a higher tax costs mean lower government consumption G_t). This allows me to isolate the effect current tax cuts will have on expectation about future monetary and fiscal policy, abstracting away from any effect on relative prices that those tax cuts may have.[11] There is no doubt that tax policy can change relative prices that these effects may be important. Those effects, however, are quite separate from the main focus of this paper.[12]

I assume a representative household so that in a symmetric equilibrium, all nominal claims held are issued by the government. It follows that the government flow budget constraint is

$$B_t + M_t = W_t + P_t(F_t - T_t) \tag{16}$$

[10] The function s(T) is assumed to be differentiable with derivatives $s'(T) > 0$ and $s''(T) > 0$ for $T > 0$.

[11] This is the key reason that I can obtain Propostion 1 in the next section even if taxation is costly.

[12] There is work in progress by Eggertsson and Woodford that considers how taxes that change relative prices can be used to affect the equilibrium allocations. That work considers labor and consumption taxes.

where B_t is the end-of-period nominal value of bonds issued by the government. Finally, market clearing implies that aggregate demand satisfies:

$$Y_t = C_t + d(\Pi_t) + F_t \tag{17}$$

I now define the set of possible equilibria that are consistent with the private sector equilibrium conditions and the technological constraints on government policy.

Definition 1 *Private Sector Equilibrium (PSE) is a collection of stochastic processes*

$\{\Pi_t, Y_t, W_t, B_t, m_t, i_t, F_t, T_t, Q_t, Z_t, G_t, C_t, n_t, h_t, \xi_t\}$ *for* $t \geq t_0$ *that satisfy equations (2)-(17) for each* $t \geq t_0$*, given* w_{t_0-1} *and the exogenous stochastic process* $\{\xi_t\}$ *that satisfies A1 for* $t \geq t_0$*.*

Having defined feasible sets of equilibrium allocations, it is now meaningful to consider how government policies affect actual outcomes in the model.

2 Equilibrium with exogenous policy expectations

According to Keynes (1936) famous analysis monetary policy loses its power when the short term nominal interest rate is zero, which is what he referred to as the liquidity trap. Others argue, most notably Friedman and Schwarts (1963) and the monetarist, that monetary expansion increases aggregate demand even under such circumstances, and this is what lies behind the "quantitative easing" policy of the BOJ since 2001.

One of Keynes better known suggestions is to increase demand in a liquidity trap by government deficit spending. Recently many have doubted the importance of this channel, pointing to Japan's mountains of nominal debt, often on the grounds of Ricardian equivalence, i.e. the principle that any decrease in government savings should be offset by an increase in private savings (to pay for higher future taxes). Yet another group of economists argue that the Ricardian equivalence argument fails if the deficit spending is financed by money creation (see e.g. Buiter (2003) and Bernanake (2000,2003)).

Here I consider whether or not "quantitative easing" or deficit spending are separate policy tools in the explicit intertemporal general equilibrium model laid out in the last section. The key result is that "quantitative easing" or deficit spending has no effect on demand *if* expectations about future money supply remain unchanged – or alternatively – expectations about future interest rate policy remain unchanged. Furthermore, this result is unchanged if these two operations are used together, so that our analysis does not support the proposition that "money financed deficit spending" increases demand independently of the expectation channel.[13] This result is a direct extension of Eggertsson and Woodford (2003) irrelevance result, extended to include fiscal policy.

It is worth stating from the outset that my contention is not that deficit spending and/or quantitative easing are irrelevant in a liquidity trap. Rather, the point is that the main effect of these policies is best

[13] As I discuss below this does not contradict Bernanke's or Buiters claims.

illustrated by analyzing how they change expectations about future policy, in particular expectations about future money supply. As we shall see the exact effect of these policy measures depends on assumptions about how monetary policy and fiscal policy are conducted in the future when the zero bound is not binding. Our proposition thus indicates that *if* future policy is set without any regard to previous decisions (or commitments) there is no effect of either deficit spending or quantitative easing.

2.1 The irrelevance of monetary and fiscal policy when policy expectations are exogenous

Here I characterize policy that allows for the possibility that the government increases money supply by "quantitative easing" when the zero bound is binding and/or engages in deficit spending. The money supply is determined by a policy function:

$$M_t = M(s_t, \xi_t) I_t \tag{18}$$

where s_t is a vector that may include any of the endogenous variables *that are determined at time t* (note that as a consequence s_t cannot include W_t that is predetermined at time t). The multiplicative factor I_t satisfies the conditions

$$I_t = 1 \text{ if } i_t > 0 \quad \text{otherwise} \tag{19}$$

$$I_t = \psi(s_t, \xi_t) \geq 1. \tag{20}$$

The rule (18) is a fairly general specification of policy (since I assume that M_t is a function of all the endogenous variables). It could for example include simple Taylor type rules, monetary targeting, and any policy that does not depend on the past values of any of the endogenous variables.[14] Following Eggertsson and Woodford (2003) I define the multiplicative factor $I_t = \psi(s_t, \xi_t)$ when the zero bound is binding. Under this policy regime a policy of "quantitative easing" is represented by a value of the function ψ that is positive. Note that I assume that the functions M and ψ are only a function of the endogenous variables and the shocks at time t. This is a way of separating the direct effect of a quantitative easing from the effect of a policy that influences expectation about future money supply. I impose the restriction on the policy rule (18) that

$$M_t \geq M^*. \tag{21}$$

This restriction says the the nominal value of the monetary base can never be smaller than some finite number M^*. This number can be arbitrarily small, so I do not view this as a very restrictive (or unrealistic)

[14] The Taylor rule is a member of this family in the following sense. TheTaylor rule is

$$i_t = \phi_\pi \Pi_t + \phi_y Y_t$$

The money demand equation (8) defines the the interest rate as a function of the monetary base, inflation and output. This relation may then be used to infer the money supply rule that would result in an indentical equilibrium outcome as a Taylor rule and would be a member of the rules we consider above.

assumption since I am not modelling any technological innovation in the payment technology (think of M^* as being equal to one cent!). I assume, for simplicity, that the central bank does quantitative easing by buying government bonds, but the model can be extended to allow for the possibility of buying a range of other long or short term financial assets (see Eggertsson and Woodford (2003) who also write out the explicit budget constraints for the both the treasury and the central bank). Also, for simplicity, I assume that the government only issues one period riskless nominal bonds so that B_t in equation (16) refer to a one period riskless nominal debt (again Eggertsson and Woodford (2003) allow for long-term real and nominal government bonds). Fiscal policy is defined by a function for real government spending:

$$F_t = F \tag{22}$$

and a policy function for deficit spending

$$T_t = T(s_t, \xi_t) \tag{23}$$

I assume that real government spending F_t is constant at all times to focus on deficit spending which is defined by the function $T(.)$ that specifies the evolution of taxes. Debt is issued the end of period t is then defined by the consolidated government budget constraint (16) and the policy specifications (18)-(23). Finally I assume that fiscal policy is run so that the government is neither a debtor or a creditor asymtotically so that

$$\lim_{T \to \infty} E_t Q_{t,T} B_T = 0 \tag{24}$$

This is a fairly weak condition on the debt accumulation of the government policy stating that asymtotically it cannot accumulate real debt at a higher rate than the real rate of interest.[15] I can now obtain the following irrelevance result for monetary and fiscal policy

Proposition 1 *The Private Sector Equilibrium consistent with the monetary and policy (18)-(24) is independent of the specification of the functions $\psi(.)$ and $T(.)$.*

The proof of this proposition is fairly simple, and the formal details are provided in the appendix. The proof is obtained by showing that I can write all the equilibrium conditions in a way that does not involve the functions T or ψ. First I use market clearing to show that the intertemporal budget constraint of the household can be written without any reference to either function. This relies on the Ricardian properties of the model. Second I show that (10) is satisfied regardless of the specification of these functions using the two restrictions we imposed on policy given by (21) and (24). Finally I show, following the proof by Eggertsson and Woodford (2003), that I can write the remaining conditions without any reference to the function $\psi(.)$.

[15] One plausible sufficient condition that would guarantee that (24) must always hold is to assume that the private sector would never hold more government debt that correpondes to expected future discounted level of some maximum tax level – that would be a sum of the maximum seignorage revenues and some technology constraint on taxation.

2.2 Discussion

Proposition 1 says that a policy of quantitative easing and/or deficit spending has no effect on the set of feasible equilibrium allocations that are consistent with the policy regimes I specified above. It may seem that our result contradicts Keynes' view that deficit spending is an effective tool to escape the liquidity trap. It may also seem to contradict the monetarist view (see e.g. Friedman and Scwartz) that increasing money supply is effective in a liquidity trap. But this would only be true if one took a narrow view of these schools of tought as for example Hicks (1933) does in his ground breaking paper "Keynes and the Classics". Hicks develops a static version of the General Theory and contrast it to the monetarist view and assumes that expectation are exogenous constants. This is the IS-LM model. But what my analysis indicates is that it is the intertemporal elements of the liquidity trap that are crucial to understand the effects of different policy actions, namely their effect on expectations (to be fair to Hick he was very explicit that he was abstracting from expectation and recognized this was a major issues). Both Keynes (1936) and many monetarist (e.g. Friedman and Schwartz (1963)) discussed the importance of expectations in some detail in their work. Trying to evaluate the theories of "Keynes and the Classics" in a static model is therefore not going to resolve the debate.

My result is that deficit spending has no effect on demand if it does not change expectations about future policy. But as we shall see in later sections, when analyzing a Markov equilibrium, deficit spending can be very effective at hanging expectation. Similarly my result that quantitative easing is ineffective also relies on that expectations about future policy remain unaltered. As we shall also see when analyzing a Markov equilibrium (a point developed better in a companion paper), if the money printed is used to buy a variety of real asset, quantitative easing may be effective at changing policy expectations. It is only when the money printed is used to by short term government bonds that quantitative easing is ineffective in a Markov equilibrium. Thus by modelling expectations explicitly I believe my result neither contradicts Friedman and Schwartz' interpretation of the "Classics" , i.e. the Quantity Theory of Money, nor Keynes' General Theory. On the contrary, it may serve to integrate the two by explicitly modelling expectations.

Proposition 1 may also seem to contradict the claims of Bernanke (2003) and Buiter (2003). Both authors indicate that money financed tax cuts increase demand. Buiter, for example, writes that "base money-financed tax cuts or transfer payments – the mundane version of Friedman's helicopter drop of money – will always boost aggregate demand." But what Buiter implicitly has in mind, is that the tax cuts permanently increases the money supply. Thus a tax cut today, in his model, increases expectations about future money supply. Thus my proposition does not disprove Buiter's or Bernanke's claims since I assume that money supply in the future is set without any reference to past policy actions. The propositions, therefore, clarifies that tax cuts will only increase demand to the extent that they change beliefs about future money supply. The higher demand equilibrium that Buiter analyses, therefore, does not at all depend on the tax cut. It relies on higher expectations about future money supply. It is the expectation about the higher money supply that matters, not the tax cut itself. A similar principle applies to Auerbach and Obstfeld's (2003) result. They argue that open-market operations will increase aggregate demand. But their assumption is that open-market operations increase expectation about future money supply. It is that belief that matters and not the open market operation itself.

An obvious criticism of the irrelevance result for fiscal policy in Proposition 1 is that it relies on Ricardian equivalence. This aspect of the model is unlikely to hold exactly in actual economies. If taxes effect relative prices, for example if I consider income or consumption taxes, changes in taxation change demand in a way that is independent of expectations about future policy. Similarly, if some households have finite-life horizons and no bequest motive, current taxing decisions affect their wealth and thus aggregate demand in a way that is also independent of expectation about future policy.[16] The assumption of Ricardian equivalence is not applied here, however, to downplay the importance of these additional policy channels. Rather, it is made to focus the attention on how fiscal policy may change policy expectations. That exercise is most clearly defined by specifying taxes so that they can *only* affect the equilibrium through expectations about future policy. Furthermore, since our model indicates that expectations about future monetary policy have large effects in equilibrium, my conjecture is that this channel is of first order in a liquidity trap and thus a good place to start.

3 Equilibrium with Endogenous Policy Expectations

The main lesson from the last section is that expectation about future monetary and fiscal policy are crucial to understand policy options in a liquidity trap. Deficit spending and quantitative easing have no effect if they do not change expectations about future policy. But does deficit spending have no effect on expectations under reasonable assumptions about how these expectation are formed? Suppose, for example, that the government prints unlimited amounts of money and drops it from helicopters, distributes it by tax cuts, or prints money and buys unlimited amounts of some real asset. Would this not alter expectations about future money supply? To answer this question I need an explicit model of how the government sets policy in the future. I address this by assuming that the government sets monetary and fiscal policy optimally at all future dates. By optimal, I mean that the government maximizes social welfare that is given by the utility of the representative agent. I analyze equilibrium under two assumptions about policy formulation. Under the first assumption, which I call the commitment equilibrium, the government can commit to future policy so that it can influence the equilibrium outcome by choosing future policy actions (at all different states of the world). Rational expectation, then, require that these commitment are fulfilled in equilibrium. Under the second assumption, the government cannot commit to future policy. In this case the government maximizes social welfare under discretion in every period, disregarding any past policy actions, except insofar as they have affected the endogenous state of the economy at that date.

[16] This is a point developed by Ireland (2003) who show that in an overlapping generation model wealth transfers increase demand at zero nominal interest rate (this of course would also be true at positive interest rate).

Thus the government can only choose its current policy instruments, it cannot directly influence future governments actions. This is what I call the Markov equilibrium. In the Markov Equilibrium, following Lucas and Stockey (1983) and a large literature that has followed, I assume that the government is capable of issuing one period riskless nominal debt and commit to paying it back with certainly. In this sense, even under discretion, the government is capable of limited commitment. The contribution of this section is methodological. I define the appropriate equilibria, proof propositions about the relevant state variables, characterize equilibrium conditions and then show how the equilibria can be approximated. The next two section apply the methods developed here and proof a series of propositions and show numerical results. The impatient reader, that is only interested in the results of this exercise, can go directly to Section 4.

3.1 Recursive representation

To analyze the commitment and Markov equilibrium it is useful to rewrite the model in a recursive form so that I can identify the endogenous state variables at each date. When the government can only issue one period nominal debt I can write the total nominal claims of the government (which in equilibrium are equal to the total nominal wealth of the representative household) as:

$$W_{t+1} = (1 + i_t)B_t + (1 + i^m)M_t$$

Substituting this into (16), defining the variable $w_t \equiv \frac{W_{t+1}}{P_t}$ and using the definition of m_t I can write the government budget constraint as:

$$w_t = (1 + i_t)(w_{t-1}\Pi_t^{-1} + (F - T_t) - \frac{i_t - i^m}{1 + i_t} m_t \Pi_t^{-1}) \tag{25}$$

Note that I use the time subscript t on w_t (even if it denotes the real claims on the government at the beginning of time $t + 1$) to emphasize that this variable is determined at time t. I assume that $F_t = F$ so that real government spending is an exogenous constant at all times. In Eggertsson (2003a) I treat F_t as a choice variable. Instead of the restrictions (21) and (24) I imposed in the last section on government policies, I impose a borrowing limit on the government that rules out Ponzi schemes:

$$u_c(C_t, \xi_t)w_t \leq \bar{w} < \infty \tag{26}$$

where $\bar{w}$ is an arbitrarily high finite number. This condition can be justified by that the government can never borrow more than corresponds to expected discounted value of its maximum tax base (e.g. discounted future value of all future output). Since this constraint is never binding in equilibrium and $\bar{w}$ can be any arbitrarily high number for the results to be obtained, I do not model in detail the endogenous value of the debt limit. It is easy to show that this limit ensures that the transversality condition of the representative household is satisfied at all times.

The policy instruments of the treasury is taxation, T_t, that determines the end-of-period government debt which is equal to $B_t + M_t$. The central bank determines how the end-of-period debt is split between bonds and money by open market operations. Thus the central banks policy instrument is $M_{t.}$ Note that since P_{t-1} is determined in the previous period I may think of $m_t \equiv \frac{M_t}{P_{t-1}}$ as the instrument of monetary policy.

It is useful to note that I can reduce the number of equations that are necessary and sufficient for a private sector equilibrium substantially from those listed in Definition 1. First, note that the equations that determine $\{Q_t,\ Z_t, G_t, C_t, n_t, h_t\}$ are redundant, i.e. each of them is only useful to determine one particular variable but has no effect on the any of the other variables. Thus I can define necessary and sufficient condition for a private sector equilibrium without specifying the stochastic process for $\{Q_t, Z_t, G_t, C_t, n_t, h_t\}$ and do not need to consider equations (3), (5), (6), (11), (15) and I use (17) to substitute out for C_t in the remaining conditions. Furthermore condition (26) ensures that the transversality condition of the representative household is satisfied at all times so I do not need to include (10) in the list of necessary and sufficient conditions.

It is useful to define the expectation variable

$$f_t^e \equiv E_t u_c(Y_{t+1} - d(\Pi_{t+1}) - F, m_{t+1}\Pi_{t+1}^{-1}, \xi_{t+1})\Pi_{t+1}^{-1} \tag{27}$$

as the part of the nominal interest rates that is determined by the expectations of the private sector formed at time t. Here I have used (17) to substitute for consumption. The IS equation can then be written as

$$1 + i_t = \frac{u_c(Y_t - d(\Pi_t) - F, m_t\Pi_t^{-1}, \xi_t)}{\beta f_t^e} \tag{28}$$

Similarly it is useful to define the expectation variable

$$S_t^e \equiv E_t u_c(Y_{t+1} - d(\Pi_{t+1}) - F, m_{t+1}\Pi_{t+1}^{-1}, \xi_{t+1})\Pi_{t+1} d'(\Pi_{t+1}) \tag{29}$$

The AS equation can now be written as:

$$\theta Y_t[\frac{\theta-1}{\theta}(1+s)u_c(Y_t - d(\Pi_t) - F, m_t\Pi_t^{-1}, \xi_t) - \tilde{v}_y(Y_t, \xi_t)] + u_c(Y_t - d(\Pi_t) - F, m_t\Pi_t^{-1}, \xi_t)\Pi_t d'(\Pi_t) - \beta S_t^e = 0 \tag{30}$$

The next two propositions are useful to characterize equilibrium outcomes. Proposition 1 follows directly from our discussion above:

Proposition 2 *A necessary and sufficient condition for a PSE at each time $t \geq t_0$ is that the variables $(\Pi_t, Y_t, w_t, m_t, i_t, T_t)$ satisfy: (i) conditions (8), (9), (25),(26), (28), (30) given w_{t-1} and the expectations f_t^e and S_t^e. (ii) in each period $t \geq t_0$, expectations are rational so that f_t^e is given by (27) and S_t^e by (29).*

Proposition 3 *The possible PSE equilibrium defined by the necessary and sufficient conditions for any date $t \geq t_0$ onwards depends only on w_{t-1} and ξ_t.*

The second proposition follows from observing that w_{t-1} is the only endogenous variable that enters with a lag in the necessary conditions specified in (i) of Proposition 1 and using the assumption that ξ_t

is Markovian (i.e. using A1) so that the conditional probability distribution of ξ_t for $t > t_0$ only depends on ξ_{t_0}. It follows from this proposition (w_{t-1}, ξ_t) are the only state variables at time t that directly affects the PSE. I may economize on notation by introducing vector notation. I define vectors

$$\Lambda_t \equiv \begin{bmatrix} \Pi_t \\ Y_t \\ m_t \\ i_t \\ T_t \end{bmatrix}, \text{ and } e_t \equiv \begin{bmatrix} f_t^e \\ S_t^e \end{bmatrix}.$$

Since Proposition 3 indicates that w_t is the only relevant endogenous state variable I prefer not to include it in either vector but keep track of it separately. I can summarize conditions (8), (25), (28), (30) in Proposition 2 (arranging every element of each equation on the left hand size so that each equation is equal to zero) by the vector valued function $\Gamma : \mathbb{R}^{16+r} \rightarrow \mathbb{R}^4$ so that

$$\Gamma(e_t, \Lambda_t, w_t, w_{t-1}, \xi_t) = 0 \tag{31}$$

(where the first element of this vector is conditions (8), the second (25) and so on. Here r is the length of the vector of shocks ξ). I summarize rational expectation conditions (27) and (29) by $\Xi : \mathbb{R}^{16+2r} \rightarrow \mathbb{R}^2$ so that

$$E_t \Xi(e_t, \Lambda_t, \Lambda_{t+1}, \xi_t, \xi_{t+1}) = 0 \tag{32}$$

and the inequalities (9) and (26) by $\Upsilon : \mathbb{R}^{7+r} \rightarrow \mathbb{R}^2$ so that

$$\Upsilon(\Lambda_t, w_t, \xi_t) \geq 0 \tag{33}$$

Finally I can write the utility function as the function $U : \mathbb{R}^{6+r} \rightarrow \mathbb{R}$

$$U_t = U(\Lambda_t, \xi_t)$$

using (15) to solve for G_t as a function of F and T_t, along with (12) and (14) to solve for $h_t(i)$ as a function of Y_t.

3.2 The Commitment Equilibrium

Definition 2 *The optimal commitment solution at date $t \geq t_0$ is the Private Sector Equilibrium that maximizes the utility of the representative household given w_{t_0-1}* and ξ_{t_0}.

To derive the optimal commitment conditions I use the vector notation defined above and form the Lagrangian:

$$L_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^t [U(\Lambda_t, \xi_t) + \phi_t' \Gamma(e_t, \Lambda_t, w_t, w_{t-1}, \xi_t) + \psi_t' \Xi(e_t, \Lambda_t, \Lambda_{t+1}, \xi_t, \xi_{t+1}) + \gamma_t' \Upsilon(\Lambda_t, w_t, \xi_t)$$

where ϕ_t is a (4×1) vector, ψ_t is (2×1) and γ_t is (2×1). The first order conditions for $t \geq 1$ are (where each of the derivatives of L are equated to zero):

$$\begin{aligned}\frac{dL}{d\Lambda_t} &= \frac{dU(\Lambda_t,\xi_t)}{d\Lambda_t} + \phi_t' \frac{d\Gamma(e_t,\Lambda_t,w_t,w_{t-1},\xi_t)}{d\Lambda_t} + \psi_t' E_t \frac{\Xi(e_t,\Lambda_t,\Lambda_{t+1},\xi_t,\xi_{t+1})}{d\Lambda_t} \\ &\quad + \beta^{-1}\psi_{t-1}' \frac{d\Xi(e_{t-1},\Lambda_{t-1},\Lambda_t,\xi_{t-1},\xi_t)}{d\Lambda_t} + \gamma_t' \frac{d\Upsilon(\Lambda_t,w_t,\xi_t)}{d\Lambda_t}\end{aligned} \tag{34}$$

$$\frac{dL}{de_t} = \phi_t' \frac{d\Gamma(e_t,\Lambda_t,w_t,w_{t-1},\xi_t)}{de_t} + \psi_t' E_t \frac{\Xi(e_t,\Lambda_t,\Lambda_{t+1},\xi_t,\xi_{t+1})}{de_t} \tag{35}$$

$$\frac{dL}{dw_t} = \phi_t' \frac{\Gamma(e_t,\Lambda_t,w_t,w_{t-1},\xi_t)}{dw_t} + \beta E_t \phi_{t+1}' \frac{\Gamma(e_{t+1},\Lambda_{t+1},w_{t+1},w_t,\xi_{t+1})}{dw_t} + \gamma_t' \frac{d\Upsilon(\Lambda_t,w_t,\xi_t)}{dw_t} \tag{36}$$

The complementary slackness conditions are:

$$\gamma_t \geq 0, \quad \Upsilon(\Lambda_t,\xi_t) \geq, 0 \quad \gamma_t'\Upsilon(\Lambda_t,\xi_t) = 0 \tag{37}$$

Here $\frac{dL}{d\Lambda_t}$ is a (1×5) Jacobian. I use the notation

$$\frac{dL}{d\Lambda_t} \equiv [\frac{\partial L}{\partial \Pi_t}, \frac{\partial L}{\partial Y_t}, \frac{\partial L}{\partial m_t}, \frac{\partial L}{\partial i_t}, \frac{\partial L}{\partial T_t}]$$

so that (34) is a vector of 5 first order conditions, (35) and (37) each are vectors of two first order conditions, and (36) is a single first order condition. The explicit algebraic expressions for this total of 10 conditions is in the appendix. For $t = 0$ I obtain the same conditions as above if I set $\psi_{t-1} = 0$.

A noteworthy feature of the first order conditions is the history dependence of Λ_t. This history dependence is brought about by the assumption that the government can control expectations that are given by the expectations S_t^e and f_t^e. This is the central feature of optimal policy under commitment as we shall see when I illustrate numerical examples.

3.3 The Markov Solution under Discretion

Now I consider equilibrium in the case that policy is conducted under discretion so that the government cannot commit to future policy. This is what I refer to as a Markov Equilibrium (it is formally defined for example by Maskin and Tirole (2001)) and has been extensively applied in the monetary literature. The basic idea behind this equilibrium concept is to restrict attention to equilibria that only depends on variables that directly affect market conditions. Proposition 3 indicates that a Markov Equilibrium requires that the variables (Λ_t, w_t) and the expectations e_t only depend on (w_{t-1}, ξ_t), since these are the minimum set of state variables that affect the private sector equilibrium. Thus, in a Markov equilibrium, there must exist policy functions $\bar{\Pi}(.), \bar{Y}(.), \bar{m}(.), \bar{\imath}(.), \bar{F}(.), \bar{T}(.)$ that I denote by the vector valued function $\bar{\Lambda}(.)$, and a function $\bar{w}(.)$, such that each period:

$$\begin{array}{c} \Lambda_t \\ w_t \end{array} \equiv \begin{bmatrix} \Pi_t \\ Y_t \\ m_t \\ i_t \\ T_t \end{bmatrix} = \begin{bmatrix} \bar{\Pi}(w_{t-1},\xi_t) \\ \bar{Y}(w_{t-1},\xi_t) \\ \bar{m}(w_{t-1},\xi_t) \\ \bar{\imath}(w_{t-1},\xi_t) \\ \bar{T}(w_{t-1},\xi_t) \end{bmatrix} \equiv \begin{array}{c} \bar{\Lambda}(w_{t-1},\xi_t) \\ \bar{w}(w_{t-1},\xi_t) \end{array} \tag{38}$$
$$w_t \qquad \bar{w}(w_{t-1},\xi_t)$$

Note that the function $\bar{\Lambda}(.)$ and $\bar{w}(.)$ will also define a set of functions of (w_{t-1},ξ_t) for $(Q_t, Z_t, G_t, C_t, n_t, h_t)$ by the redundant equations from Definition 1. Using $\bar{\Lambda}(.)$ I may also use (27) and (29) to define a function $\bar{e}(.)$ so so that

$$e_t = \begin{bmatrix} f_t^e \\ S_t^e \end{bmatrix} = \begin{bmatrix} \bar{f}^e(w_t,\xi_t) \\ \bar{S}^e(w_t,\xi_t) \end{bmatrix} = \bar{e}(w_t,\xi_t) \tag{39}$$

Rational expectations imply that these function are correct in expectation, i.e. the function $\bar{e}$ satisfies

$$\begin{aligned} &\bar{e}(w_t,\xi_t) \\ = &\begin{bmatrix} E_t u_c(\bar{C}(w_t,\xi_{t+1}),\bar{m}(w_t,\xi_{t+1})\bar{\Pi}(w_t,\xi_{t+1})^{-1};\xi_{t+1})\bar{\Pi}(w_t,\xi_{t+1})^{-1} \\ E_t u_c(\bar{C}(w_t,\xi_{t+1}),\bar{m}(w_t,\xi_{t+1})\bar{\Pi}(w_t,\xi_{t+1})^{-1};\xi_{t+1})\bar{\Pi}(w_t,\xi_{t+1})d'(\bar{\Pi}(w_t,\xi_{t+1})) \end{bmatrix} \end{aligned} \tag{40}$$

I define a value function $J(w_{t-1},\xi_t)$ as the expected discounted value of the utility of the representative household, looking forward from period t, given the evolution of the endogenous variable from period t onwards that is determined by $\Lambda(.)$ and $\{\xi_t\}$. Thus I define:

$$J(w_{t-1},\xi_t) \equiv E_t \left\{ \sum_{T=t}^{\infty} \beta^T [U(\bar{\Lambda}(w_{T-1},\xi_T),\xi_T] \right\} \tag{41}$$

The timing of events in the game is as follows: At the beginning of each period t, w_{t-1} is a predetermined state variable. At the beginning of the period, the vector of exogenous disturbances ξ_t is realized and observed by the private sector and the government. The monetary and fiscal authorities choose policy for period t given the state and the private sector forms expectations e_t. Note that I assume that the private sector may condition its expectation at time t on w_t, i.e. it observes the policy actions of the government in that period so that Λ_t and e_t are jointly determined. This is important because w_t is the relevant endogenous state variable at date $t+1$. Thus the set of possible values (Λ_t, w_t) that can be achieved by the policy decisions of the government are those that satisfy the equations given in Propositions 2 given the values of w_{t-1}, ξ_t and the expectation function (39).

The optimizing problem of the government is as follows. Given w_{t-1} and ξ_t the government chooses the values for (Λ_t, w_t) (by its choice of the policy instruments m_t and T_t) to maximize the utility of the representative household subject to the constraints in Proposition 1 summarized by (31) and (33) and (39). Thus its problem can be written as:

$$\max_{m_t,w_t} [U(\Lambda_t,\xi_t) + \beta E_t J(w_t,\xi_{t+1})] \tag{42}$$

s.t. (31), (33) and (39).

I can now define a Markov Equilibrium.

Definition 2 *A Markov Equilibrium is a collection of functions $\bar{\Lambda}(.), J(.), \bar{e}(.)$, such that (i) given the function $J(w_{t-1}, \xi_t)$ and the vector function $\bar{e}(w_t, \xi_t)$ the solution to the policy maker's optimization problem (42) is given by $\Lambda_t = \bar{\Lambda}(w_{t-1}, \xi_t)$ for each possible state $(w_{t-1,}\xi_t)$ (ii) given the vector function $\bar{\Lambda}(w_{t-1}, \xi_t)$ then $e_t = \bar{e}(w_t, \xi_t)$ is formed under rational expectations (see equation (40)). (iii) given the vector function $\bar{\Lambda}(w_{t-1}, \xi_t)$ the function $J(w_{t-1}, \xi_t)$ satisfies (41).*

I will only look for a Markov equilibrium in which the functions $\bar{\Lambda}(.)$, $J(.)$, $\bar{e}(.)$ are continuous and have well defined derivatives. I do not provide a general proof of existence or non-existence of equilibria when these functions are non-differentiable.[17] The value function satisfies the Bellman equation:

$$J(w_{t-1}, \xi_t) = \max_{m_t, w_t} [U(\Lambda_t, \xi_t) + E_t \beta J(w_t, \xi_{t+1})] \tag{43}$$

s.t. (31), (33) and (39).

Using the same vector notation as in last section I obtain the necessary conditions for a Markov equilibrium by differentiating the Lagrangian.

$$L_t = U(\Lambda_t, \xi_t) + E_t \beta J(w_t, \xi_{t+1}) + \phi_t' \Gamma(e_t, \Lambda_t, w_t, w_{t-1}, \xi_t) + \psi_t'(e_t - \bar{e}(w_t, \xi_t)) + \gamma_t' \Upsilon(\Lambda_t, w_t, \xi_t)$$

The first order conditions for $t \geq 0$ are (where each derivatives of L are equated to zero):

$$\frac{dL}{d\Lambda_t} = \frac{dU(\Lambda_t, \xi_t)}{d\Lambda_t} + \phi_t' \frac{d\Gamma(e_t, \Lambda_t, w_t, w_{t-1}, \xi_t)}{d\Lambda_t} + \gamma_t' \frac{d\Upsilon(\Lambda_t, w_t, \xi_t)}{d\Lambda_t} \tag{44}$$

$$\frac{dL}{de_t} = \phi_t' \frac{d\Gamma(e_t, \Lambda_t, w_t, w_{t-1}, \xi_t)}{de_t} + \psi_t \tag{45}$$

$$\frac{dL}{dw_t} = \beta E_t J_w(w_t, \xi_{t+1}) + \phi_t' \frac{d\Gamma(e_t, \Lambda_t, w_t, w_{t-1}, \xi_t)}{dw_t} - \psi_t' \frac{d\bar{e}(w_t, \xi_t)}{dw_t} + \gamma_t' \frac{d\Upsilon(\Lambda_t, w_t, \xi_t)}{dw_t} \tag{46}$$

$$\gamma_t \geq 0, \quad \Upsilon(\Lambda_t, w_t, \xi_t) \geq, 0 \quad \gamma_t' \Upsilon(\Lambda_t, w_t, \xi_t) \tag{47}$$

The Markov equilibrium must also satisfy an envelope condition:

$$J_w(w_{t-1}, \xi_t) = \phi_t' \frac{d\Gamma(e_t, \Lambda_t, w_t, w_{t-1}, \xi_t)}{dw_{t-1}} \tag{48}$$

Explicit algebraic solution for these first order conditions are shown in the Appendix.

The central difference between the first order conditions in a Markov solution comes from the governments inability to control expectations directly. In the Markov equilibrium the government has only indirect control of expectation through the state variable w_t. As we shall see in numerical examples w_t will be very important in a Markov equilibrium because it enables the government to manage expectation in a way that closely resembles commitment.

[17] Whether such equilibria exist is an open questions.

3.4 Equilibrium in the absence of seigniorage revenues

It simplifies the discussion to assume that the equilibrium base money small, i.e. that m_t is a small number (see Woodford (2003), chapter 2, for a detailed treatment). This simplifies the algebra and my presentation of the results. I discuss in the footnote some reasons for why I conjecture that this abstraction has no significant effect.[18]

To analyze an equilibrium with a small monetary base I parameterize the utility function by the parameter $\bar{m}$ and assume that the preferences are of the form:

$$u(C_t, m_t\Pi_t^{-1}, \xi_t) = \tilde{u}(C_t, \xi_t) + \chi(\frac{m_t}{\bar{m}}\Pi_t^{-1}C_t^{-1}, \xi_t) \tag{49}$$

As the parameter $\bar{m}$ approaches zero the equilibrium value of m_t approaches zero as well. At the same time it is possible for the value of u_m to be a nontrivial positive number, so that money demand is well defined and the government's control over the short-term nominal interest rate is still well defined (see discussion in the proofs of Propositions 4 and 5 in the Appendix). I can define $\tilde{m}_t = \frac{m_t}{\bar{m}}$ as the policy instrument of the government, and this quantity can be positive even as $\bar{m}$ and m_t approach zero. Note that even as the real monetary base approaches the cashless limit the *growth rate* of the nominal stock of money associated with different equilibria is still well defined. I can then still discuss the implied path of money supply for different policy options. To see this note that

$$\frac{\tilde{m}_t}{\tilde{m}_{t-1}} = \frac{\frac{M_t}{P_{t-1}\bar{m}}}{\frac{M_{t-1}}{P_{t-2}\bar{m}}} = \frac{M_t}{M_{t-1}}\Pi_{t-1}^{-1} \tag{50}$$

which is independent of the size of $\bar{m}$. For a given equilibrium path of inflation and $\tilde{m}_t$ I can infer the growth rate of the nominal stock of money that is required to implement this equilibrium by the money demand equation. Since much of the discussion of the zero bound is phrased in terms of the implied path of money supply, I will also devote some space to discuss how money supply adjusts in different equilibria. By assuming $\bar{m} \to 0$ I only abstract from the effect this adjustment has on the marginal utility of consumption and seigniorage revenues, both of which would be trivial in a realistic calibration (see footnote 18).

[18] First, as shown by Woodford (2003), for a realistic calibration parameters, this abstraction has trivial effect on the AS and the IS equation under normal circustances. Furthermore, at zero nominal interest rate, increasing money balances further *does nothing* to facilitate transactions since consumer are already satiated in liquidity. This was one of the key insights of Eggertsson and Woodford (2003), which showed that at zero nominal interest rate increasing money supply has no effect if expectations about future money supply do not change. It is thus of even *less* interest to consider this additional channel for monetary policy at zero nominal interest rates than if the short-term nominal interest rate was positive. Second, assuming m_t is a very small number is likely to change the government budget constraint very little in a realistic calibration. By assuming the cashless limit I am assuming no seignorage revenues so that the term $\frac{i_t - i^m}{1+i_t}m_t\Pi_t^{-1}$ in the budget constraint has no effect on the equilibrium. Given the low level of seignorage revenues in industrialized countries I do not think this is a bad assumption. Furthermore, in the case the bound on the interest rate is binding, this term is zero, making it of even *less* interest when the zero bound is binding than under normal circumstances.

3.5 Approximation Method

3.5.1 Defining a Steady State

I define a steady state as a solution in the absence of shocks were each of the variables $(\Pi_t, Y_t, m_t, i_t, T_t, w_t, f_t^e, S_t^e) = (\Pi, Y, m, i, T, w, f^e, S^e)$ are constants. In general a steady-state of a Markov equilibrium is non-trivial to compute, as emphasized by Klein et al (2003). This is because each of the steady state variables depend on the mapping between the endogenous state (i.e. debt) and the unknown functions $J(.)$ and $\bar{e}(.)$, so that one needs to know the derivative of these functions with respect to the endogenous policy state variable to calculate the steady state. Klein et al suggest an approximation method by which one may approximate this steady state numerically by using perturbation methods. In this paper I take a different approach. Below I show that a steady state may be calculated under assumptions that are fairly common in the monetary literature, without any further assumptions about the unknown functions $J(.)$ and $e(.)$.

Following Woodford (2003) I define a steady state where monetary frictions are trivial so that (i) $\bar{m} \to 0$. Furthermore I assume, following Woodford (2003), that the model equilibrium is at the efficient steady state so that (ii) $1 + s = \frac{\theta-1}{\theta}$. Finally I suppose that in steady state (iii) $i_{ss}^m = 1/\beta - 1$. To summarize:

A2 Steady state assumptions. (i) $\bar{m} \to 0$, (ii) $1 + s = \frac{\theta-1}{\theta}$ (iii) $i_{ss}^m = 1/\beta - 1$

Proposition 4 *If $\xi = 0$ at all times and (i)-(iii) hold there is a commitment equilibrium steady state that is given by $i = 1/\beta - 1$, $w = S^e = \phi_1 = \phi_3 = \phi_4 = \psi_1 = \psi_2 = \gamma_1 = \gamma_2 = 0$, $\Pi = 1$, $\phi_2 = g_G(\bar{F} - s(\bar{F}))s'(\bar{F})$, $f^e = u_c(\bar{Y}), F = \bar{F} = G = T + s(T)$ and $Y = \bar{Y}$ where $\bar{Y}$ is the unique solution to the equation $u_c(Y - F) = v_y(Y)$*

Proposition 5 *If $\xi = 0$ at all times and (i)-(iii) hold there is a Markov equilibrium steady state that is given by $i = 1/\beta - 1$, $w = S^e = \phi_1 = \phi_3 = \phi_4 = \psi_1 = \psi_2 = \gamma_1 = \gamma_2 = 0$, $\Pi = 1$, $\phi_2 = g_G(\bar{F} - s(\bar{F}))s'(\bar{F})$, $f^e = u_c(\bar{Y}), F = \bar{F} = G = T + s(T)$ and $Y = \bar{Y}$ where $\bar{Y}$ is the unique solution to the equation $u_c(Y - F) = v_y(Y)$.*

To proof these two propositions I look at the algebraic expressions of the first order conditions of the government maximization problem. The proof is in in Appendix B. A noteworthy feature of the proof is that the mapping between the endogenous state and the functions $J(.)$ and $e(.)$ does not matter (i.e. the derivatives of these functions cancel out). The reason is that the Lagrangian multipliers associated with the expectation functions are zero in steady state and I may use the envelope condition to substitute for the derivative of the value function. The intuition for these Lagrange multipliers are zero in equilibrium is simple. At the steady state the distortions associated with monopolistic competition are zero (because of A2 (ii)). This implies that there is no gain of increasing output from steady state. In the steady the real debt is zero and according to assumption (i) seigniorage revenues are zero as well. This implies that even if there is cost of taxation in the steady state, increasing inflation does not reduce taxes. It follows that all the Lagrangian multipliers are zero in the steady state apart from the one on the government budget constraint. That multiplier, i.e. ϕ_2, is positive because there are steady state tax costs. Hence it would be beneficial (in terms of utility) to relax this constraint.

Discussion Proposition 4 and 5 give a convenient point to approximate around because the commitment and Markov solution are identical in this steady state. Below, I will then relax both assumption A2(ii) and A3(iii) and investigate the behavior of the model local to this steady state. A major convenience of using A2 is that I can proof all of the key propositions in the coming sections analytically but do not need to rely on numerical simulation except to graph up the solutions.

There is by now a rich literature studying the question whether there can be multiple Markov equilibria in monetary models that are similar in many respects to the one I have described here (see e.g. Albanesi et al (2003), Dedola (2002) and King and Wolman (2003)). I will not proof the global uniqueness of the steady state in Proposition 5 here but show that it is locally unique.[19] I conjecture, however, that the steady state is unique under A2.[20] But even if I would have written the model so that it had more than one steady state, the one studied here would still be the one of principal interest as discussed in the footnote.[21]

3.5.2 Approximate system and computational method

The conditions that characterize equilibrium, in both the Markov and the commitment solution, are given by the constraints of the model and the first order conditions of the governments problem. A linearization of this system is complicated by the Kuhn-Tucker inequalities (37) and (47). I look for a solution in

[19] By locally unique I mean "stable" so that if one perturbs the endogenous state, the system converges back to the steady state.

[20] The reason for this conjecture is that in this model, as opposed to Albanesi et al and Dedola work, I assume in A2 that there are no monetary frictions. The source of the multiple equilibria in those papers, however, is the payment technology they assume. The key difference between the present model and that of King and Wolman, on the other hand, is that they assume that some firms set prices at different points in time. I assume a representative firm, thus abstacting from the main channel they emphasize in generating multiple equilibria. Finally the present model is different from all the papers cited above in that I introduce nominal debt as a state variable. Even if the model I have illustrated above would be augmented to incorporate additional elements such as montary frictions and staggering prices, I conjecture that the steady state would remain unique due to the ability of the government to use nominal debt to change its future inflation incentive. That is, however, a topic for future reasearch and there is work in progress by Eggertsson and Swanson that studies this question.

[21] Even if I had written a model in which the equilibria proofed above is not the unique global equilibria the one I illustrate here would still be the one of principal interest. Furthermore a local analysis would still be useful. The reason is twofold. First, the equilibria analyzed is identical to the commitment equilibrium (in the absence of shocks) and is thus a natural candidate for investigation. But even more importantly the work of Albanesi et al (2002) indicates that if there are non-trivial monetary frictions there are in general only two equilibria.There are also two equilibria in King and Wolman's model. In Dedola's model there are three equilibria, but the same point applies. The first is a low inflation equilibria (analogues to the one in Proposition 1) and the other is a high inflation equilibria which they calibrate to be associated with double digit inflation. In the high inflation equilibria, however, the zero bound is very unlikely ever to be binding as a result of real shocks of the type I consider in this paper (since in this equilibria the nominal interest rate is very high as I will show in the next section). And it is the distortions created by the zero bound that are the central focus of this paper, and thus even if the model had a high inflation steady state, that equilibria would be of little interest in the context of the zero bound.

which the bound on government debt is never binding, and then verify that this bound is never binding in the equilibrium I calculate. Under this conjectured the solution to the inequalities (37) and (47) can be simplified into two cases:

$$\text{Case } 1 : \gamma_t^1 = 0 \;\; \text{if } i_t > i^m \tag{51}$$

$$\text{Case } 2 : i_t = i^m \;\; \text{otherwise} \tag{52}$$

Thus in both Case 1 and 2 I have equalities that characterizing equilibrium. In the case of commitment, for example, these equations are (31),(32) and (34)-(36) and either (51) when $i_t > i^m$ or (52) otherwise. Under the condition A1(i) and A1(ii) but $i^m < \frac{1}{\beta} - 1$ then $i_t > i^m$ and Case 1 applies in the absence of shocks. In the knife edge case when $i^m = \frac{1}{\beta} - 1$,however, the equations that solve the two cases (in the absence of shocks) are identical since then both $\gamma_{1t} = 0$ and $i_t = i^m$. Thus both Case 1 and Case 2 have the same steady state in the knife edge case $i_t = i^m$. If I linearize around this steady state (which I show exists in Proposition 3 and 4) I obtain a solution that is accurate up to a residual $(||\xi||^2)$ for both Case 1 and Case 2. As a result I have one set of linear equations when the bound is binding, and another set of equations when it is not. The challenge, then, is to find a solution method that, for a given stochastic process for $\{\xi_t\}$, finds in which states of the world the interest rate bound is binding and the equilibrium has to satisfy the linear equations of Case 1, and in which states of the world it is not binding and the equilibrium has to satisfy the linear equations in Case 2. Since each of these solution are accurate to a residual $(||\xi||^2)$ the solutions can be made arbitrarily accurate by reducing the amplitude of the shocks. Eggertsson and Woodford (2003) describe a recursive solution method for a simple Markov process which results in the zero bound being temporarily binding. Note that I may also consider solutions when i^m is below the steady state nominal interest rate. A linear approximation of the equations around the steady state in Proposition 4 and 5 is still valid if the opportunity cost of holding money, i.e. $\bar{\delta} \equiv (i - i^m)/(1 + i)$, is small enough. Specifically, the result will be exact up to a residual of order $(||\xi, \bar{\delta}||^2)$. In the numerical example below I suppose that $i^m = 0$ (see Eggertsson and Woodford (2003) for further discussion about the accuracy of this approach when the zero bound is binding and Woodford (2003) for a more detailed treatment of approximation methods).

A non-trival complication of approximating the Markov equilibrium is that I do not know the unknown expectation functions $\bar{e}(.)$. I illustrate a simple way of matching coefficients to approximate this function in the proof of Propositions 9.

4 The Deflation Bias

In the last section I showed how an equilibrium with endogenous policy expectation can be defined and characterized and how one may approximate this equilibrium. I now apply these methods to show that deflation can be modeled as a credibility problem. It should be noted right from the start that the point of this section is not to absolve the government any responsibility of deflation. Rather, the point is to identify the policy constraints that result in inefficient deflation in equilibrium. The policy constraint introduced in this section, apart from inability to commit to future policy, is that I assume that government spending and taxes are constant. Money supply, by open market operations in short-term government bonds, is the only policy instrument of the government. This is equivalent to assuming that the nominal interest rate is the only policy instrument. An appealing interpretation of the results is that they apply if the central bank does not coordinate its action with the treasury, i.e. if the central bank has "narrow objective". This interpretation is discussed further in a companion paper Eggertsson (2003a) (where this model is interpreted in the context of Japan today and some historical episodes are discussed).

I assume in this section that the only instrument of the government is money supply through open market operations in short-term government bonds. This is equivalent to assuming that the governments only instrument is the nominal interest rate.

A3 *Limited instruments: Open market operations in government bonds, i.e.* $\tilde{m}_t$, *is the only policy instrument. Fiscal policy is constant so that* $w_t = 0$ *and* $T_t = F$ *at all times*

To gain insights into the solution in an approximate equilibrium, it is useful to consider the linear approximation of the private sector equilibrium constraints. The AS equation is:

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} \tag{53}$$

where $\kappa \equiv \theta \frac{(\sigma^{-1}+\lambda_2)}{d''}$. Here π_t is the inflation rate, $x_t \equiv y_t - y_t^n$ is the output gap, y_t is the percentage deviation of output from its steady state and y_t^n is the percentage deviation of the natural rate of output from its steady state. The natural rate of output is the output that would be produced if prices where completely flexible, i.e. it is the output that solves the equation[22]

$$v_y(Y_t^n, \xi_t) = \frac{\theta - 1}{\theta}(1+s)u_c(Y_t^n, \xi_t). \tag{54}$$

The "Phillips curve" in (53) has become close to standard in the literature. In a linear approximation of the equilibrium the IS equation is given by:

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1} - r_t^n) \tag{55}$$

where $\sigma \equiv \frac{u_{cc}Y}{u_c}$ and r_t^n is the natural rate of interest, i.e. the real interest rate that is consistent with the natural rate of output and is only a function of the exogenous shocks. The exact form of r_t^n is shown in the Appendix. It has been shown by Woodford (2003) that the natural rate of interest in this class of models, summarizes all the disturbances of the linearized private sector equilibrium conditions (although note that other shocks may change the government objectives, e.g. through the utility of government consumption,

[22] Note that this definition of the natural rate of output is different from the efficient level of output which is obtained if $(1+s) = \frac{\theta}{\theta-1}$ and prices are flexible. Also note that I allow for both s and i^m to be different A1 so that the AS and the IS equation is accurate to the order $o(||\xi, \bar{\delta}, 1+s-\frac{\theta}{\theta-1}||^2)$.

and that I abstract from stochastic variations in markups). I first show that if the natural rate of interest is positive at all times, and A2 and A3 hold, the commitment and the Markov solution are identical and the zero bound is never binding. To be precise, the assumption on the natural rate of interest is:

A4 *$r_t^n \in [i^m, S]$ at all times where S is a finite number greater than i^m.*

Assuming this restriction on the natural rate of interest I proof the following proposition.

Proposition 6 ***The equivalence of the Markov and the commitment equilibrium when only one policy instrument.*** *If A2, A3 and A4 and $0 \leq i^m \leq 1/\beta - 1$, at least locally to steady state and for S close enough to i^m, there is a unique bounded Markov and commitment solution given by $i_t = r_t^n \geq i^m$ and $\pi_t = x_t = 0$. The equilibrium is accurate up to an error that is only of order $o(||\xi, \bar{\delta}||^2)$*

Proof: Appendix

I proof this proposition by taking a linear approximation of the nonlinear first order conditions of the government shown in (34)-(37) and (44)-(80) and show that both of them imply an equilibrium with zero inflation and zero output gap. I only proof this locally, a global characterization is beyond the scope of this paper. Note that I allow for $i^m \leq 1/\beta - 1$ so I may consider the case when $i^m = 0$. The intuition for this result is straight forward and can be appreciated by considering the linear approximation of the IS and AS conditions in addition to a second order expansion of the representative household utility (which is the objective of the government). When fiscal policy is held constant, the utility of the representative household, to the second order, is equal to (the derivation of this is contained in the Computational Appendix[23]):

$$U_t = -[\pi_t^2 + \frac{\kappa}{\theta}(x_t - x^*)^2] + o(||\xi, \bar{\delta}, 1 + s - \frac{\theta}{\theta - 1}||^3) + t.i.p. \tag{56}$$

where $x^* = (\omega + \sigma^{-1})^{-1}(1 - \frac{\theta-1}{\theta}(1 + s))$ and $t.i.p$ is terms independent of policy. Here I have expanded this equation around the steady state in Proposition 4 and 5 and allowed for stochastic variations in ξ and also assumed that s and i^m may be deviate from the steady state I expand around (hence the error is of order $o(||\xi, \bar{\delta}, 1 + s - \frac{\theta}{\theta-1}||^3)$). Note that I assume A2 in Proposition 6 so that $(1 + s) = \frac{\theta}{\theta-1}$ an thus $x^* = 0$. One can then observe by the IS and the AS equation that the government can completely stabilize the loss function at zero inflation and zero output gap in an equilibrium where $i_t = r_t^n$ at all times. Since this policy maximizes the objective of the government at all times, there is no incentive for the government to deviate. It should then be fairly obvious that the government ability to commit has no effect on the equilibrium outcome, which is the intuition behind the proof of Proposition 6.

One should be careful to note that Proposition 6 only applied to the case when $x^* = 0$ as assumed in A2. When $x^* > 0$ the commitment and Markov solution are different because of the classic inflation bias, stemming from monopoly powers of the firms, as first shown by Kydland and Prescott (1977). I will now

[23] Available upon request.

show that even when $x^* = 0$ the commitment and Markov solution may also differ because of shocks that make the zero bound binding and the result is temporarily excessive deflation in the Markov equilibrium. This new dynamic inconsistency problem is what I call the deflation bias. In the next subsection I relax the assumption that $x^* = 0$, so that there may also be a permanent inflation bias in this model, and illustrate the connection between the inflation and the deflation bias.

The deflation bias can be shown by making some simple assumptions about the shocks that affect the natural rate of interest (recall that all the shocks that change the private sector equilibrium constraints can be captured by the natural rate of interest). Here I assume that the natural rate of interest becomes unexpectedly lower than i^m (e.g. negative) in period 0 and then reverses back to a positive steady state in every subsequent period with a some probability. Once it reverts back to steady state it stays there forever. It simplifies some of the proofs of the propositions that follow to assume that there is some finite date K after which there is no further uncertainty as in A1. This is not a very restrictive assumptions since I assume that K may be arbitrarily high. To be more precise I assume:

A5 $r_t^n = r_L^n < i^m$ *at* $t = 0$ *and* $r_t^n = r_{ss}^n = \frac{1}{\beta} - 1$ *at all* $0 < t < K$ *with probability* α *if* $r_{t-1}^n = r_L^n$ *and probability 1 if* $r_{t-1}^n = r_{ss}^n$ *at all* $t > 0$. *There is an arbitrarily large number* K *so that* $r_t^n = r_{ss}^n$ *with probability 1 for all* $t \geq K$

It should be fairly obvious that the commitment and Markov solutions derived in Proposition 6 are not feasible if I assume A5, because the solution in Proposition 6 requires that $i_t = r_t^n$ at all times. If the natural rate of interest is temporarily below i^m, as in A5, this would imply a nominal interest rate below the bound i^m for that equilibrium to be achieved. How does the solution change when the natural rate of interest is below i^m (for example negative)? Consider first the commitment solution. The commitment solution is characterized by the nonlinear equations (44)-(80) suitably adjusted by A3 so that fiscal policy is held constant. The key insight of these first order conditions is that the optimal policy is history depend so that the optimal choice of inflation, output and interest rate depends on the past values of the endogenous variables.

To gain insights into how this history dependence mattes I consider the following numerical example. Suppose that in period 0 the natural rate of interest becomes unexpectedly negative so that $r_L^n = -2\%$ and then reverts back to steady state of $r_{ss}^n = 0.02\%$ with 10% probability in each period (taken to be a quarter here). The calibration parameters I use are the same as in Eggertsson and Woodford (2003) (see details in the Appendix). Figure 1 shows (solid lines) the evolution of inflation, the output gap and the interest rate in the commitment equilibrium using the approximation method described in Section 3.5.2. The first line in each panel shows the evolution of inflation in the event the natural rate of interest returns back to the steady state in period 1, the second if it returns back in period 2 and so on.[24] The optimal commitment involves committing to a higher price level in the future. This commitment implies inflation once the zero

[24] The numerical solution reported here is exactly the same as the one shown by Eggertsson and Woodford (2003) in a model that is similar but has Calvo prices (instead of the quadratic adjustment costs I assume here). Their solution also differs in

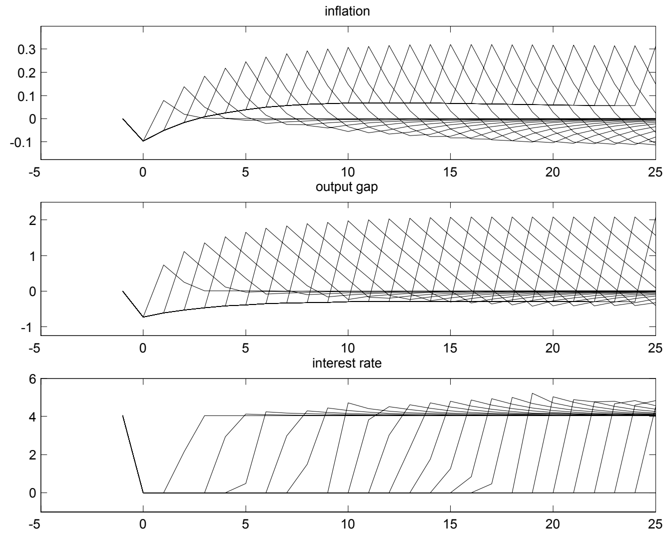

Figure 1: Inflation, the output gap, and the short-term nominal interest rate under optimal policy committment when the goverment can only use open market operations as its policy instrument. Each line represent the response of inflation, the output gap or the nominal interest rate when the natural rate of interest returns to its steady-state value in that period.

bound stops being binding, a temporary boom and a commitment to keeping the nominal interest rate low for a substantial period after the natural rate becomes positive again. This creates inflationary expectation when $r_L^n < 0$ and lowers expected long real rates which increases demand. The logic of this result is very simple and can be seen by considering the IS equation (55). Even if the nominal interest rate cannot be reduced below the 0 in period t, the real rate of return (i.e. $i_t - E_t\pi_{t+1}$) is what is important for aggregate demand and it can still be lowered by increasing inflation expectations. This is captured by the second element of the right hand side of equation (55). Furthermore, a commitment to a temporary boom, i.e. an increase in $E_t x_{t+1}$, will also stimulate demand by the permanent income hypothesis. This is represented by the first term on the right hand side of equation (55). Another way of viewing the result can also be illustrated by forwarding the IS equation to yield

$$x_t = -\sigma \sum_{T=t}^{\infty} (i_t - E_t\pi_{t+1} - r_t^n) + x_\infty \tag{57}$$

where x^∞ is a constant equal to the long run output gap. Note that aggregate demand depends on expectation of future interest rates. The optimal commitment involves keeping the nominal interest rate at zero for a substantial time, so that even though the government cannot increase demand by lowering the nominal interest rate at date t, it can increase demand by committing to keeping the nominal interest rate low in the future.

that they compute the optimal policy in a linear quadratic framework. As our numerical solution illustrates, however, the results for the commitment equilibrium are identical.

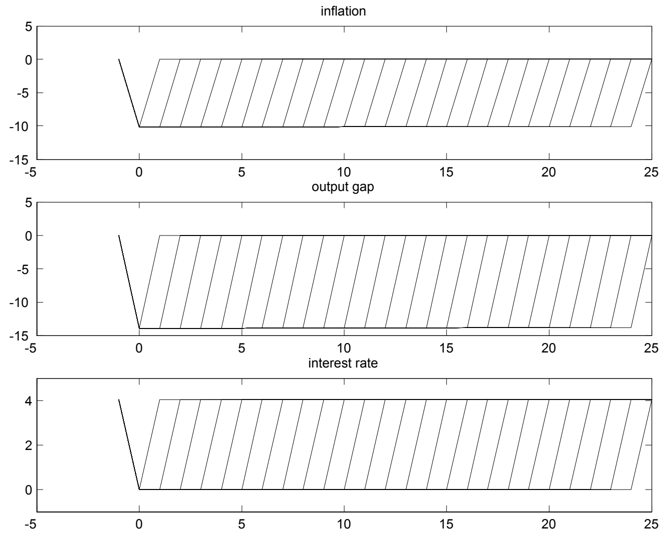

Figure 2: Inflation, the output gap, and the short-term nominal interest rate in a Markov equilibrium under discretion when the goverment can only use open market operations as its policy instrument. Each line represent the response of inflation, the output gap or the nominal interest rate when the natural rate of interest returns to its steady-state value in that period.

But is this commitment "credible"? The optimal commitment crucially depends on manipulating expectations, and it is worth considering to what extent this policy commitment is credible, i.e. if the government ever has an incentive to deviate from the optimal plan. One objection that Bank of Japan officials have commonly raised against calls for an inflation target, for example, is that setting an inflation target would not be "credible" since they cannot lower the nominal interest rate to manifest their intentions. I consider now the Markov solution that is characterized by the non-linear equations (44)-(80). The key feature of these equations is that the history dependence of the endogenous variables is only present through the state variable, w_t, i.e. the real debt. In this subsection, i.e. according to A3, I assume that $w_t = 0$ and $T_t = F$. It follows from Proposition 2 that in this case the Markov equilibrium conditions involve no history dependence. The result of this lack of history dependence is striking. Figure 2 shows the Markov Equilibrium. In contrast to the optimal commitment the Markov equilibrium mandates zero inflation and zero output gap as soon as the natural rate of interest is positive again. Thus the government cannot commit to a higher future price level as the optimal commitment implies. The result of the government inability to commit, as the figure makes clear, is excessive deflation and output gap in periods when the natural rate of interest is negative. This is the deflation bias of discretionary policy.

Proposition 7 ***The deflation bias***. *If A2, A3 and A4 then, at least local to steady state, the Markov equilibrium for $t \geq \tau$ is given by $\pi_t = x_t = 0$ and the result is excessive deflation and output gap for $t < \tau$ relative to a policy that implies $\pi_\tau > 0$ and $x_\tau > 0$ and $i_t = 0$ when $t \leq \tau$. This equilibrium, calculated by*

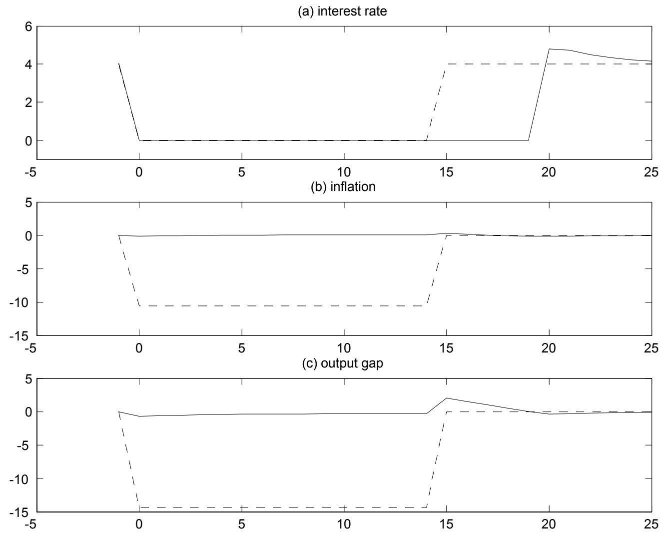

Figure 3: Response of the nominal interest rate, inflation and the output gap to a shocks that lasts for 15 quarters.

the solution method in discussed Section 3.5.2, is accurate to the order $o(||\xi, \bar{\delta}||^2)$

Proof: See Appendix

What is the logic behind the deflation bias? The logic can be clarified by considering our numerical simulation for one particular realization of the stochastic process of the natural rate of interest. Figure 3 shows the commitment and the Markov solution under A2 when the natural rate of interest returns back to steady state in quarter 15. The commitment solution involves committing to keeping the nominal interest rate low for a substantial period of time after the natural rate becomes positive again. This results in a temporary boom and modest inflation once the natural rate of interest becomes positive at time $\tau = 15$ (i.e. $x^C_{\tau=15}, \pi^C_{\tau=15} > 0$). If the government is discretionary, however, this type of commitment is not credible. In period 15, once the natural rate becomes positive again, the government raises the nominal interest rate to steady state, thus achieving zero inflation and zero output gap from period 15 onward. The result of this policy, however, is excessive deflation in period 0 to 14. This is the deflationary bias of discretionary policy. The intuition for this can be appreciated by observing the objectives of the government when $x^* = 0$. At time 15 once the natural rate of interest has become positive again, the optimal policy from that time onward is to set the nominal interest rate at the steady state and this policy will result in zero output gap and zero inflation *at that time onwards* — thus the Markov policy maximizes the objectives (56) from period 15 onwards. Thus the government has an incentive to renege on the optimal commitment since the optimal commitment results in a temporary boom and inflation in period 15 and thus implies higher utility losses in period 15 onwards relative to the Markov solution. In rational expectation, however, the private sector understands this incentive of the government, and if it is unable to commit, the result is excessive deflation and output gap in period 0 to 14 when the zero bound is binding. Note that Proposition 7 is proofed analytically without any reference to the cost of changing prices. Thus it remains true even if the cost of changing prices is made arbitrarily small.[25]

The problem of commitment when the zero bound is binding was first recognized by Krugman (1998). He assumed that the government follows a monetary policy targeting rule so that $M_t = M^*$ at all times. He then showed that at zero nominal interest rate, if expectation about future money supply are fixed by M^*, increasing money supply at time t has no effect if the government is expected to revert back to M^*. Thus if the government follows this rule of behavior, there is what Krugman calls "the inverse of the usual credibility problem" namely the need to "commit to being irresponsible" in the future by increasing expectations about future money supply. The key to effective policy, according to Krugman (and verified by our example above), is to commit to higher money supply in the future, i.e. to "commit to being irresponsible". My result here, illustrates, that the credibility problem is not isolated to a government that for some reason is expected to follow a monetary targeting rule. Even if the government maximizes social welfare, I obtain essentially the same commitment problem simply if the government cannot c*ommit to not reoptimizing in the future* and has only one policy instrument. This may be of potential practical importance because it implies that it may be hard for the government to change expectation about its future behavior, because I have shown that a deflationary solution is consistent with a *rational behavior* of the government. Indeed if the government is maximizing its objectives in any point in time the Markov equilibrium is the locally unique equilibrium. In contrast, Krugman's government, is committed to some monetary targeting policy rule, that is inconsistent with a rational government. It may, therefore, seem that it should be easy to change policy expectations, that the only problem is to find the optimal policy and then simply implement it. But as I have shown here, the solution is much harder than this, since if the government has limited credibility the commitment to the optimal policy is infeasible if the government has only one policy instrument.

[25]It is easiest to see this for a special case of A5. If $\alpha = 1$ the natural rate of interest is positive with probability 1 in period 1. Then Proposition 6 indicates that the solution in period 1 onwards is given by $\pi_t = x_t = 0$ for $t \geq 1$. The IS indicates that in period 0 the output gap is $x_0 = \sigma r_t^n$. Note that the output gap in period 0 is independent of the cost of changing prices since neither r_t^n nor σ are a function of the cost of price changes. This is because the output gap only depends on the difference between the current interest rate and the natural rate of interest and expectations about future inflation and output gap, and the latter are zero in period 1 onwards. The AS equation, however, indicates that the deflation in period 0 is going to depend on the cost of changing prices, i.e. $\pi_0 = \kappa x_0$. The lower the cost of changing prices the higher is $\kappa = \frac{\theta}{d''}(\sigma^{-1} + \omega)$ which indicates that there will be more deflation, the lower the cost of price changes (since x_0 is given by the IS equation which does not depend on d"). The intuition for this is that the lower the cost of price changes, the more prices need to adjust for the equation $x_0 = \sigma r_t^n$ to be satisfied. Thus the deflation bias is worse – in terms of actual fall in the price level – the lower the cost of changing prices. This basic intuition will also carry through to the stochastic case.

4.1 Extension: The inflation bias vs the deflation bias

An obvious questions that arises in the context of the deflation bias illustrated in the last section is how the result changes if the economy is subject to the classic inflation bias first illustrated by Kydland and Prescott (1977). What these authors showed was that if there are distortions (represented here as monopoly distortions) in the economy a government inability to commit results in chronic inflation in equilibrium. This is what has been referred to as the inflation bias of discretionary policy. In my model this incentive can be represented by $x^* > 0$ in objective (56), e.g. when $s = 0$ I have $x^* = (\omega + \sigma^{-1})^{-1}\frac{1}{\theta}$. In this case there is an average inflation bias and it is easy to show (see Proposition 8 below) that if the zero bound is never binding (e.g under A3) inflation is given by:

$$\pi_t = \bar{\pi} = \frac{1-\beta}{1-\beta+\theta\kappa}x^* \tag{58}$$

This implies the equilibrium nominal interest rate is given by

$$i_t = r_t^n + \bar{\pi}$$

Thus the zero bound is never binding if $r_t^n + \bar{\pi} \geq i^m$. If the natural rate of interest is low enough, however, I have exactly the same dynamic inconsistency problem as before, i.e. the inability of the government to commit to a higher inflation rate the $\bar{\pi}$ results in excessive deflation. To summarize:

Proposition 8 ***The inflation bias vs the deflation bias.*** *If A2(i), A2(ii), A3, A5 and* $0 \leq s < \frac{1}{\theta-1}$ *then* $\pi_t = \frac{\kappa}{1-\beta}\bar{x} = \bar{\pi}$ *for* $t \geq \tau$ *and there is excessive deflation and output gap in period* $t < \tau$ *if* $r_L^n < i^m - \bar{\pi}$ *relative to a policy that implies* $\pi_\tau > \bar{\pi}$ *and* $x_\tau > \bar{x}$ *and* $i_t = 0$ *when* $t < \tau$. *Here* $\bar{\pi}$ *is a solution to the equation* $\bar{\pi} = \frac{1-\beta}{1-\beta+\theta\kappa}x^* \geq 0$. *This equilibrium, calculated by the solution method discussed Section 3.5.2, is accurate to the order* $o(||\xi, \bar{\delta}, 1+s-\frac{\theta}{\theta-1}||^2)$

Proof: See Appendix

Figure 4 shows the evolution of inflation and the output gap for different values of x^*. Note that by equation (58) a different value of x^* translates into different inflation targets for the government in a Markov Equilibrium . The figure shows values of x^* that corresponds to 1%, 2% and 4% inflation targets respectively (I may vary this number by assuming different values for s in the expression for x^*). I use assumption A5 here but assume that the natural rate of interest takes on a value -4% in the low state and reverts back to steady state with 10 percent probability in each period. Note that it is only in the case that the inflation bias corresponds to $\bar{\pi} = 4\%$ that no deflation bias results. If $\bar{\pi} < -r_L^n = 4\%$ the result is excessive deflation. The picture also illustrates, and this is the lesson of Proposition 8, that the deflation bias is a problem, even in an economy with an average inflation bias, as long as the negative shock is large enough. The higher the average inflation bias, however, the larger the shock required for the deflation bias to be problematic. What is a realistic value for an inflation bias in an industrial economy? If I use the calibration values used for the figures above (see Computational Appendix) the implied inflation

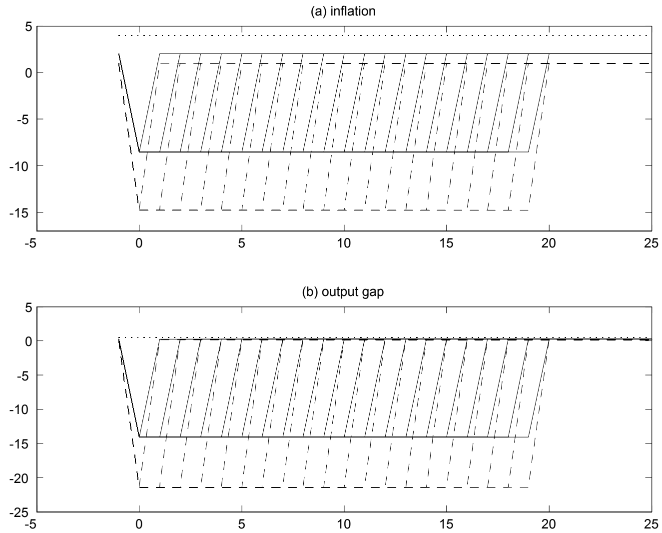

Figure 4: Inflation and the output gap under different assumption about steady state inflation bias when the natural rate of interest is temporarily -4 percent. The dotted lines correpond to a 4 percent steady state inflation bias, the solid line 2 percent and the dashed line 1 percent.

bias is 0.75 percent inflation per year. If the model is applied to Japan, this is indeed quite consistent with average inflation rates during the 80's and early 90's (before deflationary pressures emerged). Thus for these calibration values assumed here, the inflation bias is relatively low and a deflationary bias is a considerable concern. For the case of Japan, I think it is fairly realistic to assume a low inflation bias. Throughout the 80's an early 90's, for example, there was virtually no unemployment, and it thus did not seem that the government had a considerable incentive to inflate, consistent with that x^* was relatively close to zero. The assumption that $x^* = 0$ made in A3, therefore, does not seem grossly at odds with the evidence for Japan, and as argued by Rogoff (2003) the great disinflation in the world indicates that the inflation bias may be small (and shrinking) throughout the rest of the world. Since it will simplify my exposition considerably to assume that there is no inflationary bias, I will assume A2 in the remainder of this paper.[26] But it should be clear from the discussion above, that the results could be modified to include the case with an inflation bias in equilibrium. All that is needed would be larger shocks to obtain the same results.

Two aspects of a liquidity trap render the deflation bias a particularly acute problem, and possibly a more serious one for policy makers than the inflation bias analyzed by Kydland and Prescott (1977) and Barro and Gordon (1983). First, if the central bank announces a higher inflation target in a liquidity trap it involves no direct policy action - since the short-term nominal interest rate is at zero it cannot lower them any further. The central bank has therefore no means to manifest its desire for inflation. Thus

[26] Rogoff, Woodford, Dedola, Albanesi, King

announcing an inflation target in a liquidity trap may be less credible than under normal circumstances when the central bank can take direct actions to show its commitment. Second, unfavorable shocks create the deflation bias. If these shocks are infrequent (which is presumably the case given the few examples of a binding zero bound in economic history) it is hard for the central bank to acquire any reputation for dealing with them. To make matters worse, optimal policy in a liquidity trap involves committing to inflation. In an era of price stability the optimal policy under commitment is fundamentally different from what has been observed in the past.

5 Committing the Being Irresponsible

In the last section I showed that deflation can be modelled as a credibility problem if the government is unable to commit to future policy, and its only policy instrument is open market operations in short-term government bonds. I applied the approximation methods discussed in Section 3 to the nonlinear equilibrium conditions described in that section under the restriction that the government can not use deficit spending. In this section I show how the result changes if the government can use fiscal policy as an additional policy instrument to fight deflation. I first focus the attention on the case when the government can use deficit spending to increase demand. I then discuss how this solution can also be used to interpret the effect of open market operations in a large spectrum of private assets, such as foreign exchange or real assets. When the government coordinates fiscal and monetary policy I show that it can commit to future inflation and low nominal interest rate simply by cutting taxes and issuing nominal debt. Thus fiscal policy gives the government the ability to come closer to the optimal policy, even if it cannot make explicit commitment about future monetary policy.

The assumption about monetary and fiscal policy is summarized below.

A6 *Coordinated fiscal and monetary policy instruments: Open market operations in government bonds, i.e.* $\tilde{m}_t$*, and deficit spending,* $B_t - T_t$*, are the instruments of policy.*

Using this assumption about policy I proof the following proposition.

Proposition 9 ***Committing to being irresponsible.*** *If A2, A5 and A6 there is a solution at date* $t \geq \tau$ *for each of the endogenous variables given by* $\Lambda_t = \Lambda^1 w_{t-1}$*, and* $w_t = w^1 w_{t-1}$ *where* Λ^1 *and* w^1 *are constants. For a given value of* w^1 *there is a unique solution of this form for* Λ^1*. The coefficient* w^1 *must satisfy (i)* $|w^1| < \frac{1}{\beta}$ *and (ii) equation (151) in the Appendix. The solution for inflation is given by* $\pi_t = \pi^1 w_{t-1}$ *where* $\pi^1 > 0$ *so that the government can use deficit spending to increase inflation expectations, thereby curbing deflation and the output gap in period* $t < \tau$*. The equilibrium, calculated by the solution method in discussed Section 3.5.2, is accurate to the order* $o(||\xi, \bar{\delta}||^2)$

Proof: See Appendix

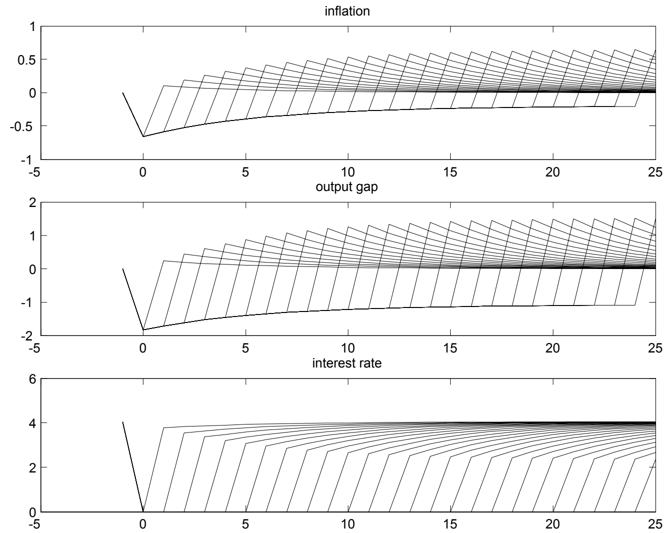

Figure 5: Inflation and output gap in a Markov equilibrium under discretion, when the government can use both monetary and fiscal policy to respond to a negative natural rate of interest.

I proof this proposition in the Appendix. This solution shows that nominal debt effectively commits the government to inflation even if it is discretionary. It is instructive to write out the algebraic expression for the inflation coefficient in the solution. I show in the Appendix that at $t \geq \tau$ the solution for inflation is

$$\pi_t = \pi^1 w_{t-1} \text{ where } \pi^1 = \frac{s' g_G}{d'' u_c}\beta^{-1} + \phi_4^1 \tag{59}$$

The government can reduce the real value of its debt (and future interest payments) by either increasing taxes or increasing inflation. Since both inflation and taxes are costly it chooses a combination of the two. The presence of debt creates inflation through two channels in our model: (1) If the government has outstanding *nominal* debt it has incentives to create inflation to reduce the real value of the debt. This incentive is captured by the term $\frac{s' g_G}{d'' u_c}\beta^{-1}$ in equation (59). The marginal cost of taxation is given by $s' g_G$ and the marginal cost of inflation is given by $d'' u_c$ (2) If the government issues debt at time t it has incentives to lower the real rate of return its pays on the debt it rolls over to time $t+1$. This incentive also translates into higher inflation.[27] This incentive is reflected in the value of the coefficient ϕ_4^1. This coefficient is the marginal value of relaxing the aggregate supply constraint, which can be beneficial because of the reduction in the real interest rate paid on debt, i.e. the government has an incentive to create a boom to lower the service on the debt it rolls over to the next period.

[27] Obstfeld (1991,1997) analyses a flexible price model with real debt (as opposed to nominal as in our model) but seignorage revenues due to money creation. He obtains a solution similar to mine (i.e. debt in his model creates inflation but is paid down over time). Calvo and Guidotti (1990) similarly illustrate a flexible price model that has a similar solution. The influence of debt on inflation these authors illustrate is closely related to the first channel we discuss above. The second channel we show, however, is not present in these papers since they assume flexible prices.

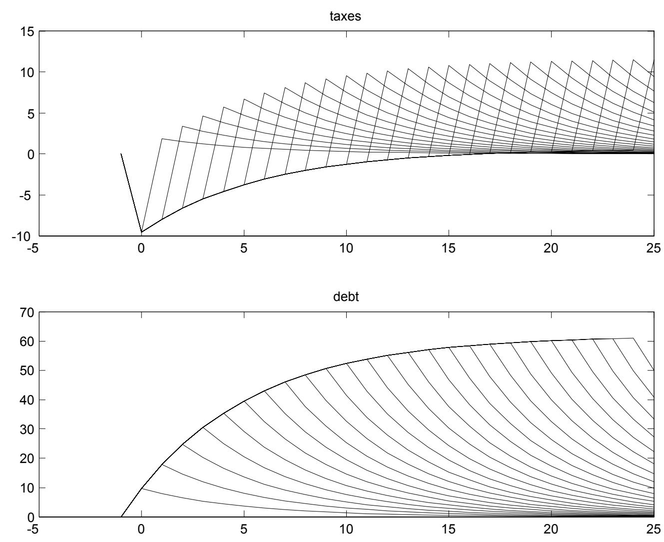

Figure 6: Taxes and debt in a Markov equilibrium under discretion, when the government can use both monetary and fiscal policy to respond to a negative natural rate of interest.

As I showed in previous sections committing to future inflation and output boom is exactly what is mandated by the optimal policy under commitment. Using the same numerical example as in previous section figures 5 and 6 show that it is indeed optimal for a discretionary government to issue debt when the zero bound is binding and thus commit to future inflation and an output boom once the zero bound stops being binding. By cutting taxes and issuing debt in a liquidity trap the government curbs deflation and increases output almost to the same level as obtained under commitment. Note that by issuing nominal debt the government commits to keeping the nominal interest rate below the steady state after the natural rate of interest returns back to steady state. This can be seen by the last panel in figure 5. That figure illustrates that the nominal interest rate rises only slowly once the zero bounds stops being binding, so that the government credibly commits itself to keep interest rates low for several periods.

The discretion solution is still slightly inferior to the commitment solution since the evolution of each variable does not still exactly match that of the optimal commitment equilibrium. Table 1 shows welfare under three different policy regimes by evaluating each of the solution by the utility function of the representative household. The first policy regime, the commitment equilibrium, shows utility (normalized to 100) associated with the full commitment equilibrium when the government can use both fiscal and monetary policy. The second policy regime shows the Markov equilibrium if the government can use both fiscal and monetary policy. The third policy regime is the one analyzed in the last section, i.e. the Markov equilibrium when the governments only policy instrument is open market operations in short term bonds. This table shows that the ability of the government to use debt as a commitment device almost eliminates all the cost of its inability to commit. The interpretation of these utility losses is that the representative household would be ready to pay only 0.02 percent in terms consumption equivalence units to move from the Markov equilibrium under fiscal and monetary discretion to the full commitment case. In contrast the monetary discretion case involves considerable utility losses.[28]

Table 1

Policy	Utility in consumption equivalence units
Optimal commitment	100
Markov equilibrium under fiscal and monetary discretion	99.98
Markov equilibrium under monetary discretion	86.52

Proposition 9, figures 5 and 6, and Table 1 summarize the central results of this paper. Even if the government cannot make commitments about future policy, it can increase the price level at zero nominal interest rates. A simple way to increase inflation expectations is to coordinate fiscal and monetary policy and run budget deficits. This increases output and prices. The channel is simple. Budget deficits generate nominal debt. Nominal debt in turn makes a higher inflation target in the future credible because the real value of the debt increases if the government reneges on the target. Higher inflation expectations lower the real rate of interest and thus stimulate aggregate demand. This policy involves direct actions by the government as opposed to only announcements about future policies (a criticism that is sometiems raised about the commintment solution as a policy suggestion is that it does not require any actions, only announcements about future intentions. See e.g. Friedman (2003)). The government can announce an inflation target and then increase budget deficits until the target is reached.

Discussion To contrast the commitment and the discretion solutions it is useful to consider the evolution of the price level. Figure 7 shows the evolution of the price level under the three policy regimes reported in Table 1. The optimal commitment solution is to commit to a higher future price level as can be seen in panel a of figure 7, although the extent to which the price level increases is small. If the government is unable to commit, however, this commitment is not credible as I discussed in the last section. Panel b shows that under monetary discretion (i.e. if the government is unable to use fiscal policy as a commitment device) there is a dramatic decline in the price level. If the natural rate of interest becomes positive in period 15, for example (this is the case I showed in figure 3), the price level declines by about 35 percent. Panel c of figure 2 shows that this large decline in the price level is largely avoided if the government uses fiscal policy to "commit to being irresponsible". As that figure illustrates this commitment involves increasing the price level once the natural rate becomes positive. In the example when the natural rate of interest reverts to steady state in period 15 the long run price level falls by less than 1 percent, compared to 35 percent decline under monetary discretion.

[28] Here I normalize the utility flow by transforming the utility stream (which is the future discounted stream of utility from private and public consumption – in all states of the world – minus the flow from the disutility of working) into a stream of a constant private consumption endowment.

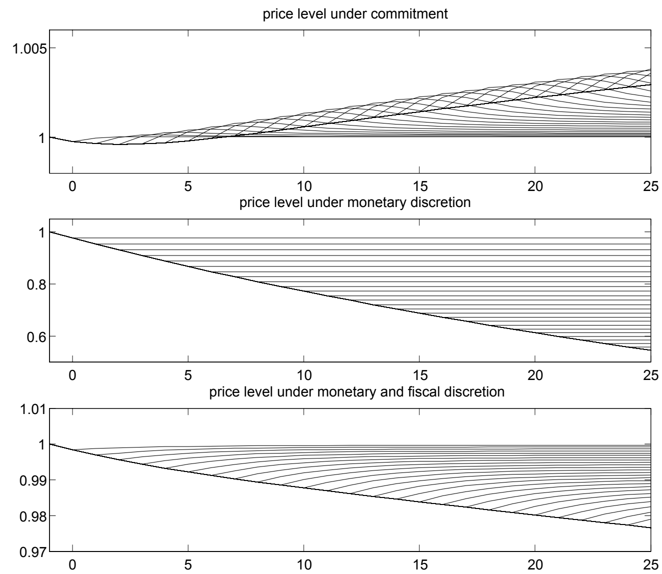

Figure 7: The evolution of the price level under different assumptions about policy.

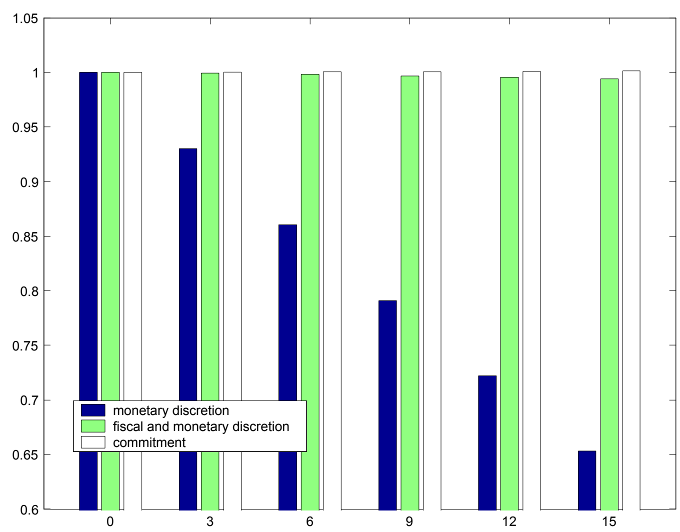

Figure 8: Long run nominal stock of money under different contingencies for the natural rate of interest.

It may be instructive to consider the evolution of money supply in these different equilibria. I have assumed here that monetary frictions are very small, but as I discussed in section (3.4) money demand is still well defined so that it remains meaningful to discuss the growth rate of money supply (even if the real monetary base relative to output is very small). The money demand equation defines the evolution for real money balances in the equilibrium, i.e the variables $\tilde{m}_t$ which is normalized by the transaction technology parameter, and the growth rate of money supply can then be inferred from equation (50). I can then calculate the money supply for each of the different equilibria. Figure 8 shows the long run nominal stock of money under each of the three different policies discussed above. In the figure I show the future level of the nominal stock of money in the cases when the natural rate of interest reverts back to steady state in periods 3, 6, 9,12 and 15. The figure shows the level of money supply under each policy once the price level has converged back to its new steady state (so I do not need to make any assumptions here about the interest rate elasticity or output elasticity of money demand. It is not very instructive to consider the evolution of the nominal stock of interest in the transition periods because the large movement in the nominal interest rate cause large swings in the nominal stock of money). I assume that the value of the money supply is 1 before the shocks hit the economy. The figure illustrates that the optimal commitment involves committing to a nominal money supply in the future that is only marginally larger than before the shock hits. In contrast the monetary discretion involves a considerable contraction in the nominal monetary base. This is because under monetary discretion the government will accommodate any given deflation by contracting the monetary base as soon as the natural rate of interest becomes positive again, in order to prevent inflation. In the case of monetary and fiscal discretion it is deficit spending that allows the government to credibly commit to a higher money supply, thus suppressing deflationary expectations. As a result the government can achieve an equilibrium outcome that is very close to the commitment solution, as illustrated in our welfare evaluation above and shown in figures 5 and 6.

An obvious question that arises if this model is interpreted to Japan today is why the high level of outstanding debt observed there today has not increased inflation expectations as the model would predict. Currently the gross national debt is over 130 percent of GDP. Eggertsson (2003a) gives two possible explanations of this. The first is that even if gross debt is high in Japan large part of this debt is actually held by public institution (and thus not creating any incentive for inflation). A better measure of the actual inflation incentive is net government debt. Net debt government debt as a fraction of GDP is not as high in Japan, about 70 percent, and only slightly above the G7 average. The other explanation given in Eggertsson (2003a) is that the Bank of Japan (BOJ) does not internalize the inflation incentive of outstanding government debt, i.e. that it has an objective that is more narrow than social welfare (that paper proofs that if the objective of BOJ is given by $\pi_t^2 + \lambda x_t^2$ deficit spending has no effect because it does not change the future incentive of the bank to inflate). Eggertsson (2003b) argues that this indicates that there may be benefits of monetary and fiscal coordination, as suggested by Bernanke (2003), and verified by our welfare evaluation.

5.1 Extension: Dropping money from helicopters and open market operations in foreign exchange as a commitment device

The model can be extended to analyze non-standard open market operations such the purchasing of foreign exchange and stocks, or even more exotically, dropping money from helicopters. Here I discuss how these extensions enrich the results (details are in Eggertsson (2003b)).

Friedman suggests that the government can always control the price level by increasing money supply, even in a liquidity trap. According to Friedman's famous *reductio ad absurdum* argument, if the government wants to increase the price level it can simply "drop money from helicopters." Eventually this should increase the price level – liquidity trap or not. Bernanke (2000) revisits this proposal and suggests that Japanese government should make "money-financed transfers to domestic households–the real-life equivalent of that hoary thought experiment, the "helicopter drop" of newly printed money." This analysis supports Friedman's and Bernanke's suggestions. The analysis suggests, however, that it is not the increase in the money supply, as such, that has this effect, rather it is the increase in government liabilities (money +bonds). Since money and bonds are equivalent in a liquidity trap dropping money from helicopters is in fact exactly equivalent to issuing nominal bonds. If the treasury and the central bank coordinate policy the effects of a helicopter drop of money will thus be exactly the same as the effects of deficit spending that I have discussed in this paper. Thus the model of this paper can be interpreted as establishing a "fiscal theory" of dropping money from helicopters.

The model can also be extended to consider the effects of the government buying assets that have some real value (or alternatively foreign exchange which buys real foreign goods and services). It is often suggested that by purchasing foreign assets the Central Bank can depreciate the exchange rate, and stimulate spending that way. As pointed out by Eggertsson and Woodford (2003), however, the interest rate parity implies that such a policy should have no effect upon the exchange rate, except in so far as it changes expectations about future policy. Will such operations have any effect on future policy? Since open market operation in real assets, or foreign exchange, would lead to a corresponding increase in public debt. This gives the government an incentives to create inflation through exactly the same channel as I have explored in this paper. An advantage of buying assets with real value is that it does not imply that the net fiscal position of the government is made any worse, only changes its inflation incentive. The exact way in which this is achieved is explored in better detail in Eggertsson (2003b).

6 Conclusion

The analysis here offers some insights into the state of the Japanese economy today. The irrelevance proposition presented in the paper implies that "quantitative easing" beyond the size of monetary base required to keep the call rate at zero may not have any effect. I suspect this may help explain the apparent ineffectiveness of "quantitative easing" as practiced by the Bank of Japan (BOJ) since May 2001.

The irrelevance proposition can also shed light on the failure of deficit spending to do more to stimulate the Japanese economy and eliminate expectations of deflation. In the model the principle of "Ricardian equivalence" holds. This aspect of the model is plainly an idealization, and one would expect Ricardian equivalence not to hold exactly in a more realistic model. Nonetheless, the essential prediction of such a model does not seem too far off in Japan: decreases in government saving (increases in government borrowing) have resulted in offsetting increases in private savings, so that little stimulus to aggregate demand has been achieved.

The analysis of the Markov equilibrium indicates that if the BOJ has limited credibility open market operations in short-term government bonds may not be enough to fight deflation. Coordinating interest rate policy with other policy instruments, however, can be effective. Government deficits are an example of an additional policy instrument that if used with interest rate policy can be used to stimulate aggregate demand, and head off deflation. If future monetary policy takes account of the distortions resulting from high taxes, then a higher nominal public debt results in more inflationary monetary policy. This does not, however, seem to match the expectations of many observers regarding the likely behavior of the BOJ in the future. In particular, the public may not anticipate that the BOJ will care much about reducing the burden of public debt when determining future monetary policy, given some statements by BOJ officials. This implies that cooperation between the Ministry of Finance and the BOJ may be useful to increase inflation expectations, as suggested by Bernanke (2003), and discussed in more detail in a companion paper Eggertsson (2003b); or at the very least that it would be useful for the BOJ to make fiscal developments an explicit concern as a way of credibly committing to a higher future price level. Given the relative high level of current public debt in Japan, I do not feel that much additional deficit spending is needed. What is needed is not more debt, but a clarification of the principles that will guide future monetary policy, of a kind that would imply that the existing fiscal incentive for inflation will actually be reflected in future monetary policy.

One policy that has apparently already had some success in Japan – although this is a subject of debate – is the policy of repeated interventions in the foreign exchange market by the Ministry of Finance during the first half of 2003. This policy may have prevented further appreciation of the yen, in which case it has played an important role in the recent improvement in real growth in Japan. As discussed in this paper (and Eggertsson and Woodford 2003)) foreign-exchange market intervention is an example (like deficit spending) of a policy that should have negligible effects on either asset prices or the economy more generally, except insofar as it is useful to change expectations about of future monetary policy. Yet intervention against an exchange rate that indicates market expectations inconsistent with the policy commitments of the government may succeed in changing those expectations. One may think of exchange rate intervention as working through the same channel as increasing government debt, since the Japanese governments purchase of foreign exchange will lead to an equal increase in its own liabilities. Thus the results here indicates that foreign exchange intervention are helpful to signal expansionary future monetary policy because they change the inflation incentives of the government.

7 Appendix A: Explicit first order conditions for commitment and discretion

7.1 Nonlinear Commitment FOC

The commitment Lagrangian is

$$
\begin{aligned}
L_t = E_{t_0} \sum_{t=t_0}^{\infty} \beta^t [& u(Y_t - d(\Pi_t) - F, m_t \Pi_t^{-1}, \xi_t)) + g(F - s(T_t), \xi_t) - \tilde{v}(Y_t) \\
& + \phi_{1t} \Big(\frac{u_m(Y_t - d(\Pi_t) - F, m_t \Pi_t^{-1}, \xi_t) \Pi_t^{-1}}{u_c(Y_t - d(\Pi_t) - F, m_t \Pi_t^{-1}, \xi_t)} - \frac{i_t - i^m}{1 + i_t} \Big) \\
& + \phi_{2t}(w_t - (1 + i_t)\Pi_t^{-1} w_{t-1} - (1 + i_t)F + (1 + i_t)T_t + (i_t - i^m) m_t \Pi_t^{-1}) + \phi_{3t}\Big(\beta f_t^e - \frac{u_c(Y_t - d(\Pi_t) - F, m_t \Pi_t^{-1}, \xi_t)}{1 + i_t}\Big) \\
& + \phi_{4t}\Big(\theta Y_t \Big[\frac{\theta - 1}{\theta}(1 + s) u_c(Y_t - d(\Pi_t) - F, m_t \Pi_t^{-1}, \xi_t) - \tilde{v}_y(Y_t, \xi_t)\Big] + u_c(Y_t - d(\Pi_t) - F, m_t \Pi_t^{-1}, \xi_t) \Pi_t d'(\Pi_t) - \beta S_t^e\Big) \\
& + \psi_{1t}(f_t^e - u_c(Y_{t+1} - d(\Pi_{t+1}) - F, m_{t+1} \Pi_{t+1}^{-1}, \xi_{t+1}) \Pi_{t+1}^{-1}) \\
& + \psi_{2t}(S_t^e - u_c(Y_{t+1} - d(\Pi_{t+1}) - F, m_{t+1} \Pi_{t+1}^{-1}, \xi_{t+1}) \Pi_{t+1} d'(\Pi_{t+1})) + \gamma_{1t}(i_t - i^m) + \gamma_{2t}(\bar{w} - w_t)]
\end{aligned}
$$

FOC (all the derivatives should be equated to zero)

$$
\begin{aligned}
\frac{\delta L_s}{\delta \Pi_t} = \; & -u_c d'(\Pi_t) - u_m m_t \Pi_t^{-2} + \phi_{1t}\Big[-\frac{u_{mc} d' \Pi_t^{-1}}{u_c} - \frac{u_{mm} m_t \Pi_t^{-2}}{u_c} - \frac{u_m \Pi_t^{-2}}{u_c} + \frac{u_m u_{cc} d' \Pi_t^{-1}}{u_c^2} + \frac{u_m u_{cm} m_t \Pi_t^{-2}}{u_c}\Big] \\
& + \phi_{2t}[(1 + i_t) w_{t-1} \Pi_t^{-2} - (i_t - i^m) m_t \Pi_t^{-2}] + \phi_{3t}\Big[\frac{u_{cc} d'}{1 + i_t} + \frac{u_{cm} m_t \Pi_t^{-2}}{(1 + i_t)}\Big] \\
& + \phi_{4t}[-Y_t(\theta - 1)(1 + s)(u_{cc} d' + m_t \Pi_t^{-2} u_{cm}) - u_{cc} \Pi_t d'^2 - u_{cm} m_t \Pi_t^{-1} d' + u_c \Pi_t d'' + u_c d'] \\
& + \beta^{-1} \psi_{1t-1}[u_{cc} d' \Pi_t + u_{cm} m_t \Pi_t^{-1} + u_c \Pi_t^{-2}] + \beta^{-1} \psi_{2t-1}[u_{cc} d'^2 \Pi_t + u_{cm} d' m_t \Pi_t^{-1} - u_c d' - u_c d'' \Pi_t]
\end{aligned}
\tag{60}
$$

$$
\begin{aligned}
\frac{\delta L_s}{\delta Y_t} = \; & u_c - \tilde{v}_y + \phi_{1t}\Big[\frac{u_{mc} \Pi_t^{-1}}{u_c} - \frac{u_m \Pi_t^{-1}}{u_c^2}\Big] - \phi_{3t} \frac{u_{cc}}{1 + i_t} \\
& + \phi_{4t}\Big[\theta\Big(\frac{\theta - 1}{\theta}(1 + s) u_c - \tilde{v}_y\Big) + \theta Y_t\Big(\frac{\theta - 1}{\theta}(1 + s) u_{cc} - \tilde{v}_{yy}\Big) + u_{cc} \Pi_t d'\Big] \\
& - \beta^{-1} \psi_{1t-1} u_{cc} \Pi_t - \beta^{-1} \psi_{2t-1} u_{cc} d' \Pi_t
\end{aligned}
\tag{61}
$$

$$
\frac{\delta L_s}{\delta i_t} = -\phi_{1t} \frac{1 + i^m}{(1 + i_t)^2} + \phi_{2t}(m_t \Pi_t^{-1} + T_t - w_{t-1} \Pi_t - F) + \phi_{3t} \frac{u_c}{(1 + i_t)^2} + \gamma_{1t} \tag{62}
$$

$$
\begin{aligned}
\frac{\delta L_s}{\delta m_t} = \; & u_m \Pi_t^{-1} + \phi_{1t}\Big[\frac{u_{mm}}{u_c} - \frac{u_m}{u_c^2} u_{cm} \Pi_t^{-1}\Big] \Pi_t^{-1} + \phi_{2t}(i_t - i^m) \Pi_t^{-1} - \phi_{3t} \frac{u_{cm}}{1 + i_t} \Pi_t^{-1} \\
& - \phi_{4t}[Y_t(\theta - 1)(1 + s) u_{cm} \Pi_t^{-1} - u_{cm} d'] - \psi_{1t-1} u_{cm} \Pi_t^{-2} - \psi_{2t-1} u_{cm} d'
\end{aligned}
\tag{63}
$$

$$
\frac{\delta L_s}{\delta T_t} = -g_G s'(T_t) + \phi_{2t}(1 + i_t) \tag{64}
$$

$$
\frac{\delta L_s}{\delta w_t} = \phi_{2t} - \beta E_t \phi_{2t+1}(1 + i_{t+1}) \Pi_{t+1}^{-1} - \gamma_{2t} \tag{65}
$$

$$
\frac{\delta L_s}{\delta f_t^e} = \beta \phi_{3t} + \psi_{1t} \tag{66}
$$

$$\frac{\delta L_s}{\delta S_t^e} = -\beta\phi_{4t} + \psi_{2t} \tag{67}$$

The complementary slackness conditions are:

$$\gamma_{1t} \geq 0,\ i_t \geq i^m,\ \ \gamma_{1t}(i_t - i^m) = 0 \tag{68}$$

$$\gamma_{2t} \geq 0,\ \ \bar{w} - w_t \geq 0,\ \ \gamma_{2t}(\ \bar{w} - w_t) = 0 \tag{69}$$

7.2 Nonlinear Markov equilibrium

Markov equilibrium period Lagrangian:

$$\begin{aligned}
L_t &= u(Y_t - d(\Pi_t) - F, m_t\Pi_t^{-1}, \xi_t)) + g(F - s(T_t), \xi_t) - \tilde{v}(Y_t) + E_t\beta J(w_t, \xi_{t+1}) \\
&+ \phi_{1t}(\frac{u_m(Y_t - d(\Pi_t) - F, m_t\Pi_t^{-1}, \xi_t)\Pi_t^{-1}}{u_c(Y_t - d(\Pi_t) - F, m_t\Pi_t^{-1}, \xi_t)} - \frac{i_t - i^m}{1 + i_t}) \\
&+ \phi_{2t}(w_t - (1 + i_t)\Pi_t^{-1}w_{t-1} - (1 + i_t)F + (1 + i_t)T_t + (i_t - i^m)m_t\Pi_t^{-1}) + \\
&+ \phi_{3t}(\beta f_t^e - \frac{u_c(Y_t - d(\Pi_t) - F, m_t\Pi_t^{-1}, \xi_t)}{1 + i_t}) \\
&+ \phi_{4t}(\theta Y_t[\frac{\theta - 1}{\theta}(1 + s)u_c(Y_t - d(\Pi_t) - F, m_t\Pi_t^{-1}, \xi_t) - \tilde{v}_y(Y_t, \xi_t)] + u_c(Y_t - d(\Pi_t) - F, m_t\Pi_t^{-1}, \xi_t)\Pi_t d'(\Pi_t) - \beta S_t^e) \\
&+ \psi_{1t}(f_t^e - \bar{f}^e(w_t, \xi_t)) + \psi_{2t}(S_t^e - \bar{S}^e(w_t, \xi_t)) + \gamma_{1t}(i_t - i^m) + \gamma_{2t}(\bar{w} - w_t)
\end{aligned}$$

FOC (all the derivative should be equated to zero)

$$\begin{aligned}
\frac{\delta L_s}{\delta \Pi_t} \quad = \quad & -u_c d'(\Pi_t) - u_m m_t\Pi_t^{-2} + \phi_{1t}[-\frac{u_{mc}d'\Pi_t^{-1}}{u_c} - \frac{u_{mm}m_t\Pi_t^{-2}}{u_c} - \frac{u_m\Pi_t^{-2}}{u_c} + \frac{u_m u_{cc}d'\Pi_t^{-1}}{u_c^2} + \frac{u_m u_{cm}m_t\Pi_t^{-2}}{u_c}] \\
& + [\phi_{2t}(1 + i_t)w_{t-1}\Pi_t^{-2} - (i_t - i^m)m_t\Pi_t^{-2}] + \phi_{3t}[\frac{u_{cc}d'}{1 + i_t} + \frac{u_{cm}m_t\Pi_t^{-2}}{(1 + i_t)}] \\
& + \phi_{4t}[-Y_t(\theta - 1)(1 + s)(u_{cc}d' + m_t\Pi_t^{-2}u_{cm}) - u_{cc}\Pi_t d'^2 - u_{cm}m_t\Pi_t^{-1}d' + u_c\Pi_t d'' + u_c d']
\end{aligned} \tag{70}$$

$$\frac{\delta L_s}{\delta Y_t} = u_c - \tilde{v}_y + \phi_{1t}[\frac{u_{mc}}{u_c} - \frac{u_m}{u_c^2}]\Pi_t^{-1} - \phi_{3t}\frac{u_{cc}}{1 + i_t} + \phi_{4t}[\theta(\frac{\theta - 1}{\theta}(1 + s)u_c - \tilde{v}_y) + \theta Y_t(\frac{\theta - 1}{\theta}(1 + s)u_{cc} - \tilde{v}_{yy}) + u_{cc}\Pi_t d'] \tag{71}$$

$$\frac{\delta L_s}{\delta i_t} = -\phi_{1t}\frac{1 + i^m}{(1 + i_t)^2} + \phi_{2t}(m_t\Pi_t^{-1} + T_t - w_{t-1}\Pi_t^{-1} - F) + \phi_{3t}\frac{u_c}{(1 + i_t)^2} + \gamma_{1t} \tag{72}$$

$$\frac{\delta L_s}{\delta m_t} = u_m\Pi_t^{-1} + \phi_{1t}[\frac{u_{mm}}{u_c} - \frac{u_m}{u_c^2}u_{cm}\Pi_t^{-1}]\Pi_t^{-1} + \phi_{2t}(i_t - i^m)\Pi_t^{-1} - \phi_{3t}\frac{u_{cm}}{1 + i_t}\Pi_t^{-1} - \phi_{4t}[Y_t(\theta - 1)(1 + s)u_{cm}\Pi_t^{-1} - u_{cm}d'] \tag{73}$$

$$\frac{\delta L_s}{\delta T_t} = -g_G s'(T_t) + \phi_{2t}(1 + i_t) \tag{74}$$

$$\frac{\delta L_s}{\delta w_t} = \beta E_t J_w(w_t, \xi_{t+1}) - \psi_{1t}f_w^e - \psi_{2t}S_w^e + \phi_{2t} - \gamma_{2t} \tag{75}$$

$$\frac{\delta L_s}{\delta f_t^e} = \beta\phi_{3t} + \psi_{1t} \tag{76}$$

$$\frac{\delta L_s}{\delta S_t^e} = -\beta\phi_{4t} + \psi_{2t} \tag{77}$$

The complementary slackness conditions are:

$$\gamma_{1t} \geq 0,\ i_t \geq i^m,\quad \gamma_{1t}(i_t - i^m) = 0 \tag{78}$$

$$\gamma_{2t} \geq 0,\quad \bar{w} - w_t \geq 0,\quad \gamma_{2t}(\ \bar{w} - w_t) = 0 \tag{79}$$

The optimal plan under discretion also satisfies an envelope condition:

$$J_w(w_{t-1}, \xi_t) = -\phi_{2t}(1 + i_t)\Pi_t^{-1} \tag{80}$$

Necessary and sufficient condition for a Markov equilibrium thus are given by the first order conditions (70) to (80) along with the constraints (8), (25), (28), (30) and the definitions (27) and (29). Note that the first order conditions imply restrictions on the unknown vector function Λ_t and the expectation functions.

7.3 Linearized solution

I here linearize the first order conditions and the constraints around the steady state in Propositions 4 and 5. I assume the form of the utility discussed in section 3.4. I allow for deviations in the vector of shocks ξ_t, the production subsidy s (the latter deviation is used in Proposition 8) so and in i^m so that the equations are accurate of order $o(||\xi, \bar{\bar{\delta}}, 1 + s - \frac{\theta}{\theta-1}||^2)$. I abstract from the effect of the shocks on the disutility of labor. Here $dz_t = z_t - z_{ss}$ The economic constraints under both commitment and discretion are:

$$\bar{u}_c d'' d\Pi_t + \theta(\bar{u}_{cc} - \bar{v}_{yy})dY_t + (\theta - 1)\bar{u}_c ds + \theta\bar{u}_{c\xi} d\xi_t - \bar{u}_c d''\beta E_t d\Pi_{t+1} = 0 \tag{81}$$

$$\bar{u}_{cc} dY_t + \bar{u}_{c\xi} d\xi_t - \beta\bar{u}_{cc} E_t dY_{t+1} - \beta\bar{u}_{c\xi} E_t d\xi_{t+1} - \beta\bar{u}_c di_t + \beta\bar{u}_c E_t d\Pi_{t+1} = 0 \tag{82}$$

$$dw_t - \frac{1}{\beta}dw_{t-1} + \frac{1}{\beta}dT_t = 0 \tag{83}$$

$$dS_t^e - \bar{u}_c d'' E_t d\Pi_{t+1} = 0 \tag{84}$$

$$df_t^e + \bar{u}_c E_t d\Pi_{t+1} - \bar{u}_{cc} E_t dY_{t+1} - \bar{u}_{c\xi} E_t \xi_{t+1} = 0 \tag{85}$$

The equation determining the natural rate of output is:

$$(v_{yy} - u_{cc})dY_t^n + (v_{y\xi} - u_{c\xi})d\xi_t - \frac{(\theta - 1)}{\theta}u_c ds = 0 \tag{86}$$

The equation determining the natural rate of interest is:

$$\beta E_t(\bar{u}_{cc} dY_{t+1}^n - \bar{u}_{c\xi} E_t d\xi_{t+1}) - (\bar{u}_{cc} dY_t^n - \bar{u}_{c\xi} d\xi_t) + \beta\bar{u}_{cc} dr_t^n = 0 \tag{87}$$

Note that the real money balances deflated by $\bar{m}$, i.e. $\tilde{m}_t$, are well defined in the cashless limit so that equation 50 is

$$d\tilde{m}_t - d\tilde{m}_{t-1} - d\frac{M_t}{M_{t-1}} + d\pi_{t-1} = 0$$

and money demand is approximated by

$$\frac{\bar{\chi}_{mm}}{u_c}d\tilde{m}_t - \frac{\bar{\chi}_{mm}}{u_c}\tilde{m}d\Pi_t - \frac{\bar{\chi}_{mm}}{u_c}\tilde{m}dY_t - \beta di_t = 0$$

The Kuhn Tucker conditions imply that

Case 1 when $i_t > i^m$

$$d\gamma_{1t} = 0 \tag{88}$$

Case 2 when $i_t = i^m$

$$di_t = 0 \tag{89}$$

I look for a solution in which case the debt limit is never binding so that $d\gamma_{2t} = 0$ at all times and verify that this is satisfied in equilibrium. The linearized FOC in a commitment equilibrium are:

$$-d''\bar{u}_c d\Pi_t + \bar{\phi}_2\beta^{-1}dw_{t-1} + d''\bar{u}_c d\phi_{4t} - \bar{u}_c d\phi_{3t-1} - d''\bar{u}_c d\phi_{4t-1} = 0 \tag{90}$$

$$(\bar{u}_{cc} - \bar{v}_{yy})dY_t + \bar{u}_{c\xi}d\xi_t - \bar{v}_{y\xi}d\xi_t - \bar{u}_{cc}\beta d\phi_{3t} + \theta(\bar{u}_{cc} - \bar{v}_{yy})d\phi_{4t} + \bar{u}_{cc}d\phi_{3t-1} = 0 \tag{91}$$

$$\bar{\phi}_2 dT_t - \bar{\phi}_2 dw_{t-1} + \bar{u}_c\beta^2 d\phi_{3t} + d\gamma_{1t} = 0 \tag{92}$$

$$\bar{g}_{GG}(s')^2 dT_t - \bar{g}_G s'' dT_t - \bar{g}_{G\xi}d\xi_t + \beta^{-1}d\phi_{2t} + \bar{\phi}_2 di_t = 0 \tag{93}$$

$$d\phi_{2t} - E_t d\phi_{2t+1} - \beta\bar{\phi}_2 E_t di_{t+1} + \bar{\phi}_2 E_t d\Pi_{t+1} - d\gamma_{2t} = 0 \tag{94}$$

Linearized FOC in a Markov Equilibrium

$$-d''\bar{u}_c d\Pi_t + \bar{\phi}_2\beta^{-1}dw_{t-1} + d''\bar{u}_c d\phi_{4t} = 0 \tag{95}$$

$$(\bar{u}_{cc} - \bar{v}_{yy})dY_t + \bar{u}_{c\xi}d\xi_t - \bar{v}_{y\xi}d\xi_t - \bar{u}_{cc}\beta d\phi_{3t} + \theta(\bar{u}_{cc} - \bar{v}_{yy})d\phi_{4t} = 0 \tag{96}$$

$$\bar{\phi}_2 dT_t - \bar{\phi}_2 dw_{t-1} + \bar{u}_c\beta^2 d\phi_{3t} + d\gamma_{1t} = 0 \tag{97}$$

$$\bar{g}_{GG}(s')^2 dT_t - \bar{g}_G s'' dT_t - \bar{g}_{G\xi}d\xi_t + \beta^{-1}d\phi_{2t} + \bar{\phi}_2 di_t = 0 \tag{98}$$

$$d\phi_{2t} - E_t d\phi_{2t+1} - \beta\bar{\phi}_2 E_t di_{t+1} + \bar{\phi}_2 E_t d\Pi_{t+1} + \beta f_w d\phi_{3t} - \beta S_w d\phi_{4t} - d\gamma_{2t} = 0 \tag{99}$$

Note that the first order condition with respect to m_t does not play any role in the cashless limit so that it is omitted above.

8 Appendix B: Proofs

8.1 Proof of Proposition 1:

I proof this proposition by showing that all the necessary and sufficient conditions for a PSE listed in Definition 1 (i.e. equation (3)-(17)) can be written without any reference to either ψ_t or T_t. I first show that the equilibrium conditions can be written without any references to the function $T(.)$. Since only one period bonds are issued I can write $W_{t+1} = (1 + i_t)B_t + (1 + i^m)M_t$ and equation (16) becomes

$$\frac{1}{1+i_t}W_{t+1} = W_t + P_tF - P_tT_t - \frac{i_t - i^m}{1+i_t}M_t \tag{100}$$

which defines W_t as a function of T_t and M_t so that I must show that I can write all the necessary condition for equilibrium without any reference to $W(.)$ as well. Using equation (100) and (10) I can write

$$W_t - E_t\sum_{T=t}^{\infty} Q_{t,T}(P_TT_T - \frac{i_T - i^m}{1+i_T}M_T) = E_t\sum_{T=t}^{\infty} Q_{t,T}P_TF$$

Furthermore I can use the expression for firms profits and the requirement of symmetric equilibrium to yield:

$$E_t\sum_{T=t}^{\infty} Q_{t,T}[\int_0^1 Z_T(i)di + \int_0^1 n_T(j)h_T(j)dj = E_t\sum_{T=t}^{\infty} Q_{t,T}P_tY_t$$

Using the last two equation I can write the intertemporal budgets constraint (25) as:

$$E_t\sum_{T=t}^{\infty} Q_{t,T}P_TC_T \leq E_t\sum_{T=t}^{\infty} Q_{t,T}[P_TY_T - P_TF] \tag{101}$$

Thus this constraint can be written without any reference to the function $T(.)$ or $W(.)$. The other condition that need to be satisfied regardless of the specification of $T(.)$ is equation (10). Using (100) this condition can be simplified to yield:

$$\lim_{T\to\infty} \beta^T[u_c(Y_T - d(\Pi_T) - F_T, M_T/P_T; \xi_T)M_T/P_T] = 0 \tag{102}$$

Note here that I have used A1(ii) to eliminate the expectation from this expression that depends neither on $T(.)$ or $W(.)$.

I now show that all the constraint of required for a private sector equilibrium can be expressed independently of the specification of $\psi(.)$. I first consider equation (102). The specification of $\psi(.)$ could be important if the zero bound is asymtotically binding. It is easy to show that for the zero bound to be asymtotically binding I must have $\Pi_t = \frac{P_t}{P_{t-1}} = \beta$ and $Y_t = \bar{Y}$. Then I can write,for all $t \geq K$ (i.e. all dates after the uncertainty is resolved) $P_t = \beta^{t-K}P_K$. Then (102) becomes

$$\lim_{T\to\infty} \beta^K[u_c(Y_T - d(\Pi_T) - F_T, M_T/P_T; \xi_T)\frac{M_T}{P_K} = 0$$

This condition is only satisfied if $M_T \to 0$. But this would violate (21) and thus an asymtotic deflationary equilibrium is not consistent with my specification of fiscal and monetary policy. It follows that the specification of $\psi(.)$ has no effect on whether or not (102) is satisfied since I have just shown that the zero bound cannot be asymtotically binding. What remains to be shown is that all the other equilibrium conditions can be written without any reference to the function $\psi(.)$. That part of the proof follows exactly the same steps as the proof of Proposition 1 in Eggertsson and Woodford (2003).

8.2 Proof of Propositions 4 and 5

In the assumption made in the proposition I assume the cashless limit and the form of the utility given by (49) so that

$$u(C_t, m_t\Pi_t^{-1}, \xi_t) = \tilde{u}(C_t, \xi_t) + \chi(\frac{m_t}{\bar{m}}\Pi_t^{-1}C_t^{-1}, \xi_t) \quad (103)$$

The partial derivatives with respect to each variable are given by

$$u_c = \tilde{u}_c - \chi' \frac{m}{\bar{m}} C^{-2}\Pi^{-1} \quad (104)$$

$$u_m = \frac{\chi'}{\bar{m}} C^{-1}\Pi^{-1} \quad (105)$$

$$u_{mm} = \frac{\chi''}{\bar{m}^2} C^{-2}\Pi^{-2} < 0 \quad (106)$$

$$u_{cm} = -\chi'' \frac{m}{\bar{m}^2} C^{-3}\Pi^{-2} - \frac{\chi'}{\bar{m}} C^{-2}\Pi^{-1} \quad (107)$$

As $\bar{m}- > 0$ I assume that for $\tilde{m} = \frac{m}{\bar{m}} > 0$ I have

$$lim_{\bar{m}\to 0}\frac{\chi\prime}{\bar{m}} \equiv \bar{\chi}' \geq 0 \quad (108)$$

$$lim_{\bar{m}\to 0}\frac{\chi\prime\prime}{\bar{m}^2} \equiv \bar{\chi}'' > 0 \quad (109)$$

This implies that there is a well defined money demand function, even as money held in equilibrium approaches zero, given by

$$\frac{\bar{\chi}\prime(\tilde{m}C_t^{-1}\Pi_t^{-1}, \xi_t)C_t^{-1}\Pi_t^{-1}}{\bar{u}_c(C_t, \xi_t)} = \frac{i_t - i^m}{1 + i_t}$$

so that $\bar{\chi}' = 0$ when $i_t = i_t^m$. From the assumptions (108)-(109) it follows that:

$$lim_{\bar{m}\to 0}\chi\prime = 0$$

$$lim_{\bar{m}\to 0}\chi\prime\prime = 0$$

Then the derivatives u_c and u_{cm} in the cashless limit are:

$$\lim_{\bar{m}\to 0} u_c = \tilde{u}_c$$

and

$$\lim_{\bar{m}\to 0} u_{cm} = \lim_{\bar{m}\to 0}[-\bar{m}\frac{\chi\prime\prime}{\bar{m}^2}\frac{m}{\bar{m}}C^{-3}\Pi^{-1} - \frac{\chi\prime}{\bar{m}}C^{-2}] = -\bar{\chi}\prime C^{-2}$$

Hence in a steady state in which $\bar{m} \to 0$ and $i_t = i^m$ I have that $\bar{\chi}\prime = 0$ so that at the steady state

$$\lim_{\bar{m}\to 0} u_{cm} = 0. \quad (110)$$

Note that this does not imply that the satiation point of holding real balances is independent of consumption. To see this note that the satiation point of real money balances is is given by some finite number $S^* = \frac{m}{\bar{m}}Y$ which implies that $\chi(S \geq S^*) = \tilde{v}(S^*)$. The value of the satiation point as $\bar{m} \to 0$ is:

$$lim_{\bar{m}\to 0} S^* \equiv \bar{S} = \tilde{m}C$$

The value of this number still depends on C even as $\bar{m} \to 0$ and even if $u_{cm} = 0$ at the satiation point.

I now show that the steady state stated in Proposition 3 and 4 satisfies all the first order conditions and the constraints. The steady state candidate solution in both proposition is:

$$i = \frac{1}{\beta} - 1, w = \phi_1 = \phi_3 = \phi_4 = \psi_1 = \psi_2 = \gamma_1 = \gamma_2 = 0, \Pi = 1, \phi_2 = g_G s', \ T = F \tag{111}$$

Note that (111) and the functional assumption about d (see footnote 5) imply that:

$$d' = 0 \tag{112}$$

Let us first consider the constraints. In the steady state the AS equation is

$$\theta Y[\frac{\theta - 1}{\theta}(1 + s)u_c - \tilde{v}_y] - u_c \Pi d'(\Pi) + \beta u_c \Pi d'(\Pi) = 0$$

Since by (112) d'=0, and according to assumption (ii) of the propositions $\frac{\theta-1}{\theta}(1 + s) = 1$ the AS equation is only satisfied in the candidate solution if

$$u_c = v_y \tag{113}$$

Evaluated in the candidate solution the IS equation is:

$$\frac{1}{1 + i} = \frac{\beta u_c}{u_c} \Pi^{-1} = \beta$$

which is always satisfied at because it simply states that $i = 1 - 1/\beta$ which is consistent with the steady state I propose in the propositions and assumption (iii). The budget constraint is:

$$w - (1 + i)\Pi^{-1}w - (1 + i)F + (1 + i)T + (1 + i)\bar{m}\tilde{m}\Pi_t^{-1} = 0$$

which is also always satisfied in our candidate solution since it states that $F = T$, $w = 0$ and $\bar{m} \to 0$. The money demand equation indicates that the candidate solutions is satisfied if

$$u_m = \Pi u_c \frac{i - i^m}{1 + i} = 0 \tag{114}$$

By (27) and (29) the expectation variables in steady state are

$$S^e = u_c \Pi d'$$

$$f^e = u_c \Pi$$

Since $\Pi = 1$ and $d' = 0$ by (112) these equations are satisfied in the candidate solution. Finally both the inequalities (9) and (26) are satisfied since $\bar{w} > w = 0$ in the candidate solution and $i = i^m$.

I now show that the first order conditions, i.e. the commitment and the Markov equilibrium first order conditions, that are given by (60)-(69) and (70)-(80) respectively, are also consistent with the steady state suggested. I first show the commitment equilibrium. The proof for the Markov equilibrium will follow along the same lines.

Commitment equilibrium steady state Let us start with (60). It is

$$
\begin{aligned}
&-u_c d' - u_m \bar{m}\tilde{m}\Pi^{-2} + \phi_1[-\frac{u_{mc}d'\Pi^{-1}}{u_c} - \frac{u_{mm}\bar{m}\tilde{m}\Pi^{-2}}{u_c} - \frac{u_m\Pi^{-2}}{u_c} + \frac{u_m u_{cc}d'\Pi^{-1}}{u_c^2} + \frac{u_m u_{cm} m\Pi^{-2}}{u_c}] \\
&+[\phi_2(1+i)w\Pi^{-2} - (i-i^m)\bar{m}\tilde{m}\Pi^{-2}] + \phi_3[\frac{u_{cc}d'}{1+i} + \frac{u_{cm}\bar{m}\tilde{m}\Pi^{-2}}{(1+i)}] \\
&+\phi_4[-Y(\theta-1)(1+s)(u_{cc}d' + \bar{m}\tilde{m}\Pi^{-2}u_{cm}) - u_{cc}\Pi d'^2 - u_{cm}\bar{m}\tilde{m}\Pi^{-1}d' + u_c\Pi d'' + u_c d'] \\
&+\beta^{-1}\psi_1[u_{cc}d'\Pi + u_{cm}\bar{m}\tilde{m}\Pi^{-1} + u_c\Pi^{-2}] + \beta^{-1}\psi_2[u_{cc}d'^2\Pi + u_{cm}d'\bar{m}\tilde{m}\Pi^{-1} - u_c d' - u_c d''\Pi] = 0
\end{aligned}
$$

By (112) and (114) the first two terms are zero. The constraints that are multiplied by ϕ_1, ϕ_3, ϕ_4, ψ_1 and ψ_2 are also zero because each of these variables are zero in our candidate solution (111). Finally, the term that is multiplied by ϕ_2 (which is positive) is also zero because $w = 0$ in our candidate solution (111) and so is $i - i^m$. Thus I have shown that the candidate solution (111) satisfies (60).

Let us now turn to (61). It is

$$
u_c - \tilde{v}_y + \phi_1[\frac{u_{mc}\Pi^{-1}}{u_c} - \frac{u_m\Pi^{-1}}{u_c^2}] - \phi_3\frac{u_{cc}}{1+i} + \phi_4[\theta(\frac{\theta-1}{\theta}(1+s)u_c - \tilde{v}_y) + \theta Y(\frac{\theta-1}{\theta}(1+s)u_{cc} - \tilde{v}_{yy}) + u_{cc}\Pi d'] - \psi_1\beta^{-1}u_{cc}\Pi - \psi_2\beta^{-1}u_{cc}d
$$

The first two terms $u_c - v_y$ are equal to zero by (113). The next terms are also all zero because they are multiplied by the terms ϕ_1, ϕ_3, ϕ_4, ψ_1 and ψ_2 which are all zero in our candidate solution (111). Hence this equation is also satisfied in our candidate solution. Let us then consider (62). It is:

$$
-\phi_1\frac{1+i^m}{(1+i)^2} + \phi_2(\bar{m}\tilde{m} + T - w\Pi^{-1} - F) + \phi_3\frac{u_c}{(1+i)^2} + \gamma_1 = 0
$$

Again this equation is satisfied in our candidate solution because $\phi_1 = \phi_3 = w = 0$, $F = T$ and $\bar{m} \to 0$ in the candidate solution. Conditions (63) in steady state is:

$$
\begin{aligned}
&\bar{m}\tilde{m}u_m\Pi^{-1} + \phi_1[\frac{u_{mm}}{u_c}\Pi^{-1} - \frac{u_m}{u_c^2}u_{cm}\Pi^{-2}] + \phi_2(i-i^m)\Pi^{-1} - \phi_3\frac{u_{cm}}{1+i}\Pi^{-1} \\
&-\phi_4[Y(\theta-1)(1+s)u_{cm}\Pi^{-1} - u_{cm}d'] - \psi_1 u_{cm}\Pi^{-2} - \psi_2 u_{cm}d' = 0
\end{aligned} \tag{115}
$$

The first term is zero by (114). All the other terms are also zero because ϕ_1, ϕ_3, ϕ_4, ψ_1 and ψ_2 are all zero in our candidate solution (111). Finally $i = i^m$ in our candidate solution so that the third term is zero as well. Condition (64) in steady state is:

$$
-g_G s'(T) + \phi_2(1+i) = 0 \tag{116}
$$

which is satisfied in the candidate solution. Condition (65) is

$$
\phi_2 - \beta\phi_2(1+i)\Pi^{-1} - \gamma_2 = 0
$$

This condition is also satisfied in our candidate solution because $\gamma_2 = 0$ and $(1+i)\Pi^{-1} = \frac{1}{\beta}$. Conditions (66) and (67) are:

$$
\beta\phi_3 + \psi_1 = 0 \tag{117}
$$

$$\beta\phi_4 + \psi_2 = 0 \tag{118}$$

Since $\phi_3 = \phi_4 = \psi_1 = \psi_2 = 0$ in our candidate solution, these conditions are also satisfied. Finally our candidate solution (111) indicates that (68) and (69) are also satisfied in steady state. I have now showed that our candidate solution satisfies all necessary and sufficient conditions for an equilibrium and Proposition 3 is thus proofed.

Markov equilibrium steady state Let us now turn to the Markov equilibrium. The first order conditions in steady state are

$$\begin{aligned}
&-u_c d' - u_m \bar{m}\tilde{m}\Pi^{-2} + \phi_1[-\frac{u_{mc}d'\Pi^{-1}}{u_c} - \frac{u_{mm}\bar{m}\tilde{m}\Pi^{-2}}{u_c} - \frac{u_m\Pi^{-2}}{u_c} + \frac{u_m u_{cc} d'\Pi^{-1}}{u_c^2} + \frac{u_m u_{cm} m\Pi^{-2}}{u_c} \\
&+[\phi_2(1+i)w\Pi^{-2} - (i-i^m)\bar{m}\tilde{m}\Pi^{-2}] + \phi_3[\frac{u_{cc}d'}{1+i} + \frac{u_{cm}\bar{m}\tilde{m}\Pi^{-2}}{(1+i)}] \\
&+\phi_4[-Y(\theta-1)(1+s)(u_{cc}d' + \bar{m}\tilde{m}\Pi^{-2}u_{cm}) - u_{cc}\Pi d'^2 - u_{cm}\bar{m}\tilde{m}\Pi^{-1}d' + u_c\Pi d'' + u_c d'] \\
&+\beta^{-1}\psi_1[u_{cc}d'\Pi + u_{cm}\bar{m}\tilde{m}\Pi^{-1} + u_c\Pi^{-2}] + \beta^{-1}\psi_2[u_{cc}d'^2\Pi + u_{cm}d'\bar{m}\tilde{m}\Pi^{-1} - u_c d' - u_c d''\Pi] = 0
\end{aligned} \tag{119}$$

$$u_c - \tilde{v}_y + \phi_1[\frac{u_{mc}}{u_c} - \frac{u_m}{u_c^2}] - \phi_3\frac{u_{cc}}{1+i} + \phi_4[\theta(\frac{\theta-1}{\theta}(1+s)u_c - \tilde{v}_y) - \theta Y(\frac{\theta-1}{\theta}(1+s)u_{cc} - \tilde{v}_{yy}) - u_{cc}\Pi d'] = 0 \tag{120}$$

$$-\phi_1\frac{1+i^m}{(1+i)^2} + \phi_2(\bar{m}\tilde{m} + T - w\Pi^{-1} - F) + \phi_3\frac{u_c}{(1+i)^2} + \gamma_1 = 0 \tag{121}$$

$$u_m\Pi^{-1} + \phi_1[\frac{u_{mm}}{u_c} - \frac{u_m}{u_c^2}u_{cm}\Pi^{-1}] + \phi_2(i-i^m)\bar{m}\tilde{m} - \phi_3\frac{u_{cm}}{1+i}\Pi^{-1} - \phi_4[Y(\theta-1)(1+s)u_{cm}\Pi^{-1} - u_{cm}d'] = 0 \tag{122}$$

$$-g_G s'(T) + \phi_2(1+i) = 0 \tag{123}$$

$$\beta J_w - \psi_1\beta f_w^e - \psi_2\beta S_w^e + \phi_2 - \gamma_2 = 0 \tag{124}$$

$$\beta\phi_3 + \psi_1 = 0 \tag{125}$$

$$\beta\phi_4 + \psi_2 = 0 \tag{126}$$

$$J_w = -\phi_2(1+i)\Pi^{-1} \tag{127}$$

Condition (119)-(123) and (125)-(126) are the same as in the commitment equilibrium, apart from the presence of ψ_1 and ψ_2 in the equations corresponding to (119) and (120). Since $\psi_1 = \psi_2 = 0$ in the candidate solution this does not change our previous proof. Thus, exactly the same arguments as I made to show that the candidate solution (111) satisfied the first order conditions in the commitment equilibrium can be used in the Markov equilibrium for equations (119)-(123) and (125)-(126). The crucial difference between the first order conditions in the Markov and the commitment equilibrium is in equation (124). This equation involves three unknown functions, J_w, f_w^e and S_w^e.I can use (127) to substitute for J_w in (124) obtaining

$$-\beta\phi_2(1+i)\Pi^{-1} - \psi_1\beta f_w^e - \psi_2\beta S_w^e + \phi_2 - \gamma_2 = 0 \tag{128}$$

In general I cannot know if this equation is satisfied without making further assumption about f_w^e and S_w^e. But note that in our candidate solution $\psi_1 = \psi_2 = 0$. Thus the terms involving these two derivatives in this equation are zero. Since $\gamma_2 = 0$, this equation is satisfied if $(1+i)\Pi^{-1} = 1/\beta$. This is indeed the case in our candidate solution. Thus I have shown that all the necessary and sufficient conditions of a Markov equilibrium are satisfied by our candidate solution (111). QED

8.3 Proof of Proposition 6

In this equilibrium there is only one policy instrument so that $dT_t = dw_t = 0$ and I may ignore the linearized first order conditions (93), (94) for commitment and (98) and (99) in the Markov equilibrium. The remaining FOC along with the constraint (81), (82) and (88) determine the equilibrium.

1. I first consider the commitment case. Equation (93) indicates that $\phi_{3t} = 0$. Then I can write (90) and (91) in terms of inflation and output gap as (using (86) to solve it in terms of the output gap):

$$\pi_t - \phi_{4t} + \phi_{4t-1} = 0$$

$$x_t + \theta\phi_{4t} = 0$$

Substituting these two equations into the AS equation (53) combined with (86) I can write the solution in terms of a second order difference equation:

$$\beta E_t x_{t+1} - (1+\beta+\kappa\theta)x_t + x_{t-1} \tag{129}$$

The characteristic polynomial

$$\beta\mu^2 - (1+\beta+\kappa\theta)\mu + 1 = 0$$

has two real roots

$$0 < \mu_1 < 1 < \beta^{-1} < \mu_2 = (\beta\mu_1)^{-1}$$

and it follows that (129) has an unique bounded solution $x_t = 0$ consistent with the the initial condition that $x_{-1} = 0$. Substituting this solution into (53) I can verify that the unique bounded solution for inflation is $\pi_t = 0$.

2. In the case of the Markov solution equation (98) indicates that $\phi_{3t} = 0$. Then I can write (95) and (96) so that (using (86) to solve it in terms of the output gap):

$$-\pi_t + \phi_{4t} = 0 \tag{130}$$

$$x_t + \theta\phi_{4t} = 0 \tag{131}$$

I can substitute these equations into the AS (53) together with (86) and write the solution in terms of the difference equation:

$$(1+\theta\kappa)x_t - \beta E_t x_{t+1} = 0 \tag{132}$$

This equation has a unique bounded solution $x_t = 0$ and it follows that the unique bounded solution for inflation is $\pi_t = 0$.

8.4 Proof of Proposition 7

1. The first part of the proposition is that $\pi_t = x_t = 0$ for $t \geq \tau$. The proof for this follows directly from the second part of the proof for Proposition 6 since for $t \geq \tau$ there are no shocks and the Markov equilibrium is the one given in that Proposition. To see this note that the first order condition for $t \geq \tau$ are again given by (130) and (131) and I can again write the difference equation (132). Since this equation involves no history dependence (i.e. initial conditions do not matter) it follows once again that the unique bounded solution when $t \geq \tau$ is $x_t^M = \pi_t^M = 0$.

2. The second part of the proposition is that the Markov solution results in excessive deflation and output gap in period $0 < t < \tau$ relative to a policy that implies $\pi_\tau^C > 0$ and $x_\tau^C > 0$. I proof this by first showing that this must hold true for $\tau = K$ and then show that this implies it must hold for any $\tau < K$. Note first that our solution for the Markov equilibrium at any date $t \geq \tau$ implies that

$$\pi_\tau^C - \pi_\tau^M > 0 \tag{133}$$

$$x_\tau^C - x_\tau^M > 0 \tag{134}$$

The IS and AS equation implies that in the Markov equilibrium at date $K-1$ is

$$\tilde{x}_{K-1}^M = \sigma \tilde{r}_{K-1}^n$$

$$\tilde{\pi}_{K-1}^M = \kappa \tilde{x}_{K-1}^M$$

where I denote each of the variables by a hat to state that it is their value conditional on the natural rate of interest being negative at that time. Compared to a solution that implies that $x_K^C > 0$ and $\pi_K^C > 0$ I can use the AS and the IS equations to write the inequalities:

$$\tilde{x}_{K-1}^C - \tilde{x}_{K-1}^M = x_K^C + \sigma \pi_K^C > 0$$

and

$$\tilde{\pi}_{K-1}^C - \tilde{\pi}_{K-1}^M = \kappa(\tilde{x}_{K-1}^M - \tilde{x}_{K-1}^C) + \beta \pi_K^C > 0$$

Using these two equation I can use the IS and AS equations at time $K-2$, (133)-(134), and the assumption about the natural rate of interest to write:

$$\tilde{x}_{K-2}^C - \tilde{x}_{K-2}^M = \alpha[(x_{K-1}^C - x_{K-1}^M) + \sigma(\pi_{K-1}^C - \pi_{K-1})] + (1-\alpha)[(\tilde{x}_{K-1}^C - \tilde{x}_{K-1}^M) + \sigma(\tilde{\pi}_{K-1}^C - \tilde{\pi}_{K-1})] > 0 \tag{135}$$

$$\tilde{\pi}_{K-2}^M - \tilde{\pi}_{K-2}^C = \kappa(\tilde{x}_{K-2}^M - \tilde{x}_{K-2}^M) + \alpha\beta\alpha(\pi_{K-1}^C - \pi_{K-1}) + (1-\alpha)\beta(\tilde{\pi}_{K-1}^C - \tilde{\pi}_{K-1}) > 0 \tag{136}$$

I can similarly solve the system backwards and write equation (135) and (136) for $K-2, K-3, ..., 0$ using at each time t the solution for $t+1$ and thereby the proposition is proofed.

8.5 Proof of Proposition 8

1. I first proof the that the solution for $t \geq \tau$ in the Markov solution is given by the one solution stated in the proposition. In the case of the Markov solution equation (98) indicates that $\phi_{3t} = 0$. When s is away from $\frac{\theta}{\theta-1}$ I can write (95) and (96) so that:

$$-\pi_t + \phi_{4t} = 0 \tag{137}$$

$$(x_t - x^*) + \theta\phi_{4t} = 0 \tag{138}$$

where $x^* = (\omega+\sigma^{-1})^{-1}(1-\frac{\theta-1}{\theta}(1+s))$. These two equation imply that $\pi_t = -\theta^{-1}(x_t - x^*)$.I can substitute this into the AS (53) equation and write the solution in terms of the difference equation:

$$(1+\theta\kappa)x_t - \beta E_t x_{t+1} = (1-\beta)x^* \tag{139}$$

This equation has a unique bounded solution given by $x_t = \frac{1-\beta}{1-\beta+\theta\kappa}x^*$ and it follows that the unique bounded solution for inflation is $\pi_t = \frac{\kappa}{1-\beta+\theta\kappa}x^*$.

2. The second part of the proposition follows exactly the same steps as the second part of Proposition 7.

8.6 Proof of Proposition 9

At time $t \geq \tau$ each of the variables evolve according to $w_t = w^1 w_{t-1}$ and $d\Lambda_t = \Lambda^1 w_{t-1}$, where the first element of the vector $d\Lambda_t$ is $d\pi_t = \pi^1 w_{t-1}$, the second $dY_t = Y^1 w_{t-1}$ and so on. To find the value of each of these coefficients I substitute this solution into the system (81)-(85) and (95)-(99) and match coefficients. For example equation (81) implies that

$$\bar{u}_c d''\pi^1 w_{t-1} + \theta(\bar{u}_{cc} - \bar{v}_{yy})Y^1 w_{t-1} - \bar{u}_c d''\beta\pi^1 w^1 w_{t-1} = 0 \tag{140}$$

where I have substituted for $d\pi_t = \pi^1 w_{t-1}$ and for $d\pi_{t+1} = \pi^1 w_t = \pi^1 w^1 w_{t-1}$. Note that I assume that $t \geq \tau$ so that there is perfect foresight and I may ignore the expectation symbol. This equation implies that the coefficients π^1, y^1 and w^1 must satisfy the equation:

$$\bar{u}_c d''\pi^1 + \theta(\bar{u}_{cc} - \bar{v}_{yy})Y^1 - \bar{u}_c d''\beta\pi^1 w^1 = 0 \tag{141}$$

I may similarly substitute the solution into each of the equation (81)-(85) and (95)-(99) to obtain a system of equation that the coefficients must satisfy:

$$\bar{u}_c d''\pi^1 + \theta(\bar{u}_{cc} - \bar{v}_{yy})Y^1 - \bar{u}_c d''\beta\pi^1 w^1 = 0 \tag{142}$$

$$\bar{u}_{cc}Y^1 - \beta\bar{u}_{cc}Y^1 w^1 - \beta\bar{u}_c i^1 + \beta\bar{u}_c\pi^1 w^1 = 0 \tag{143}$$

$$w^1 - \frac{1}{\beta} + \frac{1}{\beta}T^1 = 0 \tag{144}$$

$$S^1 - \bar{u}_c d'' \pi^1 w^1 = 0 \tag{145}$$

$$f^1 + \bar{u}_c \pi^1 w^1 - \bar{u}_{cc} Y^1 w^1 = 0 \tag{146}$$

$$-d_{\mathrm{J}} \bar{u}_c \pi^1 + \frac{s' \bar{g}_G}{\beta} + d'' \bar{u}_c \phi_4^1 = 0 \tag{147}$$

$$(\bar{u}_{cc} - \bar{v}_{yy}) Y^1 - \bar{u}_{cc} \beta \phi_3^1 + \theta(\bar{u}_{cc} - \bar{v}_{yy}) \phi_4^1 = 0 \tag{148}$$

$$s' \bar{g}_G T^1 - s' \bar{g}_G + \bar{u}_c \beta^2 \phi_3^1 = 0 \tag{149}$$

$$\bar{g}_{GG} (s')^2 T^1 - \bar{g}_G s'' T^1 + \beta^{-1} \phi_2^1 + \bar{g}_G s' i^1 = 0 \tag{150}$$

$$\phi_2^1 - \phi_2^1 w^1 - \beta \bar{g}_G s' i^1 w^1 + \bar{g}_G s' \pi^1 w^1 + \beta f^1 \phi_3^1 - \beta S^1 \phi_4^1 = 0 \tag{151}$$

There are 10 unknown coefficients in this system i.e. $\pi^1, Y^1, i^1, S^1, f^1, T^1, \phi_2^1, \phi_3^1, \phi_4^1, w^1$. For a given value of w^1, (142)-(150) is a linear system of 9 equations with 9 unknowns, and thus there is a unique value given for each of the coefficients as long as the system is non-singular (which it is easy to verify that it is for standard functional form of the utility and technology functions). This establishes the first part of the proposition. The second part of the proposition can be verified by considering conditions () and () local to the steady state. They are given by:

$$d\phi_{2t} - E_t d\phi_{2t+1} - \beta \bar{g}_G s' E_t di_{t+1} + \bar{g}_G s' E_t d\pi_{t+1} + \beta f_w d\phi_{3t} - \beta S_w d\phi_{4t} = d\gamma_{2t} \tag{152}$$

and

$$d\gamma_{2t} \geq 0 \tag{153}$$

Suppose that $w^1 > \frac{1}{\beta}$ (the economically interesting case). It can then be shown that the left hand side of (152) is negative so that (153) is violated. Finally that $\pi^1 > 0$ can be shown by manipulating (142)-(151).

9 Appendix C: Calibration for numerical results

In the numerical examples I assume the following functional forms for preferences and technology:

$$u(C, \xi) = \frac{C^{1-\sigma^{-1}} \bar{C}^{\sigma^{-1}}}{1 - \sigma^{-1}}$$

where $\bar{C}$ is a preference shock assumed to be 1 in steady state.

$$g(G, \xi) = g_1 \frac{G^{1-\sigma^{-1}} \bar{G}^{\sigma^{-1}}}{1 - \sigma^{-1}}$$

where $\bar{G}$ is a preference shock assumed to be 1 in steady state

$$v(H, \xi) = \frac{\lambda_1}{1 + \omega} H^{1+\omega} \bar{H}^{-\lambda_2}$$

where $\bar{H}$ is a preference shock assumed to be 1 in steady state

$$y = Ah^{\epsilon}$$

where A is a technology shock assumed to be 1 in steady state. I may substitute the production function into the disutility of working to obtain (assuming A=1):

$$\tilde{v}(Y,\xi_t) = \frac{\lambda_1}{1+\lambda_2} Y^{1+\lambda_2} \bar{H}^{-\omega}$$

When calibrating the shocks that generate the temporarily negative natural rate of interest I assume that it is the shock $\bar{C}$ that is driving the natural rate of interest negative (as opposed to A) since otherwise a negative natural rate of interest would be associated with a higher natural rate of output which does not seem to be the most economically interesting case. I assume that the shock $\bar{G}$ is such that the F_t would be constant in the absence of the zero bound, in order to keep the optimal size of the government (in absence of the zero bound) constant (see Eggertsson (2003) for details)). The cost of price adjustment is assumed to take the form:

$$d(\Pi) = d_1 \Pi^2$$

The cost of taxes is assumed to take to form:

$$s(T) = s_1 T^2$$

Aggregate demand implies$Y = C + F = C + G + s(F)$. I normalize $Y = 1$ in steady state and assume that the share of the government in production is $F = 0.3$. Tax collection as a share of government spending is assumed to be $\gamma = 5\%$ of government spending. This implies

$$0.1 = \frac{s(F)}{F} = s_1 F$$

so that $s_1 = \frac{\gamma}{F}$. The result for the inflation and output gap response are not very sensitive to varying γ under either commitment or discretion. The size of the public debt issued in the Markov equilibrium, however, crucially depends on this variable. In particular if γ is reduced the size of the debt issued rises substantially. For example if $\gamma = 0.5\%$ the public debt issued is about ten times bigger than reported in the figure in the paper. I assume that government spending are set at their optimal level in steady state giving me the relationship (see Eggertsson 2003b for details on how this is derived)

$$g_2 = \frac{u_c}{g_G - s' g_G} = \frac{C^{-\sigma^{-1}}}{G^{-\sigma^{-1}}(1-s')} = (\frac{G}{C})^{\sigma^{-1}} \frac{1}{1-s'} = (\frac{G}{C})^{\sigma^{-1}} \frac{1}{1-2s_1 F}$$

The IS equation and the AS equation are

$$x_t = E_t x_{t+1} - \tilde{\sigma}(i_t - E_t \pi_{t+1} - r_t^n)$$

$$\pi_t = k\pi_t + \beta E_t \pi_{t+1}$$

I assume, as Eggertsson and Woodford, that the interest rate elasticity, $\tilde{\sigma}$, is 0.5. The relationship between σ and $\tilde{\sigma}$ is

$$\sigma = \tilde{\sigma} \frac{Y}{C}$$

I assume that κ is 0.02 as in Eggertsson and Woodford (2003). The relationship between κ and the other parameters of the model is$\kappa = \theta\frac{(\tilde{\sigma}^{-1}+\lambda_2)}{d''}$. I scale hours worked so that $Y = 1$ in steady state which implies

$$v_y = \lambda_1$$

Since $u_c = \tilde{v}_y$ in steady state I have that

$$\theta = 7.87$$

Finally I assume that $\theta = 7.89$ as in Rotemberg and Woodford and that $\lambda_2 = 2$. The calibration value for the parameters are summarized in the table below:

Table 2

σ	0.71
g_1	0.33
λ_1	1.65
λ_2	2
d_1	787
s_1	0.17
θ	7.87

10 References (to be completed)

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