

Default Risk and Aggregate Fluctuations in Emerging Economies

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Abstract

Sovereign default in emerging countries is accompanied by interest rates spikes, deep recessions, and sharp real exchange rate devaluations. This paper develops a two sector small open economy model to study default risk and its interaction with output, consumption and real exchange rates. Default probabilities and interest rates depend on incentives for repayment. Default occurs in equilibrium because asset markets are incomplete. The model predicts that default incentives and default premia are higher in recessions, as observed in the data. The reason is that in a recession, a risk averse borrower finds it more costly to repay non-contingent debt and is more likely to default. In a quantitative exercise, the model matches several features of the Argentinian economy. The model can account for the recent default episode and the dynamics observed prior to default: higher interest rate premia, capital outflows, real exchange rate depreciation, and collapses in consumption. An important implications of the model is that economies with relatively small tradable sectors have higher incentives to default on dollar denominated debt and thus have larger default probabilities.

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1 Introduction

Emerging markets tend to have volatile business cycles and experience economic crises more frequently than developed economies. Recent evidence suggests that this may be related to cyclical changes in the access to international credit. In particular, emerging market economies face volatile and highly counter-cyclical interest rates, usually attributed to counter-cyclical default risk.¹ In addition, interest rates and default risk are also systematically correlated with real exchange rate devaluations: devaluations increase the probability of default and default increases the probability of devaluations. The exposure of the economy to such devaluations is magnified by high levels of liability dollarization - that is, "dollar" denominated debt, leveraged to a large extent on domestic income and assets.² Figure (1) illustrates these correlations by plotting aggregate consumption, real exchange rates, interest rates and the current account for Argentina.³ In December 2001, Argentina defaulted on its international debt and fell into a deep economic crisis. During the crisis, consumption and real exchange rates collapsed, interest rates increased, and the current account experienced sharp reversals.⁴ This evidence indicates that a priority for theoretical work in emerging markets macroeconomics must be understanding markets for international credit and in particular the joint analysis of default risk, interest rates and real exchange rates.

This paper develops a stochastic general equilibrium model with endogenous default risk. The model allows the study of the relation between interest rates, real exchange rates and output, shedding light on potential mechanisms generating the co-movements described above. The terms of international loans are endogenous to domestic fundamentals and depend on incentives to default. The paper extends the approach developed by Eaton and Gersovitz (1981) in their seminal study on international lending, by analyzing how endogenous default probabilities and fluctuations in output are related. In addition, the model considers a default penalty that is limited to temporary financial autarky, and introduces a nontradable sector to analyze dynamics of interest rates and real exchange rates. The paper characterizes the equilibrium country interest rate schedule and its relation with output, and provides conditions under which default can be an equilibrium outcome. In a quantitative exercise the model is applied to analyze the default experience of Argentina. The model can account for

¹Neumeyer and Perri (2001) and Uribe and Yuen (2003) document the counter-cyclicality of country interest rates for Argentina, Brazil, Ecuador, Mexico, Peru, Philippines, and South Africa.

²Reinhart (2002), Calvo and Reinhart (2000), Hausmann et. al (2001) document the high levels of liability dollarization in emerging markets foreign debt and its effect on crises and devaluations.

³Consumption and real exchange rate data are log, current account data are reported as percentage of output. Consumption and current account data are filtered with a linear trend.

⁴Mendoza (2002) documents similar dynamics for all the emerging market crises in the 1990's.

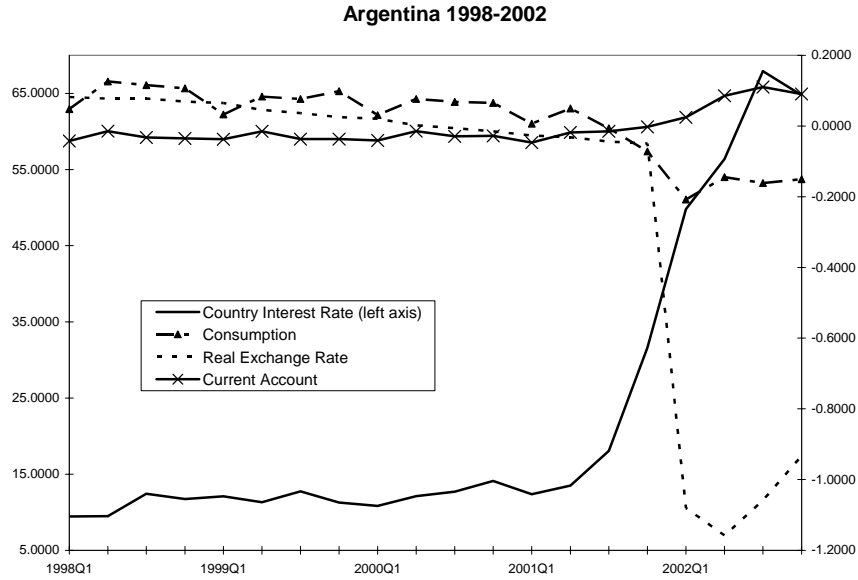


Figure 1:

the recent default in Argentina and for the sharp aggregate dynamics in the default episode.

The model in this paper accounts for the empirical regularities in emerging markets as an equilibrium outcome of the interaction between risk-neutral competitive creditors and a risk averse borrower that has the option to default. The borrower is a benevolent government of a small open economy with two sectors, tradables and nontradables. The government in this economy can buy or sell tradable denominated bonds with foreign creditors. The model starts from the premise that default probabilities are endogenous to the economy's incentives to default and they affect the equilibrium interest rate the economy faces. Default is an equilibrium outcome of the model because the asset structure is incomplete since it includes only one period discount bonds that pay a non-contingent face value. Asset incompleteness is necessary in this framework to study time-varying default premia due to equilibrium default. With non-contingent assets, risk neutral competitive lenders are willing to provide debt contracts that in some states will result in default by charging a higher premium on these contracts. In addition to more closely reflecting the actual terms of international contracts where foreign debt is largely dollar denominated and contracted at non-contingent interest rates, this market structure has the potential to deliver counter-cyclical default risk, since repayment of non-contingent, nonnegotiable loans in low output, low consumption times is more costly than repayment in boom times. The existence of nontradable goods is important because even though foreign debt cannot be used to smooth consumption of nontradable

goods, nontradable output fluctuations affect repayment incentives and equilibrium interest rates through changes in the real exchange rate.

In the first part of the paper, a simpler version of the model with *i.i.d.* shocks is considered in order to characterize analytically the equilibrium properties of credit markets and in particular the conditions under which default is an equilibrium outcome. It is shown that default is more likely the lower the output of tradables. This result contrasts with standard participation constraint models that have a complete set of assets, which have the feature that default incentives are higher in good times. The key intuition for why asset market incompleteness can reverse this outcome is that after a long series of low endowment shocks, an economy with incomplete markets can experience capital outflows in bad times. Risk averse agents with large debt holdings that can experience capital outflows have more incentives to default in times of low shocks. In essence, in times of low output, the asset available to the economy does not help agents smooth consumption, thus it is not very valued and default can be preferable than repayment. A similar intuition holds for the relationship between the level of nontradable output and default, although here it depends on how nontradable shocks affect the marginal utility of tradable consumption. We find that default is more tempting for low nontradable shocks when a low nontradable shock increases the marginal utility of tradable consumption. When this condition holds, capital outflows are more costly for lower nontradable shocks and thus default more likely.

In the quantitative part of the paper, the model is calibrated to Argentina to study its recent default episode. The model can account for the recent default of Argentina and can match the observed business cycle correlations. Specifically, it can account for the negative correlations between output and consumption with interest rates. It also matches the positive correlation between real exchange rate devaluations and interest rates. The model matches the data in that before a default occurs, the economy faces high interest rates and features sharp reductions in capital inflows, sharp reversals from large current account deficits into much smaller deficits or even surpluses, and collapses in consumption and real exchange rates. The relative sizes of the tradable and nontradable sectors are also very important. We find that relatively large nontradable sectors reduce the economy's commitment for debt repayment and further limit the financial integration of economies. A model with only a tradable sector would face much lower equilibrium interest rates and default would be a much less likely outcome.

The paper is related to several studies from different strands of the macroeconomics and finance literatures. Some papers have looked at the relation between interest rates and business cycles. Neumeyer and Perri (2001) model the effect that exogenous interest rate

fluctuations have on business cycles and find that interest rate shocks can account for 50% of the volatility of output. Uribe and Yue (2003) construct an empirical VAR to uncover the relationship between country interest rates and output, and then estimate with a theoretical model this relationship. They find that country spreads explain 12% of movements in output, and that output explains 12% in the movements of country interest rates. These papers, however, do not model endogenous country spreads responding to default probabilities.

The debt contractual arrangement in this paper is related to the optimal contract arrangements in the presence of commitment problems such as the analysis by Kehoe and Levine (1993) and Kocherlakota (1996). These studies, however, assume that a complete set of contingent assets are available and search for allocations that are efficient subject to a lack of enforceability. Alvarez and Jermann (2000) show that in this framework each state-contingent asset is associated with a state-contingent borrowing limit. This limit is such that in each state the borrowing country prefers to repay loans rather than default. While it is useful to be able to characterize allocations under the efficiency benchmark, this market structure may not be useful as a framework for understanding actual emerging markets. First, default never arises in equilibrium, so default risk premia are never observed. Second, default incentives in this class of models are typically higher in periods of high output, which is when efficiency dictates loan repayment. These features put these models at odds with the empirical evidence regarding default risk in emerging markets where bond yields are counter-cyclical and where debt prices reflect largely the risk of default. This paper delivers the correct empirical prediction because it assumes an incomplete set of assets where default occurs with a positive probability. In this regard the paper is specially related to the analysis on unsecured consumer credit with the risk of default by Chatterjee, Corbae, Nakajima and Rios Rull (2002) where they model equilibrium default in an incomplete markets setting. In some recent work Aguiar and Gopinath (2004) introduce shocks to the growth rate to a model similar to one developed here and find that these shocks can help account for the positive correlation between current account and interest rates at the cost of generating acyclical interest rates.

Other authors have developed models to study liability dollarization and have concentrated on how the level and volatility of output and relative price of nontradables affect the ability to pay dollar debt. Calvo (1998) shows how collapses in the price of nontradables due to constraints on tradable denominated debt can lead to crises by limiting the ability for debt repayment. Mendoza (2002) examines a stochastic environment where households face a liquidity requirement borrowing constraint (in which debt cannot be larger than an exogenous fraction of current income) to study emerging markets crises. He shows how crises can be the outcome of the equilibrium dynamics of an economy with imperfect credit mar-

kets. In these models though, the borrowing constraints do not guarantee that debtors have incentives to repay debts. In addition, they do not model the endogeneity of interest rates and the relation between interest rates and liability dollarization.

The focus in this paper is on understanding how the level and volatility of tradable and nontradable output affect incentives to default and thus equilibrium country interest rates in an environment of liability dollarization. Results match the empirical facts in that default incentives are higher when the economy has large debt positions, is in recession, and has relatively small tradable sectors.

The paper is organized as follows: section 2 presents the theoretical model, section 3 characterizes the equilibrium, section 4 assesses the quantitative implications of the model in explaining the data, and section 5 concludes.

2 The Model Economy

Consider small open economy with two sectors, a tradable sector and a nontradable sector. A set of one period discount bonds is available to the government of the small open economy at contingent prices. Debt contracts are not enforceable as the government can choose to default in its debt contracts if it finds it optimal. If the government defaults in its debts, it is assumed to be temporarily excluded from international financial markets and that a portion of the aggregate endowment is lost during the autarky periods. The economy trades discount bonds with risk neutral, competitive foreign creditors.

Households are identical and have preferences given by

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t^T, c_t^N) \tag{1}$$

where $0 < \beta < 1$ is the discount factor, c^T and c^N are consumption of the tradable and nontradable goods, and $u(\cdot)$ is strictly concave and increasing, and twice continuously differentiable. Households receive stochastic endowment streams of tradable y^T and nontradable y^N goods. The exogenous state vector of sector-endowment shocks is defined as $y \equiv (y^T, y^N) \in Y$. Shocks are assumed to have a compact support and to be a Markov process drawn from a probability space $(y, B(y))$ with a transition function $f(y'|y)$. In addition households receive a transfer of tradable goods from the government in a lump sum fashion. Households trade in the spot market tradable and nontradable goods with p^N being the relative price of nontradable goods.

The government is benevolent in that its objective is to maximize the utility of households. The government has access to international financial markets, where it can buy one period discount bonds B' denominated in terms of tradables at price q . The government also decides whether to repay or default on its debt. The bond price function $q(B', y)$ depends on the size of the bond B' and on the aggregate shock y because default probabilities depend on both. A purchase of a discount bond with a positive value for B' means that the government has entered into a contract where it saves $q(B', y) B'$ units of period t tradable goods to receive $B' \geq 0$ units of tradable goods the next period. A purchase of a discount bond with negative face value for B' means that households have entered into a contract where they receive $q(B', y) B'$ units of period t tradable goods, and promise to deliver, conditional on not declaring default, B' units of tradable goods the next period. The government rebates back to the households all the proceedings from its international credit operations in a lump sum fashion.

When the government chooses to repay its debts, the tradable sector market clearing condition is the following:

$$c^T = y^T + B - q(B', y)B'$$

The market clearing condition for the nontradable sector, requires that nontradable consumption equals nontradable output at all times.

$$c^N = y^N$$

Given that the government is benevolent, it effectively uses international borrowing to smooth consumption. But the idiosyncratic income uncertainty induced by y^T and y^N cannot be insured away with the set of bonds available to the economy, which pay a time and state invariant amount of tradable goods only. That is, asset markets in this model are incomplete not only because of the endogenous default risk, but also because of the set of assets available.

If the government defaults, it is assumed that current debts are erased and that it is not able to save or borrow. The government will remain in financial autarky for a stochastic number of periods. There is an exogenous constant probability θ that the government can reenter the market. This is a simple way to model that governments that default on their international debt lose access to international financial markets only temporarily. In addition, in the period when the government defaults, endowments fall a proportion $(1 - \lambda)$, where $\lambda \leq 1$. The assumption that default may reduce output can be rationalized by the common

view that after default there is a disruption in the countries' ability to engage in international trade, and this reduces the value of output (Cole and Kehoe 2000, Conklin 1998, Obstfeld and Rogoff 1989).

When the government chooses to default market clearing conditions require that consumption equals output :

$$c^T = \lambda y^T$$

$$c^N = \lambda y^N$$

Foreign creditors have access to an international credit market in which they can borrow or lend as much as needed at a constant international interest rate $r > 0$. Creditors have perfect information regarding the economy's endowment processes and can observe each period the endowment levels.⁵ Creditors are assumed to be risk neutral and behave competitively. Creditors engage in Bertrand competition, offering contracts to the government that gives them expected zero profits.

They choose loans B' to maximize expected profits ϕ , taking as given prices:

$$\phi = -qB' + \frac{(1 - \delta)}{1 + r}B' \quad (2)$$

where δ is the probability of default.

For positive levels of foreign asset holdings, $B' \geq 0$, the probability of default is zero and thus the price of a discounted bond will be equal to the opportunity cost for investors. For negative asset holdings, $B' < 0$, the equilibrium price accounts for the risk of default that creditors are facing, that is, the price of a discount bond will be equal to risk adjusted opportunity cost.⁶ This requires that bond prices satisfy:

$$q = \frac{(1 - \delta)}{1 + r} \quad (3)$$

The probability of default δ is endogenous to the model and depends on the government incentives to repay debt. Since $0 \leq \delta \leq 1$, the zero profit requirement implies that bond prices q lie in the closed interval $[0, (1 + r)^{-1}]$. The country gross interest rate can be interpreted

⁵We assume for simplicity that each period the government enters into a debt contract with only one creditor. This can be generalized by assuming that creditors can observe previously written contracts and contracts that are written first have to be honored first.

⁶Risk adjustment in this framework is not due to compensation for risk aversion as lenders are risk neutral. It reflects the risk neutral compensation for a lower expected payoff.

as the inverse of the discount bond price, $1 + r^c = \frac{1}{q}$.

The timing of decisions within each period is as follows. The government starts with initial assets B , observes the endowment shock y , and decides whether to repay its debt obligations or default. If the government decides to repay, then taking as given the schedule bond price $q(B', y)$ the government chooses B' subject to the resource constraints. Then creditors taking as given q choose B' . Finally consumption of tradable and nontradable goods c^T, c^N takes place.

3 Recursive Equilibrium

We define a recursive equilibrium in which the government does not have commitment and in which the government, foreign creditors and households act sequentially. Given aggregate states $s = (B, y)$, the policy functions for the government B' , the price function for bonds q and the policy functions for the consumers c^T, c^N together with the relative price p^N determine the equilibrium.

Households' problem is static. They observe their endowment shocks and taking as given the aggregate states, government transfers and the nontradable price they choose optimal consumption. Their first order condition equates the marginal rate of substitution between nontradable and tradable consumption to the relative price.

$$\frac{u_{c^N}(c^T, c^N)}{u_{c^T}(c^T, c^N)} = p^N \quad (4)$$

Given that purchasing power parity (PPP) is assumed to hold for the tradable sector, the real exchange rate for this small economy is the domestic consumption price index p^c which is a function of the nontradable price, p^N , and the tradable price, which is normalized to 1.

Foreign creditors in the model are risk neutral and behave competitively. They lend the amount of bonds wanted by the government as long as the price satisfies

$$q(B', y) = \frac{(1 - \delta(B', y))}{1 + r}$$

Foreign creditors in the model are pretty passive, they lend bonds as long as the gross return on the bonds equals $1 + r$.

The government observes the aggregate endowment shocks y , and given initial foreign assets B , chooses whether to repay or to default. If the government chooses to repay its debt obligations and remain in the contract, then it chooses the new level of foreign assets B' .

The government understands that the price of new borrowing $q(B', y)$ depends on the states y and on its choice of B' . The government also understands that its choices will affect the households choices c^T , c^N and p^N and internalizes the domestic market clearing conditions. The government objective is to maximize the lifetime utility of households.

Define $v^o(B, y)$ as the value function for the government who has the option to default, and that start the current period with assets B and endowment shocks y . The government decides on whether to default or repay its debts, to maximize the welfare of households. Note that only when the government has debt (i.e. negative assets) could the default option be optimal.

Given the option to default, $v^o(B, y)$ satisfies:

$$v^o(B, y) = \max_{\{c, d\}} \{v^c(B, y), v^d(y)\} \quad (5)$$

where $v^c(B, y)$ is the value associated with not defaulting and staying in the contract and $v^d(y)$ is the value associated with default.

When the government defaults, the economy will be in temporary financial autarky; θ is the probability that it will regain access to international credit markets. The value of default is given by the following:

$$v^d(y) = u(\lambda y^T, \lambda y^N) + \beta \int_{y'} [\theta v^o(0, y') + (1 - \theta)v^d(y')] f(y'|y) dy' \quad (6)$$

If the government defaults endowments fall, and consumption equal output.

When the government chooses to remain in the credit relation, the value conditional on not defaulting is the following

$$v^c(B, y) = \max_{(B')} \left\{ u(y^T - q(B', y)B' + B, y^N) + \beta \int_{y'} v^o(B', y') f(y'|y) dy' \right\} \quad (7)$$

The government decides on optimal policies B' to maximize utility. The decision to remain in the credit contract and not default is a period by period decision. That is, the expected value from next period forward incorporates the fact that the government could choose to default in the future. The government also faces a lower bound on debt, $B' \geq -Z$, which prevents the government from running ponzi schemes but is otherwise not binding in equilibrium.

The government default policy can be characterized by default sets and repayment sets. Let $A(B)$ be the set of y 's for which repayment is optimal when assets are B such that⁷:

⁷We assume that if the government is indifferent between repayment and default, default is chosen.

$$A(B) = \{y \in Y : v^c(B, y) > v^d(y)\}$$

and $D(B) = \tilde{A}(B)$ be the set of y 's for which default is optimal for a level of assets B , such that:

$$D(B) = \{y \in Y : v^c(B, y) \leq v^d(y)\} \quad (8)$$

The value function can then be represented more precisely by the following dynamic problem where the government decides on optimal borrowing taking into account that its choice on assets may imply defaulting in some states.

$$v^c(B, y) = \max_{(B')} \left\{ u(y^T - q(B', y)B' + B, y^N) + \beta \left[\int_{A(B')} v^c(B', y') f(y'|y) dy' + \int_{D(B')} v^d(y') f(y'|y) dy' \right] \right\}$$

The above centralized government borrowing problem can be decentralized in multiple ways, with the simplest being lump sum taxation as presented here. In a separate appendix, it is shown that the above problem can also be decentralized as in Kehoe and Perri (2004) by letting households borrow directly from foreign creditors. The government in this case makes the economy wide default decision, and levies a savings tax that gives households the right incentives for holding the optimal level of debt. For simplicity in the exposition we have assumed here that the government has access to lump sum taxes.

Having developed the problem for each of the agents in the economy, the equilibrium is defined. Let $s = \{B, y\}$ be the the aggregate states for the economy.

Definition 1. *The recursive equilibrium for this economy is defined as a set of policy functions for (i) consumption $c^T(s)$ and $c^N(s)$, nontradable price $p^N(s)$ (ii) government's asset holdings $B'(s)$, repayment sets $A(B)$ and default sets $D(B)$, and (iii) the price for bonds $q(s, B')$, such that:*

1. *Taking as given the government policies, policy functions for households $c^T(s)$, $c^N(s)$, and the relative price $p^N(s)$ satisfy the households optimization problem and domestic market clearing conditions hold.*
2. *Taking as given the bond price function $q(B', y)$, the government's policy functions $B'(s)$, repayment sets $A(B)$ and default sets $D(B)$, satisfy the government optimization problem.*

3. Bonds prices $q(B', y)$ are such that they reflect the government default probabilities and they are consistent with creditor's expected zero profits such that the loan market clears.

The equilibrium bond price function $q(B', y)$ has to be consistent with government's optimization and with expected zero profits for foreign creditors. That is, q correctly assesses the probability of default of the government.⁸

Default probabilities $\delta(B', y)$ and default sets $D(B')$ are then related in the following way:

$$\delta(B', y) = \int_{D(B')} f(y'|y) dy' \quad (9)$$

When default sets are empty, $D(B') = \emptyset$, equilibrium default probabilities $\delta(B', y)$ are equal to 0. That is, the economy with assets B' never chooses to default for all realization of the endowment shocks. When $D(B') = Y$, default probabilities $\delta(B', y)$ are equal to 1. More generally, default sets are shrinking in assets, as the following proposition shows:

Proposition 1. (Default sets are shrinking in assets) *For all $B^1 \leq B^2$, if default is optimal for B^2 , in some states y , then default will be optimal for B^1 for the same states y , that is $D(B^2) \subseteq D(B^1)$.*

Proof. See Appendix.

This result is similar to Eaton and Gersovitz (1981) and Chatterjee, et al. (2002). Intuitively, the result follows from the property that the value of staying in the contract is increasing in B , and that the value of default is independent of B . As assets decrease, the value of the contract monotonically decreases, while the value of default is constant. Thus, if default is preferred for some level of assets B , for a given state y , the value of the contract is less than the value of default. As assets decrease, the value of the contract will be even lower than before, and so default will continued to be preferred.

Since stochastic shocks are assumed to have a bounded support and the value of the contract is monotonically decreasing as assets fall, there exists a level of B that is low enough, such that for all endowment shocks default is optimal and default sets are equal to the entire endowment set. On the other hand, given that default can only be preferable when assets have a negative value (i.e. when the government is holding debts), there exists a level of assets $B \leq 0$, such that default sets are empty. These two properties of default sets can be summarized as follows.

⁸Chatterjee et al (2001) prove the existence of an equilibrium price schedule in a similar environment for their work on consumer default risk. We conjecture that the existence proof for this model follows that of Theorem 4 in their paper.

Definition 2. Denote \underline{B} the upper bound of assets for which the default set constitutes the entire set and let \overline{B} be the lower bound of assets for which default sets are empty, where $\underline{B} \leq \overline{B} \leq 0$ due to Proposition 1.

$$\underline{B} = \sup \{B : D(B) = Y\}$$

$$\overline{B} = \inf \{B : D(B) = \emptyset\}$$

For asset holdings greater than \overline{B} , default is never optimal for all y and equilibrium bond prices are equal to $(1+r)^{-1}$ because default probabilities are zero.⁹ For asset holdings $B \leq \underline{B}$ default is always optimal and equilibrium prices for these bonds are zero because default probabilities are one. Given that default sets are shrinking in the level of assets, condition (9) implies that equilibrium default probabilities are decreasing in B' , and the equilibrium price function $q(B', y)$ is an increasing function of B' . Lower levels of assets will be associated with larger default probabilities, and thus discount prices for those bonds will be lower to compensate risk neutral investors for a lower expected payoff. That is, larger loans are generally more expensive. Equilibrium bond prices are also contingent on the endowment shocks, because the probability distribution from which shocks are drawn the next period depends on today's shocks. Since the risk of default varies with the level of debt and depends on the stochastic structure of the endowment shocks, competitive risk neutral pricing requires that equilibrium bond prices depend on both B' and y .

3.1 Case of *i.i.d.* Tradable Endowment Shocks

This section characterizes the equilibrium bond price function and the default decision for the case of a constant nontradable endowment and *i.i.d.* tradable endowment shocks. When endowment shocks are *i.i.d.*, equilibrium bond prices are independent of the endowment realization and are only a function of the level of assets $q(B')$ because today's shock gives no information on the likelihood of tomorrow's shock. We will assume that $\lambda = 1$, no output loss in autarky, and $\theta = 0$, financial autarky is permanent after default. The results can be generalized for other parameters of λ and θ .

Proposition 2. *Default incentives are stronger the lower the tradable endowment. For all $y_1^T < y_2^T$, if $y_2^T \in D(B)$, then $y_1^T \in D(B)$.*

⁹Zhang (1997) introduced this debt limit as the no default debt limit in his work on participation constraints under incomplete markets.

Proof. See Appendix.

This result comes from the property that utility is increasing and concave in tradable consumption and by noting that default can be optimal only if under no default the economy experiences net capital outflows ($B - q(B')B' < 0$). In fact, when for some state default is optimal, there are no contracts available to the government such that the economy can experience capital inflows given that level of debt for all states. Given that utility is increasing and concave in consumption, and that the economy is not able to borrow more when it has the low endowment, repayment is more costly in this low endowment state and thus default more likely.

Intuitively, the asset available to the economy is not a very useful insurance instrument for a highly indebted economy, because in times of a low endowment it cannot raise enough resources to smooth consumption. Thus, the asset the economy is giving up is not very valuable and default may be preferable in times of low endowments.

Endowment shocks have generally two opposing effects on default incentives. When output is high, the value of default is relatively high, increasing default incentives. But at the same time, when output is high, the value of repayment is relatively higher too, decreasing default incentives. With an incomplete set of assets, and *i.i.d.* shocks, the latter effect dominates and thus default is more likely the lower the tradable endowment.

This result contrasts with the participation constraint models that have a complete set of contingent assets. These models have the feature that default incentives are higher in times of good endowment shocks. In fact, for small open economy models with participation constraints and a complete set of contingent assets, default incentives are always higher in the good states because only autarky is affected by the current endowment, as the value of staying in the contract is constant in the long run and independent of the economy's specific endowment.

Proposition 3. *If default sets are non-empty, then they are closed intervals where only the upper bound depends on the level of assets:*

$$D(B) = [\underline{y}, y^*(B)] \quad \text{for } B \leq \bar{B} \quad (10)$$

where $y^*(B)$ is a continuous, non-increasing function of B , such that:

$$y^*(B) = \left\{ \begin{array}{ll} y^*(B) : v^d(y^*(B)) = v^c(B, y^*(B)) & \text{for } \underline{B} \leq B \leq \bar{B} \\ \bar{y} & \text{for } B < \underline{B} \end{array} \right\} \quad (11)$$

Proof. See Appendix.

Proposition 3 proves that for $B \in [\underline{B}, \overline{B}]$, there will be a unique y^* where the contract value and continuation value cross. Default sets can then be characterized solely by a closed interval where only the upper bound is a function of the level of assets. For a given a level of assets $B \in [\underline{B}, \overline{B}]$, default will be optimal for endowment levels less than or equal to $y^*(B)$, and repayment will be optimal for endowment levels greater than $y^*(B)$. The function $y^*(B)$ is the default boundary that divides the y, B space into the repayment and default regions.¹⁰

The first order condition of the government with respect to asset holdings can be presented more sharply in this case by the following condition:

$$\frac{\partial[q(B')B']}{\partial B'} u_{c^T} = \beta \int_{A(B')} v_{B'}^c(B', y') f(y') dy' \quad (12)$$

Equation (12) equates the marginal utility of consumption today to the expected marginal value of wealth tomorrow for the states where repayment is optimal. The marginal cost from borrowing a loan of size B' in the current period is the expected marginal disutility of consumption from repaying that loan the following period. As opposed to standard intertemporal conditions for models without default, here that cost is only experienced if in the following period the government choose to repay. That is, the cost of repaying is realized for the set of y 's, $A(B') = (y^*(B'), \overline{y}]$, for which repayment is optimal when the economy has assets equal to B' .

Given that default sets are such that only the upper bound depends on the level of debt, the equilibrium price function $q(B)$ can then be characterized by the following condition:

$$q(B) = \frac{1}{1+r} [1 - F(y^*(B))] \quad (13)$$

where F is the cumulative probability distribution of the stochastic endowments.

Equilibrium bond prices will fall in three ranges. For asset levels above \overline{B} prices are equal to the inverse of the risk free rate. For asset levels less than \underline{B} , prices are zero. For intermediate asset levels, $B \in [\underline{B}, \overline{B}]$ prices will be increasing in the level of assets because $y^*(B)$ is decreasing in this range. Note that the bond price function will be crucially dependent upon the probability distribution of the endowment shocks.

If initial bond values are B then the probability of default at every point of the state space is given by $F(y^*(B))$, which is greater than zero in for $B \in [-\infty, \overline{B}]$. The next issue to be addressed is whether the economy would ever choose a B' such that $D(B') \neq \emptyset$. That is if in

¹⁰In continuous time optimal stopping problems, it has been shown that under special cases not only the function is continuous at the boundary, but present a 'smooth pasting' condition. Which would imply that the derivatives with respect to y of the continuation and default values are equal at the boundary.

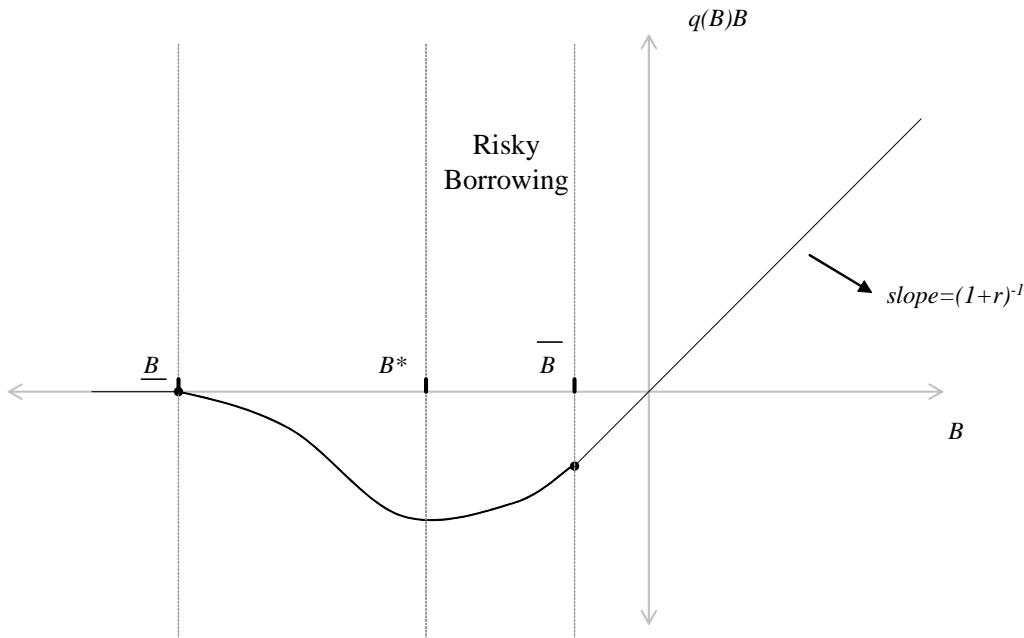


Figure 2: Total Resources Borrowed

the ergodic distribution of assets a point exists where default has a positive probability, such that if initial bond holdings are larger than \bar{B} , the model can have default as an equilibrium outcome. To clarify this issue, imagine the economy happens to start at $B \leq \underline{B}$, then in that period the economy would default with probability one. But given that discount prices are zero in this range, if the economy's initial bond holdings are greater than \bar{B} , the economy would never choose as optimal asset holdings levels of $B' \leq \underline{B}$, because it would get nothing this period, and would incur a liability the following period. The range of B' for which default can potentially be an equilibrium outcome is limited to $(\underline{B}, \bar{B}]$, because here is where default sets are non-empty and equilibrium prices are different from zero.

Intuitively, the necessary condition for default to be an equilibrium outcome of the model is that the equilibrium price function does not decrease "too fast" as assets decrease. For default to potentially be an equilibrium outcome, the economy must be able to find it optimal to borrow bonds less than or equal to \bar{B} such that the economy is exposed to the risk of default. And for this to ever be an optimal decision, the economy should be able to increase total resources borrowed $q(B)B$, that is, have a higher level of consumption, by choosing a lower level of assets at a lower price. This means that the equilibrium total resources borrowed $q(B)B$ needs to be increasing for some $B \in (\underline{B}, \bar{B})$. Figure (2) helps visualize this issue.

The slope of $q(B)B$ for $B > \bar{B}$ equals $(1+r)^{-1}$ because bond prices are constant and default probabilities are zero. The slope of $q(B)B$ for $B < \underline{B}$ is equal to zero, because bond prices are zero. For the intermediate range (\underline{B}, \bar{B}) the slope at equilibrium prices is:

$$\frac{\partial[q(B)B]}{\partial B} = \frac{1}{(1+r)} \left\{ [1 - F(y^*(B))] - f(y^*(B)) \cdot B \frac{\partial y^*}{\partial B} \right\} \quad (14)$$

Note that the sign of this derivative is ambiguous because bond positions $B \in (\underline{B}, \bar{B})$ are negative, and $y^*(B)$ is decreasing in B .¹¹ In fact if there exists some $B^* \in (\underline{B}, \bar{B})$ for which $\frac{\partial[q(B^*)B^*]}{\partial B^*} = 0$ this level of assets corresponds to the endogenous borrowing limit in the model. The government would never find optimal to choose a level of assets below B^* because it can always find some other contract such that consumption the current period increases by the same amount while incurring a smaller liability for next period. The region that is relevant for risky borrowing and thus for default to be an equilibrium outcome of the model is then $B \in (B^*, \bar{B})$.

The necessary condition for the above derivative to be positive within a range depends on the hazard function (i.e. $\frac{f(y)}{[1 - F(y)]}$) of the probability distribution in the neighborhood of \underline{y} relative to how fast the upper bound on the default sets increase with debt. The following proposition summarizes these findings:

Proposition 4. *Default can be an equilibrium outcome of the model for all probability distributions over $[\underline{y}, \bar{y}]$ satisfying the property:*

$$\lim_{B \rightarrow \bar{B}^-} \frac{1}{h(y^*(B))} > \lim_{B \rightarrow \bar{B}^-} B \frac{\partial y^*(B)}{\partial B}$$

where $h(y)$ is the hazard function of the distribution.

Proof. When the above condition holds, $\frac{\partial[q(B)B]}{\partial B} > 0$ in the neighborhood of \bar{B} from the left. Given that $y^*(B)$ is continuous by Proposition 3, total resources borrowed increase for lower levels of assets in the region where assets carry a default premium. \square

The hazard function $h(y^*(B))$ represents the instantaneous probability of default at B for $B \leq \bar{B}$. The above condition requires that the instantaneous probability of default as B approaches \bar{B} from the left is sufficiently small such that as B decreases (debt increases), the total resources borrowed increase. Due to Proposition 3, $y^*(\bar{B}) = \underline{y}$, and thus the condition is a restriction on the probability of the bad endowment shock. The government might then

¹¹In general $q(B)B$ may not be differentiable at the points \underline{B} and \bar{B} .

be willing to take on a risky loan, because in periods of low endowments it can increase consumption.

A sufficient condition for default to be an equilibrium outcome of the model is the following:

Corollary 4.1. *Default can be an equilibrium outcome of the model for all probability distributions over $[y, \bar{y}]$ satisfying the property:*

$$\lim_{y \rightarrow \underline{y}} h(y) = 0$$

where $h(y)$ is the hazard function of the distribution.

Proof. See Appendix.

For all probability distributions satisfying the above property, the model will present a region in the state space where engaging in risky borrowing can increase consumption or $\frac{\partial[q(B)B]}{\partial B} > 0$, making default a positive probability event.

3.2 Case of *i.i.d* Nontradable Endowments

Now the role of volatile nontradable endowment is explored and its effect on default incentives. Here it is assumed that nontradable endowments follow an *i.i.d.* stochastic process, and without loss of generality that $\lambda = 1$, and $\theta = 0$.

Proposition 5. *Default incentives are stronger for low nontradable endowments if the cross derivative of the utility function is negative. For all $y_1^N \leq y_2^N$, if $y_2^N \in D(B)$, then $y_1^N \in D(B)$ if $u_{c^T c^N} < 0$.*

Proof. See Appendix.

Given that assets are tradable denominated only, nontradable fluctuations affect default decisions by their effect on the utility of tradable consumption. When $u_{c^T c^N} < 0$, a low nontradable shock will tend to increase the marginal utility of tradable consumption, giving the government incentives for borrowing. In the region of default the economy experiences capital outflows and the fact that households cannot borrow as much as desired because of high interest rates and debt limits is relatively more costly for households who experience a low nontradable endowment if the above condition holds. Thus, default is more likely in the low nontradable states because repayment of tradable denominated loans is more costly.

For CES preferences it is well known that sign of this cross derivative depends on the relative magnitudes of the elasticity of intratemporal substitution between tradables and nontradables versus the intertemporal elasticity of substitution (see for example Obstfeld and Rogoff 1996 textbook). When the elasticity of intratemporal substitution between tradables and nontradables is greater than the intertemporal elasticity of substitution the cross derivative is negative, and otherwise its positive. Intuitively, decisions on optimal asset holdings depend on nontradable output fluctuations due to the effect on the relative price of nontradables. The intertemporal elasticity through time and intratemporal elasticity between tradable and nontradable consumption pull the time path of nontradable prices in opposite directions. When the intratemporal elasticity is greater than the intertemporal elasticity a low nontradable shock today tends to produce a decreasing time path in the nontradable price, which gives incentives for borrowing in the current period.

The analytical characterization of the equilibrium in this section is for i.i.d. shocks, with bond prices not depending on the endowments' realization and only increasing in the level of assets demanded. However given that debt is used for insurance purposes, the policy function for assets is increasing in the endowments as in Hugget (1993), so that when the economy is hit by negative endowment shocks, more debt is demanded. This generates in the time series a negative correlation between interest rates and endowments even for i.i.d. shocks because higher debt is associated with higher interest rates. The following section analyzes the relation between interest rates and output for a persistent endowment process and where the negative relation between output and interest rates remains.

4 Simulations

4.1 Data for Argentina

In December of 2001 Argentina experienced one of the largest defaults in recent history, defaulting on \$100 billion dollars of their external government debt which represented 37% of Argentina 2001 GDP. It also experienced a severe economic crises with output decreasing about 20% percent at the time of the default. This section documents some statistics of the Argentinian economy corresponding to the period of default.

Table 2 presents business cycle statistics for Argentinian data that are quarterly real series seasonally adjusted taken from the Ministry of Finance (MECON) from 1993 to 2001 and filtered with a linear trend¹². The interest rate series are the EMBI index for Argentina

¹²The series are constructed such that all variables are consistent with the model's statistics. Specifically,

which is taken from the dataset from Neumeyer and Perri (2004) and MECON. Output and consumption data are log, and the current account data are reported as a percentage of output. Real exchange rates are constructed from dollar nominal exchange rates (dollar per peso) using the ratio of the consumer price index for Argentina and the US. The table also presents the percent deviation of the variables in Q1 2002, the default period.

Table 1. Statistics for Argentina

	Default episode	Prior default Q1 1993 - Q4 2001		
	$x : Q1 - 2002$	$std(x)$	$corr(x, y)$	$corr(x, r)$
Interest Rates Spread	20.55	4.11	-0.58	
Consumption	-13.24	5.13	0.87	-0.77
Tradable Output	-15.79	5.72	0.76	-0.67
Nontradable Output	-13.17	4.88	0.87	-0.74
Aggregate Output	-13.25	4.68		-0.57
Current Account	2.45	1.61	-0.74	0.59
Real Exchange Rate	-72.94	2.44	0.63	-0.15

Aggregate output, tradable output and nontradable output are all negatively correlated with interest rates. This negative relations are much stronger in the default episode because in the crisis output plummeted and interest rates skyrocketed. Consumption is more volatile than nontradable output, less volatile than tradable output, and negatively correlated with interest rates. This negative relation is also magnified in the default episode. Real exchange rates prior to the default episode were extremely stable in this period, and slightly negatively correlated with interest rates. But in the default episode real exchange rates collapsed from 1 dollar per peso, to 0.4 dollars per peso or over 70% while interest rates spike by 20% giving a negative relation between devaluations an interest rates.

The current account is countercyclical which is accentuated in the time of the default because Argentina experienced a sharp current account reversal during the crisis as foreign credit dried up. The countercyclicity of the current account is due to the negative relation between the current account and nontradable output. The correlation between nontradable output and the current account is negative and equal to -0.73, while the correlation between the current account and tradable output is positive and equal to 0.12.

Interest rates in Argentina are slightly less volatile than nontradable output, and negatively correlated with output and consumption. The mean spread in Argentina defined as

aggregate output is denominated in terms of tradables and sectoral output and aggregate consumption is denominated in real terms deflated by the appropriate price deflator.

the difference in the US annual 5 year maturity bond yield and the Argentina EMBI is 7.38% in this period.

As the table shows all variables experienced very dramatic deviations at the time of the default.

4.2 Calibration and functional forms

The model is solved numerically to evaluate its predictions regarding accounting for default episodes and aggregate dynamics in crisis, and its quantitative implications for the business cycle properties of interest rates, real exchange rates and consumption. The main issue of interest in the quantitative analysis is to address whether adding an endogenous default decision to a very simple endowment model can help account for the real dynamics observed in emerging markets in times of default and crises. The benchmark model is calibrated to match certain features of the Argentinian economy.¹³

The following utility function is used in the numerical simulations:

$$u(c_t^T, c_t^N) = \frac{c(c_t^T, c_t^N)^{1-\sigma}}{1-\sigma},$$

where $c(c_t^T, c_t^N)$ is the constant elasticity of substitution aggregator

$$\left[\omega (c_t^T)^{-\eta} + (1 - \omega) (c_t^N)^{-\eta} \right]^{-\frac{1}{\eta}}.$$

The parameters of the benchmark model are calibrated to mimic some of the empirical regularities in the Argentinian economy or taken from other emerging markets studies. For the preference parameters, the risk aversion coefficient is set to 5 which is a common value use in real business cycles studies for emerging markets. The elasticity of substitution between tradable and nontradable consumption $1/(1+\mu)$ is taken from Gonzales and Neumeyer (2003), where they estimate the elasticity for Argentina to be equal to 0.48.

To calibrate the relative sizes of the tradable and nontradable sectors in Argentina, we follow the standard methodology of assessing the degree of tradability of goods by computing the share of total trade (exports plus imports) of each sector as a percentage of total sectoral gross output. We find that the agricultural, manufacturing and energy sectors have a high degree of tradability, with an average share in this period of 0.38, 0.78 and 0.34 respectively.

¹³The model is solved by a value function iteration algorithm that allows for the bond price vector to be endogenous. Specifically, endogenous and exogenous states are discretized, and the model is solved by iterating on the value function for an initial guess of the bond price vector. The bond price vector is updated with a Gauss-Seidel algorithm, using the creditors equilibrium zero profit condition. The procedure is repeated until the convergence criterion is met for the bond price vector.

Nontradable output includes construction and all service sectors, which have a degree of tradability of less than 5%. An interesting fact to note is that in Argentina the size of the tradable sector is small, constituting only 26% of output. We normalize mean $y^T = 1$, and then set mean y^N equal to 2.78.

The weight on tradable consumption in the CES aggregator, ω , is set to normalize the relative price of nontradables to be equal to one in the steady state when the economy is in autarky. The probability of reentering financial markets after default, θ is set to 0.5 which is in line to the estimates of Gelos et. al. (2002). They find that during the default episodes of the 1990's economies were excluded from the credit markets on average less than 1 year. For the benchmark calibration, the fraction of output lost in times of default, $(1 - \lambda)$, is set equal to 0.02, which is the percent in output contraction estimated by Puhon and Sturzenegger (2002) following the default episodes in the 1980's in Latin America. In the sensitivity analysis we explore the effects of other default penalties.

The time preference parameter β is set to 0.84 which is lower than standard business cycle studies. We need a relatively low β for default to arise in equilibrium. Although lower β 's allows somewhat higher default probabilities, the relation is not monotonic. If for example $\beta = 0$, no debt could be allowed in equilibrium and thus in the limiting distribution the default probability will be equal to zero. Table 2 summarizes the parameter values.

Table 2.

Parameter Values

Elasticity of Substitution	$1/(1 + \eta) = 0.48$	Gonzales and Neumeyer (2003)
Weight on CES	$\omega = 0.1061$	Normalization $p^N = 1$ in autarky steady state
Tradable Share	$y^N/y^T = 2.78$	$y^N/y^T = 2.78$ Argentina
Risk Free Interest Rate	$r = 0.01$	US quarterly interest rate
Output loss in default	$1 - \lambda = 0.02$	Sturzenegger (2003)
Probability of Re-entry	$\theta = 0.5$	Gelos et al. (2002)
Discount Factor	$\beta = 0.84$	
Risk Aversion	$\sigma = 5$	

The stochastic processes for the sectoral output are estimated jointly, from the Argentinian linearly detrended data, as AR(1) processes where the innovations to the shocks are allowed to be correlated.

$$\begin{bmatrix} y_t^T \\ y_t^N \end{bmatrix} = \begin{bmatrix} \rho^T & \rho^{TN} \\ \rho^{NT} & \rho^N \end{bmatrix} \begin{bmatrix} y_{t-1}^T \\ y_{t-1}^N \end{bmatrix} + \begin{bmatrix} \varepsilon_t^T \\ \varepsilon_t^N \end{bmatrix}$$

$E[\varepsilon^T] = E[\varepsilon^N] = 0$ and the variance-covariance matrix of the error terms is the following:

$$E[\varepsilon^{T'} \varepsilon^N] = \begin{bmatrix} \sigma^T & \sigma^{TN} \\ \sigma^{TN} & \sigma^N \end{bmatrix}$$

The above VAR process is estimated by a seemingly unrelated regression method using a two step procedure to get the GLS estimates. The following are the estimated coefficients: $\rho^N = 0.6578$, $\rho^{NT} = 0.4$, $\rho^{TN} = 0.36$, $\rho^T = 0.34$, $\sigma^T = 0.0005$, $\sigma^N = 0.0012$, $\sigma^{TN} = 0.0001$.

Each shock is then discretized into a 7 state Markov chain by using a quadrature based procedure (Hussey and Tauchen 1991) from their joint distribution.

4.3 Simulation Results

The model can predict the recent default in Argentina. We feed into the model the sectoral shocks from the data and the model predicts a default in the first quarter of 2003, which is within year of the default in Argentina. Figures (4) and (3) show the time series of the sectoral output fluctuations and interest rates in Argentina and in the model. The model predicts the relatively higher interest rates experienced in Argentina in 1995-1996 and in 2001-2002 which ended in a default. The main significant discrepancy between the model and the data is the magnitude of the interest rate spread. The model matches the qualitatively the increase in interest rates starting in 2001, but the magnitude is much smaller. In the data interest rates increase from 12% to 30% just prior the default, whereas in the model interest rates increase only by 1.2%. This small spread is the main anomaly of the model and this issue is further explored below.

Figure (5) shows that bond price schedule and the equilibrium bond price, faced by the economy in the model, as a function of assets for the highest and lowest endowment shocks of both sectors. The left panel of figure (5) plots the price schedule which shows the set of contracts $(q(B', y), B')$ the economy can choose from every period. Bond prices are an increasing function of assets. That is, larger levels of debt are associated with higher interest rates. For values of debt of up to 36% of mean tradable output, the economy faces low interest rates. Effectively, it is charged the risk free interest rate for these loans. At this level, default incentives start to increase in the economy giving rise to higher interest rates. At debt levels of about 40% of tradable output, households refuse to pay any liabilities for all endowment shocks and thus the probability of default for this debt level or higher is 1.

The right panel of the figure shows the actual bond price $q(B', y)$ the economy pays along the equilibrium path for a given state B, y by its choice of borrowing $B'(B, y)$. Along the equilibrium for a given level of initial assets when the economy is in a recession it chooses

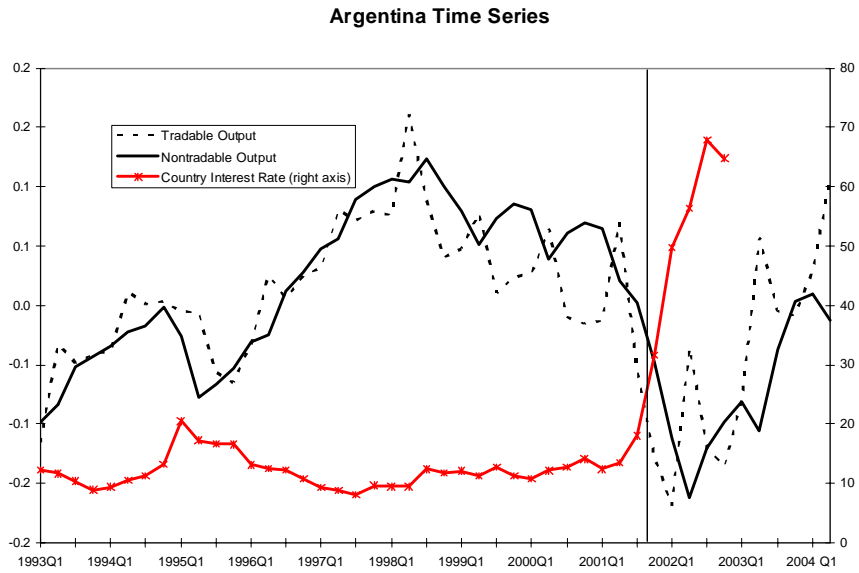


Figure 3: Argentina Default and Sectoral Output

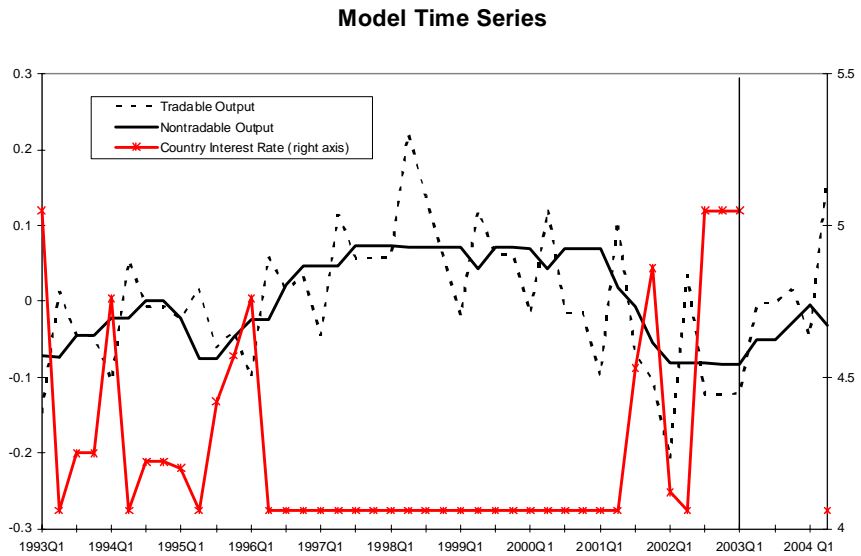


Figure 4: Model Default and Sectoral Output

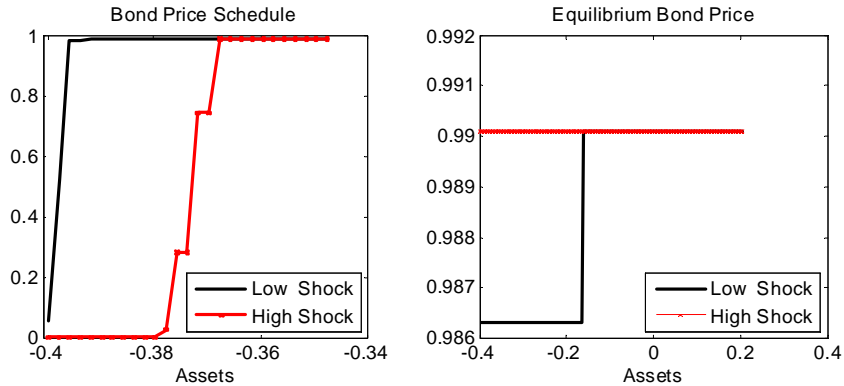


Figure 5: Interest Rates and Debt

higher levels of debt holdings. Thus the bond price is lower and interest rates are higher in recessions.

A feature of the model is that it produces a narrow range for asset positions that carry positive but finite risk premia (i.e. the range $B \in (\underline{B}, \bar{B}]$). The endogenous borrowing limit B^* for this economy is equal to -39.6% percent of tradable output or 10.5% percent of total output. The upper sloping portion of the function $q(B)B$ in the region with positive default probabilities is very small with the range $(B^*, \bar{B}]$ being equal to (-39.6,-36.7) percent of tradable output. This means that the set of y for which the economy finds it optimal to default expands fast once the economy hits the risk free debt limit. The reason is that for a given level of negative assets (debt) B the difference between $v^c(B, y)$ and $v^d(y)$ does not change too much across exogenous states y . In the case analyzed in the section 3.1 for i.i.d. tradable shocks, this is equivalent than saying that the threshold boundary $y^*(B)$ is very steep.

As the figure shows, for a given level of debt holdings demanded B' the price of those loans are lower in high endowment shocks. The reason is that with the benchmark calibration default is more likely for intermediate endowment shocks. For *i.i.d.* shocks with $\lambda = 1$ and $\theta = 0$, section 3.1 showed that default incentives are higher for low tradable endowments and low nontradable endowment when the cross derivative of the utility is negative (as is with this calibration). However in the calibration shocks are persistent and the parameters for default penalties are different.

The main parameter that seems to change this relation is the output loss λ . When the economy loses a portion of output during default, capital outflows are no longer uniformly

more costly in a recession relative to autarky, which was the main mechanism for default to be more attractive in recessions. The reason is that $\lambda < 1$ increases the difference in the autarky utility across shocks given utility is concave and this can increase the cost of recessions in autarky as well.

The persistence of shocks is another important determinant of the relation between default and output but only for the case of $\lambda < 1$. When $\lambda = 1$, for all levels of persistence of shocks ($0 < \rho < 1$), simulations of the model maintained the negative relation between output and default. However for $\lambda < 1$, higher persistence in shocks can increase default incentives in booms. Intuitively, even though the costs of defaulting are larger in recession because outflows are more costly in bad times with concave utility, the relative benefits from defaulting, i.e. autarky, increase also with persistent shocks, and this mechanism seems to dominate for cases where $\lambda < 1$.

Table 3 presents business cycle statistics for the benchmark calibration of the model. The business cycle statistics are mean values from 100 simulations of 40 observations each. The model was simulated for 200,000 periods and the time series statistics chosen were the ones containing a default to compare the model with the data in Argentina.¹⁴ The model does not have predictions for interest rates after default, because the economy is assumed to be excluded from financial markets. But the model predicts that in expectation of a default, interest rates should compensate investors for a positive default probability. Thus the time series chosen were the 40 observations right before a default occurs. The simulated data are log and filtered equally as the data and the denomination of all the statistics in the model and data are consistent.

Overall the model can match several features of the Argentinian economy. Interest rates are negatively correlated with aggregate, tradable and nontradable output, and consumption in the business cycle. This negative correlation is driven by the fact that in recessions the economy wants to increase its debt holdings and larger loans carry a larger default premium. The correlations between consumption and sectoral and aggregate output are positive and in line with the data. The model does not match the correlations of the current account with output and interest rates observed. Debt is used in the model to smooth output variations. Households generally want to run down their assets in periods of low output, and engage in precautionary savings in periods of relatively high output. Thus, as in any insuring type model of debt without investment, current account and output should be positively correlated. However, when looking at the correlation between the current account

¹⁴Business cycles statistics from the limiting distribution of assets (conditional on not defaulting) are almost identical to the ones presented in the table containing a default event.

and sectoral output, the model matches qualitatively the data in Argentina. The correlation between the current account and tradable output is positive as in the data and equal to 0.61. The correlation between the current account and nontradable output is negative as in the data and equal to -0.16. In addition, as it will be shown below, in the region of default the model presents a negative relation between current account and aggregate output.

The real exchange rate is the most volatile variable in the model, being more volatile than aggregate output. The volatility of the real exchange rate comes from the volatility and covariation of the exogenous nontradable output and the endogenous tradable consumption through equation (4). Endogenous time varying interest rates and debt limits make tradable consumption much more volatile in this model, which is an important driving mechanism of the high volatility of the real exchange rate. The comparison of the behavior of real exchange rates between the model and the data prior to default should be done with caution because in this period Argentina was under the convertibility plan, thus the low volatility observed. The real exchange rate is positively correlated with output and negatively correlated with interest rates as in the data and this negative correlation between interest rates and exchange rates seems a regularity in emerging economies.¹⁵ The model produces real exchange rate devaluations if tradable consumption is low or if nontradable consumption is high. The reason why real exchange rates are negatively related to interest rates is because along the equilibrium path interest rates are more strongly negatively related to tradable consumption than to nontradable consumption.

The table also presents mean percentage deviations in the period of default for statistics of the model. In periods of default the economy experiences significant collapses in consumption, real exchange rates, reversals in the current account, and higher interest rates. Defaults occur on average when the economy has large levels of debt to be repaid and this happens after a sequence of bad shocks. This is why on average default happens in a recession. The model matches qualitatively the data in that default events are accompanied by contractions in economic activity, high interest rates and depreciations in real exchange rates. However the model underestimates the massive collapse experienced in Argentina.

¹⁵The contemporaneous correlation of real exchange rates and interest rates for the same time period is -0.35 in Korea, -0.84 in Mexico, and -0.07 in Brazil.

Table 3.

Business Cycle Statistics in the Model

	Default Episode	$std(x)$	$corr(x, y)$	$corr(x, r^c)$
Interest Rates Spread	0.53	0.31	-0.72	
Consumption	-3.66	4.04	0.47	-0.67
Nontradable Output	-2.7	4.17	0.12	-0.46
Tradable Output	-6.92	7.43	0.96	-0.71
Aggregate Output	-9.33	10.69		-0.65
Current Account	8.72	0.57	0.56	-0.26
Real Exchange Rate	-7.5	12.49	0.92	-0.46
Other Statistics				
Debt Limit (% tradable output)	39.6	Mean Spread		0.24%
Default Probability	0.2%	Max Spread		1.53%

The mean level of assets in the limiting distribution, conditional on not defaulting, is -36.7% of tradable output (-9.7% of total output). Thus the economy is on average a net debtor. These endogenous borrowing constraints affect equilibrium allocations because they limit the ability of the economy to share risk. The borrowing limits are tight in the model because the relative benefit from participating in the international financial markets are small.¹⁶

Default is a rare outcome in the limiting distribution of the model. Annualized default probabilities are 0.2%. This is lower than the case for Argentina that defaulted 3 times in the past 100 years. Nevertheless, default is a positive probability event that affects equilibrium allocations and prices. And even though in the limiting distribution default is a rare outcome, the model is nevertheless successful in predicting Argentina's default. The reason why default is so rare has to do with the fact that the default boundary is very steep making the range of risky assets narrow.

The mean annual spread (defined as the difference between the country interest rate and the risk free interest rate, $r^c - r$) in the limiting distribution of assets, conditional on not defaulting, is 0.24% which is very similar to the average default probability of the model. The maximum spread that the model can deliver is 1.53%. These are very small numbers compared the average spread for Argentina which equals to 8.7% for the period 1993-2001. Thus the model falls short in this regard. Note that in the model there is a one

¹⁶The fact that the costs in terms of lifetime utility of being in autarky is small, is related to the small costs of business cycles in Lucas' 1987. Gourinchas et. al. (2002) document on the marginal benefits of financial integration relative to autarky.

to one matching from default probabilities to interest rates due to risk neutral lenders, and thus without some other specification for lenders the model cannot obtain average default probabilities very different from average interest rate spreads which is a feature of the data.

The fact that default probabilities do not account for all the yield spread in bonds is a well known puzzle in the finance literature on corporate defaultable bonds, even in the presence of risk averse lenders. Huang and Huang (2003) find that in investment grade bonds, default probabilities and credit risk account only for 19% of the yield observed in defaultable bonds. And for shorter maturity bonds, they find that default risk accounts for almost nothing of the yield.¹⁷

The small range of defaultable debt together with the steepness of the boundary threshold is also the reason why the volatility of interest rates in the model is much smaller than that for Argentina. Previous small open economy models have generate zero endogenous volatility of country spreads, and so in this regard the model improves over existing models. However it falls short of matching quantitatively the data. The only mechanism in the model for volatile interest rates are varying default probabilities due to volatile endowments. The model does not address other sources of interest rate volatility that are part of the volatility of interest rates in the data such as the volatility in the international interest rate, volatility in risk premia due to lender's risk aversion, and the feedback and magnifying effects that volatile interest rates can have on output.

The small spread and low volatility of interest rates is the main anomaly of the model in relation with the data. However no other models of endogenous default are able to generate substantially different interest rates spreads and volatilities, and this remains an open challenge. Below it is explored alternative mechanisms that can generate higher spreads and volatility.

An interesting feature of the model is that capital outflows (*i.e.* $y^T - c^T$) can occur in a recession because default probabilities are high. When interest rates are low, debt is a good insurance instrument: capital outflows are large in good times because the economy saves, and are low in bad times because the economy borrows. The standard deviation of tradable consumption is 6.42 which is lower than the standard deviation of tradable output. However, when the economy is highly indebted and interest rates are high due to high default probabilities, debt becomes a less good insurance instrument and in fact the economy can face capital outflows in a recession. This result is similar to Atkeson's (1991) result, where he shows that in an insurance model of debt that features moral hazard and unenforceability

¹⁷The majority of the finance studies on defaultable bonds use reduce form models that take as exogenous the process for the default probabilities or for firm value.

of debt contracts, the optimal debt contract will feature capital outflows in a recession. Here, what drives the result is the incompleteness of assets and the fact that default is an equilibrium outcome. The mean drop in tradable consumption at the time of default in half of the default episodes considered is greater than the drop in tradable output which implies a capital outflow in a recession. This feature of the model matches the empirical regularity that emerging markets in crises are not able to use the international markets for smoothing and experience net capital outflows.

4.4 Experiments

4.4.1 The Role of Nontradables

In this experiment the benchmark model is compared with a one sector endowment model where all output is composed of tradable goods. The mean, volatility and persistence of output in the one sector model is such that it equals the two sector benchmark aggregate output. We find that this economy would have no default in the long run distribution with the benchmark calibration. This economy faces a looser debt limit of 15% of aggregate output, whereas in the benchmark calibration the debt limit is 10% of aggregate output. The mean asset position is -1% of aggregate output, whereas in the two sector model the mean asset position was -9%. That is, the economy is able to access greater credit when output is composed of tradable goods only and it uses the international financial markets much more heavily for insurance engaging in greater precautionary savings. The intuition of this result is that, from an incentives perspective, an economy with larger tradable sectors will benefit more from greater access to international borrowing to smooth fluctuations. This tends to suggest that economies with relatively larger tradable sectors would have greater access to international credit markets.

In addition, nontradable goods increase default probabilities because volatile nontradable shocks increase the relative range for risky borrowing. Nontradable goods act in essence as preference shocks given the endowment nature of the economy considered and this extra volatility makes the difference between the continuation and autarky value a more smooth function of the endowment shocks.

4.4.2 Trade Costs

It has been documented by Rose (2002) that another reason why countries repay their debts is because trade decreases by 8% a year over and above any decrease in output after default. In this final section we explore how trade penalties might affect incentives to default within

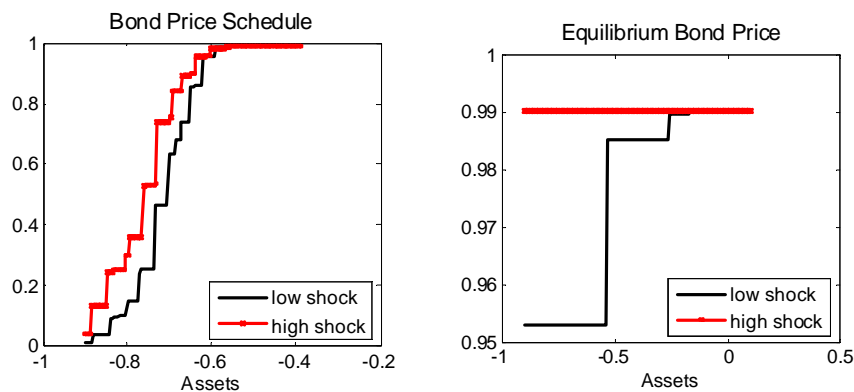


Figure 6: Interest Rates and Debt with Trade Costs

the context of our model. For this purpose we modify the tradable consumption to account for an importable and exportable sector.

$$c^T = \left[\alpha (c^H)^{-\eta} + (1 - \alpha) (c^F)^{-\eta} \right]^{-\frac{1}{\eta}}$$

We assume that the terms of trade are constant and equal to 1 and thus absent of default this specification is exactly equal than the specification presented for the benchmark model. It is assumed that $\alpha = 0.5$ and the elasticity of consumption equals the elasticity of tradable and nontradable goods above. All other parameters are equal to the benchmark model. To model trade penalties it is assumed that c^F decreases by 8% a year in the periods when the country is in financial autarky after a default.

The main feature of this economy is that it increases the range for risky borrowing (\underline{B}, \bar{B}) as it is evident from figure (6). The bond price schedule that the economy faces in this case is a much more smooth function of debt.

The reason why trade costs increase the range of risky borrowing is that the difference between the value of staying in the contract and the value of autarky is more sensitive to the shock. Autarky is essentially more equal across states for the case of trade penalties because the drop in c^F is independent of the shock as c^F decreases by 8% from its mean level prior default.

Table 4 shows that these type of penalties for defaulting can have significant effects on default probabilities and equilibrium interest rates. With the benchmark calibration, the default probability increases to 1.2% , interest volatility increases to 2.7 , debt limits are much looser and equal about 100% of tradable output, and the correlation of output and interest rates is negative. Overall, this specification for penalties gives the model greater

flexibility in being able to capture the high spreads observed in emerging markets foreign debt.

Table 4.

Trade Penalties

Mean($r^c - r^*$)	Default Rate	Debt limit	$std(r^c)$	$corr(y, r^c)$	$corr(c, r^c)$	$corr(p^c, r^c)$
1.37	1.2%	75%	2.7	-0.7727	-0.8416	-0.2822

This experiment suggests that more evidence is needed regarding the type of costs economies encounter after defaulting on their international debt. In addition more theoretical work is needed in understanding the interdependence of credit and trade relations.

5 Conclusion

This paper models endogenous default risk in a stochastic dynamic framework of a small open economy that has two sectors and that features liability dollarization. The paper presents a model where interest rates respond to output fluctuations through endogenous time-varying default probabilities. The main contributions of the paper are three. First, it studies analytically the relationship between default and output in an environment of incomplete markets and provides conditions for default to be an equilibrium outcome. Second, it explores quantitatively the predictions of the model in explaining the real dynamics observed in default episodes. The model predicts the recent Argentina default and can match several features of the data such as the negative correlation between output and consumption with interest rates. In times of default the model presents sharp declines in output and consumption and significant devaluations in real exchange rates. Third, it explores the effects that fluctuations and relative sizes of nontradable sectors have on incentives to default on tradable denominated debt. The model matches the data in that real exchange rate devaluations are correlated with high interest rates. An empirical implication of the model is that economies with relatively small tradable sectors have higher incentives to default on tradable denominated debt and thus face higher interest rates.

Even though the model is able to explore default episodes and can match interest rate counter-cyclicality, it cannot match the magnitude of the interest rate spread and volatility observed in the data of emerging markets. This remains an open challenge. Interesting extensions of the model that can potentially address this anomaly are exploring alternative specifications for creditors, modeling default penalties as a bargaining outcome (see Yue

2005 for some recent work studying this issue), and adding a feedback from interest rates to production.

References

- [1] Aiyagari, S. R. (1993), "Explaining Financial Markets Facts: The Importance of Incomplete Markets and Transactions Costs," Federal Reserve Bank of Minneapolis Quarterly Review 17, 17-31.
- [2] Aguiar, M. and G. Gopinath (2004), "Defaultable Debt, Interest Rates and the Current Account," Chicago GSB Working Paper.
- [3] Alvarez, F., and U. J. Jermann (2000) "Efficiency, Equilibrium, and Asset Pricing with Risk of Default," *Econometrica*, v. 68(4), 775-798.
- [4] Atkeson, A. (1991), "International Lending with Moral Hazard and Risk of Repudiation," *Econometrica*, v. 59 (4), 1069-89.
- [5] Betts, C., and T. J. Kehoe (2001), "Tradability of Goods and Real Exchange Rate Fluctuations," Working Paper, University of Minnesota
- [6] Beim, D., and C. Calomiris (2001). *Emerging Financial Markets*. New York: McGraw-Hill, Irvin.
- [7] Bulow, J., and K. Rogoff (1989), "A Constant Recontracting Model of Sovereign Debt," *Journal of Political Economy*, v. 97, 155-178.
- [8] Caballero, R., and A. Krishnamurthy (2000), "Dollarization of Liabilities: Underinsurance and Domestic Financial Underdevelopment," NBER WP #7740.
- [9] Calvo, G. (1998), "Capital Flows and Capital-Market Crises: The Simple Economics of Sudden Stops," *Journal of Applied Economics*, v.1, 35-54.
- [10] —, A. Izquierdo, and E. Talvi (2002), "Sudden Stops, the Real Exchange Rate and Fiscal Sustainability: Argentina's Lessons," Working Paper, Research Department, Inter-American Development Bank.
- [11] —, and C. Reinhart (2002), "Fear of Floating," *Quarterly Journal of Economics*, Vol. CXVII, No. 2, 379-408.
- [12] Chatterjee, S., D. Corbae, M. Nakajima, and J. Rios Rull (2002), "A Quantitative Theory of Unsecured Consumer Credit with Risk of Default," Working Paper, University of Pennsylvania.

- [13] Chuhan, P., and F. Sturzenegger (2003), "Default Episodes in the 1990's: What have we learned?," Working Paper, Universidad Torcuato Di Tella.
- [14] Cole, H., and T. Kehoe (2000), "Self-Fulfilling Debt Crises," *Review of Economic Studies*, 67, 91-116.
- [15] Conklin J. (1998), "The Theory of Sovereign Debt and Spain under Philip II," *Journal of Political Economy*, , vol. 106 (3), 483-513
- [16] Eaton, J., and M. Gersovitz (1981), "Debt with Potential Repudiation: Theoretical and Empirical Analysis," *Review of Economic Studies*, v. XLVII, 289-309.
- [17] Gibson, R., and S. Sundaresan (2001), "A Model of Sovereign Borrowing and Sovereign Yield Spreads," Working Paper, Columbia University.
- [18] Gonzales-Rosada, M., and P. A. Neumeyer (2003), "The Elasticity of Substitution in Demand for Non-tradable Goods in Latin America. Case Study: Argentina," Working Paper, Universidad Torcuato Di Tella.
- [19] Gourinchas, P. and O. Jeanne (2003), "The Elusive Benefits from International Financial Integration," Berkeley Working Paper
- [20] Hamann, F. (2002), "Sovereign Risk and Real Business Cycles in a Small Open Economy," Working Paper, North Carolina State University.
- [21] Hausmann, R., U. Panizza, and E. Stein (2001), "Why countries float the way they float?" *Journal of Development Economics*, v.66 (2), 387-414.
- [22] Huang, J., and M. Huang (2003), "How Much of the Corporate-Treasury Yield-Spread is due to Credit Risk?" Working Paper, Stanford University.
- [23] Huggett, M. (1993), "The risk free rate in heterogenous-agents incomplete-insurance economies," *Journal of Economic Dynamics and Control* v.17, 953-969.
- [24] Hussey, R., and G. Tauchen (1991), "Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models," *Econometrica*, v 59 (2), 371-396.
- [25] Kehoe, T. J., and D. K. Levine (1993), "Debt-Constrained Asset Markets." *Review of Economic Studies* 60 868-88.
- [26] Kehoe, P. and F. Perri (2004), "Competitive Equilibria With Limited Enforcement," forthcoming in *Journal of Economic Theory*.

- [27] Kletzer, K. M., and B. D. Wright (2000), "Sovereign Debt as Intertemporal Barter," *American Economic Review*. 90(3), 621-639.
- [28] Mendoza, E. G. (2002), "Credit, Prices, and Crashes: Business Cycles with a Sudden Stop." In *Preventing Currency Crises in Emerging Markets*, Frankel, Jeffrey and Sebastian Edwards eds. Chicago: University of Chicago Press.
- [29] ——— (2001), "The Benefits of Dollarization when Stabilization Policy Lacks Credibility and Financial Markets are Imperfect," *Journal of Money, Credit and Banking* v33.
- [30] Neumeyer, P., and F. Perri (2001), "Business Cycles in Emerging Economies: The Role of Interest Rates," Working Paper, Department of Economics, New York University.
- [31] Obstfeld M., and Rogoff K.(1989), "Sovereign Debt: Is to Forgive to Forget?" *American Economic Review* 79, 43-50.
- [32] ——— (1996), *Foundations of International Macroeconomics*, Cambridge, Massachusetts, MIT Press.
- [33] Oviedo, M. (2003) "Macroeconomic Risk and Banking Crises in Emerging Market Countries: Business Cycles with Financial Crashes ," Working Paper, Iowa State University.
- [34] Reinhart, C. (2002), "Default, Currency Crises and Sovereign Credit Ratings," NBER Working Paper #8738.
- [35] Rose, C. (2002), "One Reason Countries Pay their Debts: Renegotiation and International Trade," Berkeley Working Paper
- [36] Uribe, M., and Yue V. (2003), "Country Spreads and Emerging Markets: Who Drives Whom?" Working Paper, University of Pennsylvania.
- [37] Yue, V. (2005), "Sovereign Default and Debt Renegotiation," University of Pennsylvania Working Paper.
- [38] Zame, W. (1993), "Efficiency and the Role of Default When Security Markets are Incomplete," *American Economic Review*, 83, No. 5, 1142-1164.
- [39] Zhang, H. (1997) "Endogenous Borrowing Constraints with Incomplete Markets," *The Journal of Finance*, 52, 2187-2209.

Appendix A

Proposition 1. *Default is more likely the higher the level of debt: For $B^1 < B^2$, if default is optimal for B^2 , in some states y , then default will be optimal for B^1 for the same states y , that is $D(B^2) \subset D(B^1)$.*

This result is similar than Eaton and Gersovitz (1981) and Corbae et al (2001).

For all $\{y^T, y^N\} \in D(B^2)$, $u(y^T, y^N) + \beta Ev^d(y') > u(y^T + B^2 - qB', y^N) + \beta Ev^o(B', y')$. Since $y^T + B^2 - qB' > y^T + B^1 - qB'$ for all B' , thus $u(y^T + B^2 - qB', y^N) + \beta Ev^o(B', y') > u(y^T + B^1 - qB', y^N) + \beta Ev^o(B', y')$. That is the value of the contract under no default is increasing in foreign asset holdings. Hence $u(y^T, y^N) + \beta Ev^d(y') > u(y^T + B^1 - qB', y^N) + \beta Ev^o(B', y')$, which implies that $\{y^T, y^N\} \in D(B^1)$ ■

The Case of i.i.d. Endowment Shocks

Proposition 2. *Default incentives are stronger the lower the tradable endowment. For all $y_1^T < y_2^T$, if $y_2^T \in D(B)$, then $y_1^T \in D(B)$.*

In order to prove proposition 2, we will first prove the Lemma 2.1

Lemma 2.1 . *If for some B , the default set is non empty $D(B) \neq \emptyset$, then there are no contracts available for the economy $\{q(B'), B'\}$ such that the economy can experience capital inflows, $B - q(B')B' > 0$*

This is a proof by contradiction.

Suppose there are contracts $\{q(B'), B'\}$ available to the economy such that $B - q(B')B' > 0$. But that the economy choose under the contract utility some \widehat{B} to maximize utility such that $B - q(\widehat{B})\widehat{B} < 0$ and then finds default to be the optimal option because $u(y^T, y^N) + \beta Ev^d(y') > u(y^T + B - q(\widehat{B})\widehat{B}, y^N) + \beta Ev^o(\widehat{B}, y')$.

Now note that under all contracts $\{q(B'), B'\}$ such that $B - q(B')B' > 0$ staying in the contract is always preferable to default because $Ev^o(B', y') \geq Ev^d(y')$, and $u(y^T + B - q(B')B', y^N) > u(y^T, y^N)$. This implies that \widehat{B} cannot be the maximizing level of assets and then find default to be optimal, because it is a contradiction.

Thus, if $D(B) \neq \emptyset$, then \exists some $y \in Y$, such that $v^d(y) \geq v^c(B, y)$, or equivalently, $u(y) + \beta Ev^d(y') \geq u(y + B - q(B')B') + \beta Ev^o(B', y')$.

Given that B' is chosen to maximize the value of the contract, then if default is preferable, it must be the case that not only $B - q(B')B' < 0$, but that \nexists a contract available $\{q(B'), B'\}$ such that $B - q(B')B' > 0$ □

Now we prove proposition 2.

If $y^2 \in D(B)$ then by definition $u(y_2^T, y^N) + \beta Ev^d(y') \geq u(y_2^T + B - q(B')B', y^N) + \beta Ev^o(B', y')$

If

$$\begin{aligned} & u(y_2^T + B - q(B^2)B^2, y^N) + \beta Ev^o(B^2, y') - \\ & \{u(y_1^T + B - q(B^1)B^1, y^N) + \beta Ev^o(B^1, y')\} > \\ & u(y_2^T, y^N) + \beta Ev^d(y') - \{u(y_1^T, y^N) + \beta Ev^d(y')\} \end{aligned} \quad (15)$$

then $y^2 \in D(B)$, implies $y^1 \in D(B)$,

Now it is necessary to show that expression (15) holds.

Given that shocks are iid, the right hand side of equation (15) simplifies to

$$[u(y_2^T, y^N)] - [u(y_1^T, y^N)]$$

Because of utility maximization:

$$u(y_2^T + B - q(B^2)B^2, y^N) + \beta Ev^o(B^2, y') \geq u(y_2^T + B - q(B^1)B^1, y^N) + \beta Ev^o(B^1, y')$$

Thus if

$$\begin{aligned} & u(y_2^T + B - q(B^1)B^1, y^N) + \beta Ev^o(B^1, y') - \\ & \{u(y_1^T + B - q(B^1)B^1, y^N) + \beta Ev^o(B^1, y')\} > \\ & \{[u(y_2^T, y^N)] - u(y_1^T, y^N)\} \end{aligned} \quad (16)$$

holds then through transitivity expression (15) holds.

Simplifying (16):

$$u(y_2^T + B - q(B^1)B^1, y^N) - u(y_1^T + B - q(B^1)B^1, y^N) > [u(y_2^T, y^N)] - [u(y_1^T, y^N)]$$

Now, note that due to Lemma 2.1, if $y^2 \in D(B)$ then $B - q(B')B' < 0$ for all available $\{q(B'), B'\}$ thus $B - q(B^1)B^1 < 0$.

Hence, given that utility is increasing and strictly concave in both arguments, then (16) holds, which implies that $y^1 \in D(B)$. \square

Proposition 3. *If default sets are non-empty, then they are closed intervals where only the upper bound depends on the level of debt:*

$$D(B) = [\underline{y}, y^*(B)] \quad \text{for } B \leq \bar{B}$$

where $y^*(B)$ is a continuous, non-increasing function of B , such that:

$$y^*(B) = \begin{cases} y^*(B) : v^d(y^*(B)) = v^c(B, y^*(B)) & \text{for } \underline{B} \leq B \leq \bar{B} \\ \bar{y} & \text{for } B < \underline{B} \end{cases}$$

Proof:

For $B < \underline{B}$, $D(B) = Y$, so that $y^*(B) = \bar{y}$

For $\underline{B} \leq B \leq \bar{B}$ let $\Psi(B, y) \equiv v^d(y) - v^c(B, y)$

$$\Psi(B, y) = u(y) + \beta E v^d(y') - u(y^T + B - q(B')B'(B, y), y^N) - \beta E v^o(B'(y, B), y')$$

$$\begin{aligned} \frac{\partial \Psi}{\partial y^T} &= u_{c^T}(y^T, y^N) - u_{c^T}(y^T + B - q(B')B'(B, y), y^N) + \\ &\quad \frac{\partial B'(B, y)}{\partial y^T} \left[\frac{\partial [q(B')B']}{\partial B'} u_{c^T}(y^T + B - q(B')B'(B, y), y^N) - \beta \frac{\partial [E v^o(B'(y, B), y')]}{\partial B'} \right] \end{aligned}$$

The term in brackets is exactly the first order condition of the government's problem and thus it is equal to zero. Thus,

$$\frac{\partial \Psi}{\partial y^T} = u_{c^T}(y^T, y^N) - u_{c^T}(y^T + B - q(B')B'(B, y), y^N)$$

The sign of the above derivative depends on whether $B - q(B')B'$ is greater or less than zero. In general, this can be positive or negative because of the insurance type use of debt. But for all $B \in [\underline{B}, \bar{B}]$, default sets are non-empty, and so $B - q(B')B' < 0$ due to Lemma 2.1

Thus for $B \in [\underline{B}, \bar{B}]$, $\frac{\partial \Psi(B, y)}{\partial y^T} < 0$.

Given that default sets are non-empty and strictly less than the entire endowment space for $B \in (\underline{B}, \bar{B}]$, then for some y default is preferable, and for some y repayment is preferable

within this range. But given that $\Psi(B, y)$ is monotonically decreasing in y for all $B \in [\underline{B}, \overline{B}]$, then there exists a unique y^* such that for value $y \leq y^*(B)$ default is preferable, and for $y > y^*(B)$ repayment is preferable, where $v^c(B, y^*(B)) = v^d(y^*(B))$. And thus default sets can be characterized by closed intervals where only the upper bound depends on the level of debt.

Now using the implicit function theorem :

$$\frac{\partial y^T}{\partial B} = \frac{v_B^c(B, y)}{v_{y^T}^d(y) - v_{y^T}^c(B, y)} = \frac{v_B^c(B, y)}{u_{c^T}(y^T, y^N) - u_{c^T}(y^T + B - q(B')B'(B, y), y^N)} < 0.$$

which says that for $B \in [\underline{B}, \overline{B}]$, $y^*(B)$ is decreasing in B . \square

Corollary 4.1. *Default can be an equilibrium outcome of the model for all probability distributions over $[\underline{y}, \overline{y}]$ satisfying the property:*

$$\lim_{y \rightarrow \underline{y}} h(y) = 0$$

where $h(y)$ is the hazard function of the distribution.

Proof.

The condition that the slope of $q(B) \cdot B$ be positive stated in terms of the hazard function of the distribution:

$$\frac{\partial [q(B)B]}{\partial B} = [1 - F(y^*(B))] - f(y^*(B)) B \frac{\partial y^*}{\partial B} > 0$$

or

$$\frac{1}{h(y^*(B))} > B \frac{\partial y^*}{\partial B}$$

Note that

$$\lim_{B \rightarrow \overline{B}^-} y^*(B) = \underline{y}$$

due to Proposition 3. Thus for distributions satisfying the above condition:

$$\lim_{B \rightarrow \overline{B}^-} \frac{1}{h(y^*(B))} = \infty$$

The only thing we need to prove now is that the $\lim_{B \rightarrow \overline{B}^-} \frac{\partial y^*(B)}{\partial B}$ is finite

$$\lim_{B \rightarrow \bar{B}^-} \frac{\partial y^*(B)}{\partial B} = \lim_{B \rightarrow \bar{B}^-} \frac{v_B^c(B, y)}{u'(y^T, y^N)_{c_{c^T}} - u'(y^T + B - q(B')B'(B, y), y^N)_{c_{c^T}}}$$

The numerator of the above expression is finite and positive for finite B . Note that the only y which we need to consider the limit, is \underline{y} . And specifically that

$$\lim_{B \rightarrow \bar{B}^-} B - q(B'(B))B'(B, \underline{y}) < 0.$$

which holds by continuity due to Lemma 2.1. \square

Proposition 5. *Default incentives are stronger for low nontradable endowments if the cross derivative of the utility function is negative. For all $y_1^N \leq y_2^N$, if $y_2^N \in D(B)$, then $y_1^N \in D(B)$ if $u_{c^T c^N} < 0$.*

Using the exact same strategy than the one used in proposition for proposition 2 the condition needed to prove the proposition simplifies to depending only on period utility. That is if :

$$[u(y^T + B - q(B^1)B^1, y_2^N)] - [u(y^T + B - q(B^1)B^1, y_1^N)] > [u(y^T, y_2^N)] - [u(y^T, y_1^N)] \quad (17)$$

then for all $y_1^N < y_2^N$, if $y_2^N \in D(B)$, then $y_1^N \in D(B)$

Rearranging:

$$[u(y^T, y_1^N)] - [u(y^T + B - q(B^1)B^1, y_1^N)] > [u(y^T, y_2^N)] - [u(y^T + B - q(B^1)B^1, y_2^N)] \quad (18)$$

where $B - q(B')B' < 0$ for $B \in [\underline{B}, \bar{B}]$.

Let:

$$\Psi(y, B) = [u(y^T, y^N)] - [u(y^T + B - q(B^1)B^1, y^N)]$$

$$\frac{\partial \Psi(y, B)}{\partial y^N} = \frac{\partial u(y^T, y^N)}{\partial y^N} - \frac{\partial u(y^T + B - q(B^1)B^1, y^N)}{\partial y^N}$$

If $\frac{\partial^2 u(c^T, c^N)}{\partial c^T \partial c^N} < 0$ then $\frac{\partial \Psi(y, B)}{\partial y^N} < 0$, which then makes equation (18) hold. \square