Sharing Cost Information in Dynamic Oligopoly*

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Abstract

We study the effect of sharing cost information in dynamic oligopoly. Firms can agree to verifiably share information about common costs, as with the aggregation of input costs by an industry trade association. Cost information that is not directly shared is revealed through observed prices. We show that information sharing agreements lead to higher prices and reduce the incentive to acquire firm specific cost information. Information sharing decreases consumer surplus when either demand is sufficiently inelastic or goods are sufficiently substitutable. For markets with a large number of firms, information sharing has a minimal impact on expected prices, and can increase both consumer and producer surplus when goods are sufficiently differentiated.

1 Introduction

We consider the effect of sharing information about common costs in the context of dynamic price competition. Information that is not shared may otherwise be revealed through observed prices. Aggregation of individual firms' information changes pricing behavior directly, as there is more information available to each firm, and indirectly, as the information content of prices changes. We show that information sharing increases prices and reduces incentives to acquire information, but welfare effects depend on the substitutability of goods. When goods are relatively substitutable, sharing information strictly reduces consumer surplus.

Information sharing is one key service of trade associations. In practice, trade associations can increase producer surplus by aggregating information regarding demand or costs, or

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by explicitly coordinating strategic decisions.[1] Improved producer surplus frequently comes at the cost of consumer surplus, and price coordination and private information sharing are generally considered anti-competitive by antitrust authorities.[2] An "honest" trade association, barred from sharing strategic plans or firm-specific information, may still share information about industry trends. For example, it may generate a market forecast of raw input prices. We model such a trade association, which aggregates only information about costs which are shared by all firms, and show that when goods are substitutable even this innocuous trade association will be anti-competitive.

Our main contribution is the identification of a novel and endogenous inference channel which is affected by information sharing. As discussed above, an honest trade association is restricted to sharing industry-wide information, and firms within a trade association may remain ignorant of each others' costs. In this context, sharing information provides increased precision regarding common costs (good), which in turn increases the incentive to soften subsequent competition (bad). This tradeoff cannot exist without multidimensional uncertainty. Our analysis assumes a multidimensional cost structure (vs. e.g., Jeitschko et al. [2018]), perfect observations of market outcomes (vs. e.g., Mirman et al. [1993]), and inference from these observations (vs. e.g., Raith [1996]); see below for a more detailed literature review.

Before an overview of our results we give a basic statement of our model. We study a dynamic Bertrand competition model in which demand is common knowledge but firms have private information about costs. The structure of costs allows for both specific costs incurred by a particular firm (for example, a labor contract), and common costs that are shared by all firms in an industry (for example, raw input prices). In the first period each firm has imperfect information regarding its own costs, as well as those of its opponents. Firms may share information about common costs through a trade association. If they do so, they will have identical information about common costs but remain uninformed about their competitors' private costs. In the second period each firm knows its own costs. To infer the costs of competitors, firms interpret first-period prices as signals of costs, and update their beliefs accordingly before making second-period pricing decisions.

We give an implicit characterization of equilibrium in linear pricing strategies and show

[1] Examples of the former include *Case AT.40136 — Capacitors*, and *American Column & Lumber Co. v. United States*. Examples of the latter include *United States v. Nippon Paper Indus. Co., Ltd*, and *United States v. Archer Daniels Midland Co.*. For an overview of different forms of information sharing, see Kühn [2001] and Marshall and Marx [2014].

[2] The extent to which information sharing is permissible depends on the antitrust authority. In the U.S., regulators judge information sharing agreements between competitors by the *rule of reason* but note "...the sharing of information relating to price, output, costs, or strategic planning is more likely to raise competitive concern than...less competitively sensitive variables" (April 2000 FTC/DOJ Guildelines for Collaborations Among Competitors). In Europe, regulators have stricter principles declaring that sharing of information relevant to future prices is a restriction of competition by object (2011 Guidelines for Article 101 of TFEU).

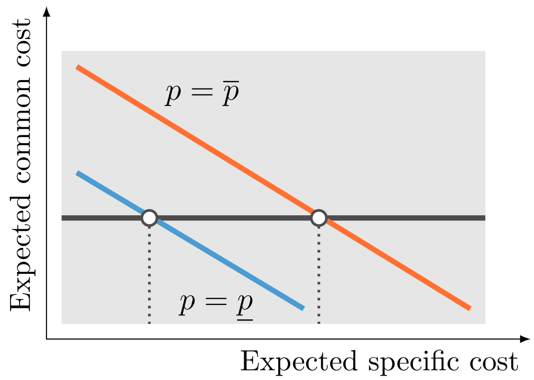

Figure 1: A high price (orange line) can be indicative of a high expected common cost or a high expected specific cost, while a low price (blue line) can be indicative of a low expected common cost or a low expected specific cost. When the opponent believes all signals are possible (shaded area) its information after witnessing the firm's price is that signals fell somewhere on the iso-price curve. When the opponent has full knowledge of the firm's expectation of common costs (dark gray line) it can uniquely identify the firm's expected specific cost (dotted lines).

that it is unique (Theorem 1). In this equilibrium we analyze the effects of sharing information about common costs. Sharing information increases the precision of firms' beliefs. The immediate effect is that firms can set prices closer to the known-cost optimum, which increases producer surplus. Tying pricing decisions more tightly to actual costs increases the variance of each firm's price as well as the covariance of the firms' prices. Increased price variance improves ex ante consumer surplus, while the effect of covariance depends on the substitutability of the firms' products.[3]

A full accounting of surplus requires an analysis of not only the direct effect of information sharing, but also the strategic effects. Information sharing indirectly impacts pricing strategies by changing the informativeness of prices with regard to firm-specific costs. If common cost information is not shared, firms have two-dimensional private information: signals of their specific costs, and signals of their common costs. As illustrated in Figure 1, first period prices pool information, and a high first period price may be indicative of high expected specific costs, high expected common costs, or both. If common cost information is shared each firm's private information is reduced to a single dimension, and its first period price will be more informative of its specific costs. Firms face an incentive to increase first period prices in hopes of indicating high costs and softening subsequent competition. When prices are more informative about firm-specific costs firms have a stronger incentive to

[3]This same logic is at play in the rich literature on sharing information about uncertain demand. See the early work of Clarke [1983], Vives [1984], and Kirby [1988], among others. These results focus on static competition and do not consider dynamic incentives to hide private information.

over-represent these costs. We show that information sharing increases expected first period prices and does not impact expected second period prices (Theorem 2).

The overall effect of information sharing on consumer welfare is determined by the relative impacts of increased precision and incentives to soften competition. We consider three cases. First, when demand is relatively inelastic strategic effects dominate. Demand does not respond strongly to increased prices, significantly reducing consumer surplus. Second, when goods are relatively substitutable the price increase induced by information sharing outweighs any benefits of increased variance, which are in turn outweighed by increased covariance of prices (Proposition 4). Third, when the market is large strategic considerations essentially vanish.[4] In this case, information sharing always improves producer surplus. Information sharing harms consumer surplus when goods are fairly substitutable, improves consumer surplus when goods are complements, and can have an ambiguous effect when goods are only weakly substitutable (Corollary 2).

We additionally consider the effect of information sharing on incentives to acquire information about specific costs. When firms share information about common costs, first period prices are less sensitive to information about firm-specific costs due to the stronger incentive to over-represent these costs (Proposition 6). Therefore, costly informational investments are not fully utilized, and firms are better off acquiring less information than when common cost information is not shared. This reduces the variance of prices and further lowers consumer surplus.

Lastly, we consider partial information sharing agreements, where firms share information that consists of an arbitrary portion of each of their signals. In this more general setting we show that expected prices in the first period are ranked by the equilibrium informativeness of the first period price, while expected second period prices are unaffected by the information sharing agreement (Proposition 7). Moreover, we show that sharing any amount of information on common costs will increase price informativeness while sharing information about firm specific costs reduces informativeness (Proposition 8).

Although our analysis maintains the assumption that firms are imperfectly informed of their own specific costs, this is inessential to any of our results. All equilibrium objects are continuous in a firm's awareness of its own, and its opponent's, costs, and approach standard values as information becomes perfect. An implication is that, while our model extends existing work on information sharing in dynamic oligopoly (see, e.g., Jeitschko et al. [2018]), the inference problem faced by firms is novel. Without multidimensional uncertainty,

[4]The bulk of our analysis is carried out in the context of a duopoly. We show that the basic structure of equilibrium applies to any number of firms and provide a welfare analysis as the number of firms goes to infinity.

firms cannot exploit imprecise estimates in one dimension to confound opponent inference in another. Sharing information about common cost parameters is qualitatively different from sharing information about specific cost parameters. Emphasizing this point, our comparative statics must be performed via implicit analysis, as closed-form expressions are not possible;[5] this contrasts with the literature on linear-normal oligopoly models, in which equilibrium price parameters may be explicitly computed. Nonetheless, many of our welfare results are familiar from the literature; see our literature review below.

Our results imply that even a relatively innocuous trade association has the potential to be anti-competitive. In our model the trade association exists only to pool information about common costs, and not to share strategic plans or firm-specific cost information. Pooling information about common costs allows firms to price more accurately, which is potentially beneficial for consumers. However, this effect is dampened by increased expected prices, and is completely overwhelmed by incentives to misrepresent costs when demand is inelastic or goods are sufficiently substitutable.[6]

Although our analysis is motived by information sharing by trade associations, our results are not sensitive to the source of aggregated information. Consumer surplus can be harmed by any public source of common cost information. For example, firms may reduce costs by eliminating in-house research teams and outsourcing market research to a consultant. If this consultant is used by all firms in an industry (where goods are substitutes) consumer surplus will be lower than if firms generate market forecasts independently. On the other hand, our results suggest that consumers are better off if firms across complementary industries share a common source of input market data.

1.1 Related literature

Our results relate to past work on the competitive effects of information sharing, and the dynamic revelation of information. Two properties of our model distinguish our results from the existing literature: firms have multidimensional private information about costs, which are partially common; and competition is dynamic, so observed prices signal information that affects subsequent competition.

Dynamic oligopoly models with incomplete information allow firms to distort strategies to signal information that is beneficial to the firm in later stages of competition. In Bertrand

[5] We prove in the appendix that computing equilibrium strategies is equivalent to solving a fifth-order polynomial with generic coefficients. Our claim that closed-form expressions are not (generically) feasible rests on the inability to solve generic quintic equations. On the other hand, obtaining computational results is straightforward.

[6] As is standard in the literature, our welfare results change signs when firms engage in Cournot (rather than Bertrand) competition. See Remark 2.

competition with private information about costs, Mailath [1989] and Mester [1992] identify the incentive to soften competition by over-representing cost through a choice of a price that is higher than is stage optimal. In Cournot competition with observed prices, firms can signal jam when selecting unobserved output quantities. Mirman et al. [1993] look at the case where firms have private information about individual demand curves. Bonatti et al. [2017] characterize the dynamics of signal jamming and learning when firms begin with private information about (only) specific costs. In our Bertrand framework, firms have the familiar incentive to soften competition by over-representing costs, and a rich strategy space by which to achieve this goal. Signal jamming relates to the weight the firm places on each source of information when choosing price. By reducing the weight on one source of information the firm reduces the informativeness of the price about this source;[7] this is counter to the single-dimensional information setting, where a reduced sensitivity of price to information means that difference from average is more indicative of extreme information.

In models with specific and common costs, firms put different weights on information about different cost components. When firms are selling substitutes they weigh common cost information more than specific cost information and vice versa when selling complements. This follows the basic intuition of Angeletos and Pavan [2007] where agents have both private and public signals about a single parameter and weigh private (public) signals more when actions are strategic substitutes (complements). When firms compete in supply schedules, Bernhardt and Taub [2015] shows that firms will weigh private signals over common signals. Moreover, the ability to signal jam in this setting increases the difference in the relative use of information. In a Bertrand setting, Myatt and Wallace [2015b] show that too much public information is used from the perspective of consumers and a less than efficient amount is used from the firms' perspective. The opposite efficiency results are identified in Myatt and Wallace [2015a] for Cournot competition.

Incentives to share private cost information in Bertrand competition depend on the structure of firms' information. For example, firms may prefer to share no information about perfectly-known private costs [Gal-Or, 1986], but it may be profitable to share affiliated noisy signals of cost parameters [Sakai, 1986].[8] Jeitschko et al. [2018] examine the impact of dynamic competition on sharing noisy signals of a single private-valued cost component. When costs are one-dimensional information sharing eliminates all private information, and along with it all incentive to soften competition. While firms directly benefit from the in-

[7]Mirman et al. [1994] identify that an increase in the choice of quantity will reduce the informativeness of price in the Cournot setting where incomplete information is symmetric. **?** show that increased incentives to achieve a better rating can increase gaming and reduce the informativeness of the rating's parameter of interest.

[8]See further discussions in Raith [1996] and Vives [2001].

creased precision of shared information, the competitive effects of more precise information reduce expected prices and information sharing is not generally profitable. Our model offers a stark comparison: sharing common cost information *increases* the incentive to soften competition and can therefore have a positive effect on profits.

Results are mixed concerning the impact of information sharing on consumer welfare in oligopoly competition with private cost information. This has lead to differing conclusions about the competitive nature of these agreements; see discussions in Kühn and Vives [1995] and Vives [2001]. Under monopolistic competition, Vives [1990] shows that information sharing harms total surplus under price setting while improving it under quantity setting. Under Bertrand competition, prices are strategic complements and information sharing increases the covariance of prices leading to larger variance of quantity and lower expected surplus. Under Cournot competition, quantities are strategic substitutes, and information sharing leads to lower variance of aggregate quantity and therefore higher expected surplus.[9] In our setting expected prices increase when information is shared, and this reduces the likelihood that information sharing yields a welfare improvement. When the number of firms is large signal jamming incentives are minimized and welfare improvement is possible: both profits and consumer surplus increase when there is significant differentiation between products.[10]

Lastly our paper is related to the literature of information acquisition from multiple heterogeneous sources. The presence of an information sharing agreement promises a more precise public signal which lowers the value more precise signal about specific costs. This crowding out effect is identified in Colombo et al. [2014] where a more precise public signals reduces investment in precision of a private signal when coordination of strategies is beneficial.[11]

The rest of the paper is organized as follows. Sections 2 and 3, respectively, introduce and analyze the model of two stage price competition with and without information sharing. Section 4 studies the welfare impact of information sharing in the two firm case and in the case when the number of firms is large while extending the equilibrium characterization to n firms. Section 5 examines the value of information with and without information sharing. Section 6 concludes. Most proofs are given in the Appendixes.

[9] In Cournot competition, Shapiro [1986] shows that information sharing of cost information increases total welfare and Sakai and Yamato [1989] show consumer surplus improves when products are differentiated enough and costs are positively correlated, similar to our conditions for an increase in consumer welfare.

[10] In the case of procurement auctions, Asker et al. [2019] show that sharing information about bidder competitiveness can improve bidder and consumer surplus at the expense of auctioneer revenue.

[11] Overuse of public information in the beauty contest coordination game is identified in Morris and Shin [2002], and Myatt and Wallace [2011] shows that endogenously acquired information in this setting will be increasingly public as player's actions are more complementary.

2 Model

Two firms, i and j, compete for market share over two periods $t = 1, 2$. Demand is linear in prices, symmetric across firms, and time-independent. Firm i's demand is given by

$$q_{it} = a - bp_{it} + ep_{jt}.^{12}$$

The demand parameters a and b are strictly positive, while the sign of e determines whether goods are complements ($e < 0$) or subsitutes ($e > 0$). We assume that demand is weakly more sensitive to a firm's own price than to its opponent's, so that $-b \leq e \leq b$. Throughout, we let $r \equiv e/b$ denote the relative dependency of firm i's demand on firm j's price. Each firm faces a constant marginal cost c_i that is the same in each period, so profits are

$$\pi_{it} = (p_{it} - c_i)\, q_{it}.$$

Firms are initially uncertain about their marginal costs of production, but know that costs are comprised of an idiosyncratic component θ_i and a common component ρ; their constant marginal cost is the sum of the two components, $c_i = \rho + \theta_i$. We assume that cost components are joint-normally distributed with zero covariance, so that $\theta_i, \theta_j \sim N(\mu_\theta, \sigma_\theta^2)$ and $\rho \sim N(\mu_\rho, \sigma_\rho^2)$. Throughout, we will denote the precision of θ_i, θ_j by $\tau_\theta = 1/\sigma_\theta^2$ and the precision of ρ by $\tau_\rho = 1/\sigma_\rho^2$.

Play proceeds in two stages. In the first stage, each firm receives two noisy signals, $s_{i\theta}$ and $s_{i\rho}$, of the values of their idiosyncratic and common costs, respectively.[13] These signals are normally distributed with uncorrelated error terms, and the error terms are uncorrelated between firms. We model these signals as $s_{i\theta} = \theta_i + \varepsilon_{i\theta}$ and $s_{i\rho} = \rho + \varepsilon_{i\rho}$, where $\varepsilon_{i\theta}$ and $\varepsilon_{i\rho}$ are independent and normally distributed with mean zero and variance $\sigma_{i\theta}^2$ and $\sigma_{i\rho}^2$, respectively.[14] Upon the realization of their private signals, firms simultaneously select prices p_{i1} and obtain stage profits π_{i1}.

After first-stage profits are obtained, firms learn both the common and their (individual) idiosyncratic cost components. Each also witnesses its opponent's first-stage price, but remains unaware of its opponent's idiosyncratic cost component.[15] Firms then compete in a

[12] Unless otherwise specified, our equations and inequalities should be taken to be symmetric for agent j.

[13] It is not essential that firm i has imperfect information regarding its specific cost θ_i, since our results depend only on what firm j can learn about firm i's specific costs. We assume noisy signals of specific costs for consistency with noisy signals of common costs; we discuss this assumption further in Section 3.

[14] For the majority of our results we will assume symmetry, so that $\sigma_{i\theta}^2 = \sigma_{j\theta}^2$ and $\sigma_{i\rho}^2 = \sigma_{j\rho}^2$. Allowing for heterogeneity in the variance of the error term is essential when we discuss value of information in Section 5.

[15] Since demand is a deterministic function of firm prices, the assumption that firms witness their own profits and each others' prices is sufficient to imply that they are perfectly informed of their own private cost

second stage by simultaneously selecting prices and obtain stage profits π_{i2}.

The game ends after the second stage, and ex post profits are the (undiscounted) sum of stage profits,

$$\pi_i\left(p_i, p_j\right) = \pi_{i1}\left(p_{i1}, p_{j1}\right) + \pi_{i2}\left(p_{i2}, p_{j2}\right).$$

We restrict attention to *subgame perfect equilibria in linear strategies.*

Definition 1. *A* subgame perfect equilibrium in linear strategies *is given by parameters* $(p_{it0}, p_{it\theta_i}, p_{it\theta_j}, p_{it\rho})_{i,t=1}^{2}$, *such that*

1. *Second-stage prices maximize profits, conditional on information* (s_i, θ_i, ρ) *and first-stage prices* $\mathbf{p}_1$*:*

$$\begin{aligned}
&p_{i20} + p_{i2\theta_i}\mathbb{E}\left[\theta_i|\, s_i, \mathbf{p}_1, \rho, \theta_i\right] + p_{i2\rho}\mathbb{E}\left[\rho|\, s_i, \mathbf{p}_1, \rho, \theta_i\right] + p_{i2\theta_j}\mathbb{E}\left[\theta_j|\, s_i, \mathbf{p}_1, \rho, \theta_i\right] \\
&= p_{i20} + p_{i2\theta_i}\theta_i + p_{i2\rho}\rho + p_{i2\theta_j}\mathbb{E}\left[\theta_j|\, \mathbf{p}_1, \rho\right] \\
&\in \operatorname*{argmax}_{\tilde{p}} \mathbb{E}\left[\left(a - b\tilde{p} + ep_{j2}\right)\left(\tilde{p} - [\theta_i + \rho]\right)|\, s_i, \mathbf{p}_1, \rho, \theta_i\right];
\end{aligned}$$

2. *First-stage prices maximize profits, conditional on information* s_i*:*

$$\begin{aligned}
&p_{i10} + p_{i1\theta_i}\mathbb{E}\left[\theta|\, s_i\right] + p_{i1\rho}\mathbb{E}\left[\rho|\, s_i\right] \\
&\in \operatorname*{argmax}_{\tilde{p}} \mathbb{E}\left[\left(a - b\tilde{p} + ep_{j1}\right)\left(\tilde{p} - [\theta_i + \rho]\right) + \pi_{i2}\left(p_{i2}, p_{j2}\right)|\, s_i\right];
\end{aligned}$$

3. $p_{i1\theta_j} = 0.$[16]

When equilibria are in linear strategies, prices are an affine function of expected common and specific costs. The equilibrium we find is an equilibrium without constraint to symmetric linear strategies, but we do not address the potential existence of equilibria in asymmetric or nonlinear strategies.

Two expositional notes are in order. First, following our initial equilibrium analysis, we consider an informational regime in which firms share their signals of the common cost ρ. Throughout, we use variables decorated with * (e.g., $\pi^\star$) to indicate values in the no-sharing equilibrium, and we use variables decorated with ^{c} (e.g., π^c) to indicate values in the equilibrium which arises following the sharing of common cost information. Second, for most of our analysis we focus on symmetric equilibria, and on the effects of information-sharing on first-period pricing. For space and (we hope) legibility we therefore abbreviate $p_{i10} \equiv p_0$, $p_{i1\theta} \equiv p_\theta$, and $p_{i1\rho} \equiv p_\rho$, where we do not believe it will create confusion.

c_i. Alternatively, if they witness their own sales volume they will be perfectly aware of their opponent's price. That they obtain perfect knowledge of each of the components of $c_i = \rho + \theta_i$ is an additional assumption.

[16]This constraint exists because $\mathbb{E}[\theta_j|s_i] = \mu_\theta$ is constant, hence p_{i10} cannot be identified from $p_{i1\theta_j}$.

3 Equilibrium

We compute the pricing equilibrium in the two stage model by backwards induction. In a subgame perfect equilibrium second period prices are best responses to available information. However, even in an equilibrium where first period prices are strictly monotone in each signal, it is impossible for private information to be fully-revealed as private information is two-dimensional while actions are one-dimensional. Residual uncertainty in the second stage is an important feature in our model, affecting firms' first period pricing through their ability to distort publicly available information about their costs.

3.1 Second period pricing

In the second period, each firm knows its own marginal costs exactly, but knows only the distribution over its opponent's costs. Letting $\mathbf{p_1} \equiv (p_{i1}, p_{j1})$ and $F^j(\cdot;\rho,\mathbf{p}_1) \equiv F^j$ be the distribution of firm j's second period price conditional on firm i's available information,[17] the profit maximization problem is

$$\max_p \int (p - c_i)(a - bp + ex)\, dF^j(x).$$

Lemma 1. *Firm i's optimal second period price is*

$$p_{i2}^\star = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right]\right).$$

Firm i's maximum second period expected profit is

$$\mathbb{E}\left[\pi_{i2}^\star \middle| \rho, \mathbf{p}_1\right] = \frac{1}{4b}\left(a - bc_i + e\mathbb{E}\left[p_{j2}^\star \middle| \rho, \mathbf{p}_1\right]\right)^2.$$

Because the game ends in the second period, optimal second period prices are identical to those in a single-stage duopoly model where the opponent's price is distributed according to F^j. Firm i's second period price is an affine function of the demand intercept, its (known) cost $c_i = \rho + \theta_i$, and its expectation over firm j's second period price. Profits then have a standard quadratic form.

Lemma 2. *In any equilibrium, expected second period prices of a firm given publicly available*

[17] Firm i also knows θ_i, $s_{i\theta}$, and $s_{i\rho}$, but these offer no information in the second stage about firm j's pricing strategy. Equivalently, $(\rho, \mathbf{p}_1)$ is firm i's knowledge of information common to both firms.

information are

$$\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]=\frac{1}{4b^2-e^2}\left(\left(2b+e\right)a+2b^2\mathbb{E}\left[c_j|\,\rho,p_{j1}\right]+be\mathbb{E}\left[c_i|\,\rho,p_{i1}\right]\right),$$

which result in the following expected second period profits:

$$\mathbb{E}\left[\pi_{i2}^{\star}|\,\rho,\mathbf{p}_1\right]=\frac{1}{4}\left(\frac{1}{4b^2-e^2}\right)^2\left(\left(4b+2e\right)a-4b^2c_i+\left(\mathbb{E}\left[c_i|\,\rho,\mathbf{p}_1\right]-c_i\right)e^2+2be\mathbb{E}\left[c_j|\,\rho,\mathbf{p}_1\right]\right)^2.$$

Note that the expression for expected profits in Lemma 2 is written in terms of expected costs, conditional only on the information relevant to forming an expectation of each firm's costs. Although firm i's first period price p_{i1} is informative regarding the common cost ρ and may be useful to make inferences from firm j's price p_{j1}, once in the second period firms have full knowledge of ρ and firm i's first period price p_{i1} no longer provides further information regarding firm j's specific cost θ_j.

3.2 First period pricing

First period prices are set to optimize the sum of profits over two periods. Although first period prices have no direct effect on second period profits, firm i's price affects firm j's beliefs regarding firm i's costs. This is shown directly in Lemma 2, where p_{i1} enters only through $\mathbb{E}[c_i|\rho,p_{i1}]$, which is the expectation of firm i's cost given information available to firm j in the second period. Firm i has an incentive to over-represent its cost, leading firm j to increase its second period price, softening competition for firm i.[18] The first period profit maximization problem is

$$\max_p \mathbb{E}\left[\left(a-bp+ep_{j1}\right)\left(p-c_i\right)+\pi_{i2}^{\star}|\,s_{i\rho},s_{i\theta}\right].$$

A marginal increase in first period price affects first period profits in a standard way, and has an additional effect on second period profits by manipulation of the opposing firm's second period beliefs. The first order condition is given in Lemma 3.

Lemma 3. *For a given first period pricing strategy p_{j1} of firm j, firm i's optimal first period*

[18]This describes the reaction of firm j when the firms are selling substitutes, $e>0$. When $e<0$, a higher value of $\mathbb{E}[c_i|\rho,p_{i1}]$ leads to a lower $p_{j2}^{\star}$, which still increases $\pi_{i2}^{\star}$. When $e=0$, the price and profit equations reduce to the standard monopoly model.

price is given by

$$\hat{p}_{i1} = \left(\frac{1}{2b}\right) \mathbb{E}\left[bc_i + a + ep_{j1} \middle| s_{i\rho}, s_{i\theta}\right]$$
$$+ e\left(\frac{1}{2b}\right)^2 \mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p^\star_{j2}\middle| \rho, \hat{p}_{i1}\right]\right) \frac{\partial}{\partial p_{i,1}} \mathbb{E}\left[p^\star_{j2}\middle| \rho, \hat{p}_{i1}\right] \middle| s_{i\rho}, s_{i\theta}\right].$$

We constrain attention to equilibria in pricing strategies that are linear in the expected value of each cost component.[19] In the Gaussian model we use, this is equivalent to prices being linear in signal. A linear first period price can be expressed as

$$p_{i1} = p_{i0} + \mathbb{E}\left[\theta_i\middle| s_{i\theta}\right] p_{i\theta} + \mathbb{E}\left[\rho\middle| s_{i\rho}\right] p_{i\rho}.$$

Under linear strategies, each firm's first period price choice is a normally distributed random variable from the perspective of the other firm. Therefore, (c_i, ρ, p_{i1}) are distributed joint-normally and $\mathbb{E}[c_i|\rho, p_{i1}]$ is linear in p_{i1}. Additionally, the effect of an increase in firm i's first period price on firm j's second period beliefs, and hence second period price, is constant and independent of the level of price. Conditioning beliefs on this relationship gives Lemma 4.

Lemma 4. *The marginal effect of firm i's first period price on firm j's expected second period price is*

$$\frac{\partial}{\partial p_{i1}} \mathbb{E}\left[p^\star_{j2}\middle| \rho, p_{i1}\right] = \frac{be}{4b^2 - e^2} \kappa_i,$$
$$\text{where } \kappa_i \equiv \frac{\partial}{\partial p_{i1}} \mathbb{E}\left[c_i\middle| \rho, p_{i1}\right] = \frac{\sigma^2_\theta \bar{\tau}_{i\theta} p_\theta}{\sigma^2_\rho \left(1 - \bar{\tau}_{i\rho}\right) \bar{\tau}_{i\rho} p^2_\rho + \sigma^2_\theta \bar{\tau}_{i\theta} p^2_\theta} \text{ and } \bar{\tau}_{ix} = \frac{\tau_{ix}}{\tau_x + \tau_{ix}}.$$

The dependence of firm j's second period price on firm i's first period price can be further manipulated to $d\mathbb{E}[p^\star_{j2}|\rho, p_{i1}]/dp_{i1} = r\kappa_i/(4 - r^2)$. This form makes clear that the response of firm j's second period price to firm i's first period price depends only on the informativeness of firm i's first period price and the relative substitutability (or complementarity) r of the two firms' goods. Although the optimal price will depend on the absolute demand responses b and e to firm prices, holding fixed their relative values the absolute magnitudes of these parameters do not affect a firm's response to its opponent's first period price.

The term κ_i captures the relative informativeness of firm i's first period price regarding

[19] This forces each firm to commit to using information about each cost component at a fixed level for all possible signals $(s_{i\theta}, s_{i\rho})$ it may receive. As we show, these strategies are best responses to the opponent's linear pricing rule, even allowing for nonlinear pricing rules, when the firm has received its private signals. We do not address the possible existence of equilibria in nonlinear pricing rules.

its specific cost component θ_i, the remaining source of asymmetric information in the second period once ρ is commonly known. Despite observing ρ, firms do not observe each other's first period signal on the common cost component, $s_{i\rho}$. Because the first period price depends on the realization of $s_{i\rho}$, price is a noisy signal of $s_{i\theta}$. Therefore the informativeness of the price with respect to θ_i depends not only on the variance of the price relative to $s_{i\theta}$ but also relative to $s_{i\rho}$.

For $x \in \{\rho, \theta\}$, $\bar{\tau}_{ix}$ is the relative contribution of the noise in firm i's signal around the true parameter x to the precision of the signal s_{ix}. When signals are very noisy, $\bar{\tau}_{ix}$ will be close to zero; when signals give a more precise prediction of the true cost parameter, $\bar{\tau}_{ix}$ will be close to one. Fixing price coefficients, increasing the precision of the specific cost signal ($\bar{\tau}_{i\theta}$) its role in the formation of second stage expectations of specific costs, and will increase κ_i. Increasing the precision of the common cost signal ($\bar{\tau}_{i\rho}$) may increase or decrease κ_i, depending on its magnitude. When $\bar{\tau}_{i\rho}$ is low, firms are equally ignorant and have essentially common information about common costs; increasing $\bar{\tau}_{i\rho}$ makes firms' information more idiosyncratic and reduces the quality of price as a signal of specific costs. The opposite is true when $\bar{\tau}_{i\rho}$ is large.

Similar comparative statics hold with respect to p_{ix}. The term $\bar{\tau}_{ix}p_{ix}$ is the derivative of first period price with respect to s_{ix}, and affects the informativeness of the first period price about the firm's cost. Therefore the choice of strategy in the first period for a given level of information precision will directly impact the value of κ_i. Specifically, κ_i decreases as either $p_{i\theta}$ or $p_{i\rho}$ increases. If a firm increases p_{ix} while signal precisions remain constant, it is increasing the variance of price and therefore changes in price will be less informative of the firm's information. Moreover, the incentive constraints of the equilibrium strategy in the first period depend on the value of κ_i.[20,21] This fixed point problem is expressed in the single variable equation in Theorem 1.

Theorem 1. *There exists a unique symmetric equilibrium in linear pricing strategies. The equilibrium strategies are determined by the value of κ in equilibrium which satisfies the*

[20] A price deviation does not indicate a specific misreport of marginal cost but rather an iso-price curve of feasible $(s_{i\rho}, s_{i\theta})$. Because prices are affine, iso-price curves are affine. The slope of the iso-price curves determines the value of κ_i.

[21] Since pricing strategies are not observed, κ_i is not affected by firm i's selection of price; it is determined by the pricing strategy the firm is believed to be following. The fact features in to our analysis of the value of information in Section 5.

following single variable equation:

$$\kappa^\star = \frac{\sigma_\theta^2 \bar{\tau}_\theta p_\theta^\star}{\sigma_\rho^2 \left(1-\bar{\tau}_\rho\right) \bar{\tau}_\rho p_\rho^{\star 2} + \sigma_\theta^2 \bar{\tau}_\theta p_\theta^{\star 2}},$$

$$\textit{subject to } p_\theta^\star = \frac{1}{2+\beta\kappa^\star} \textit{ and } p_\rho^\star = \frac{1-\left(\frac{b-e}{2b-e}\right)\beta\kappa^\star}{2-\frac{e}{b}\bar{\tau}_\rho - \frac{1}{2}\left(1-\bar{\tau}_\rho\right)\beta^2\kappa^{\star 2}},$$

where $\beta = \frac{e^2}{4b^2-e^2}$.

Note that all expressions in Theorem 1 can be written in terms of $r \equiv e/b$, and b and e can be eliminated. As is the case with second period prices, firms' equilibrium responses to information depend only on the relative substitutability (or complementarity) of their goods, and not the magnitude of demand response to prices.[22]

Remark 1. *With the exception of comparative statics on $\bar{\tau}_\theta$, our results are essentially unaffected by the assumption that firm i has imperfect information of its specific costs θ_i. Letting firms have perfect knowledge of their specific costs while remaining uncertain of their opponent's costs is equivalent to letting $\bar{\tau}_\theta = 1$. In this case firm j remains uncertain of firm i's specific costs, and firm i's incentive to soften future competition is qualitatively unchanged. As a modeling assumption, we retain imperfect information regarding specific costs for consistency with imperfect information regarding common costs.*

Our results also remain valid when specific costs are common knowledge ($\sigma_\theta = 0$). In this case, there is no private information in the second round of competition, hence there is no incentive to soften future competition. Equilibrium price coeffecients are standard, $p_\theta^\star = 1/2$ and $p_\rho^\star = 1/(2 - r\bar{\tau}_\rho)$.

There are two strategic effects we can identify in first period prices. First, due to the correlation of one cost signal and the independence of the other signal, firms may want to act more heavily on one of these signals than the other if they prefer to have their prices correlated in the first period. Additionally, firms benefit from having private information in the second period and therefore prefer to not reveal precise information about their idiosyncratic cost term. The implications of the first effect are in Proposition 1 and the implications of the second effect are in Proposition 2.

Proposition 1. *When goods are complements ($e < 0$), $p_\rho^\star < p_\theta^\star$; when goods are substitutes ($e > 0$), $p_\rho^\star > p_\theta^\star$; and when goods are independent ($e = 0$), $p_\rho^\star = p_\theta^\star$.*

[22] The absolute magnitudes of b and e will factor in to p_0, the additive component of the linear price structure.

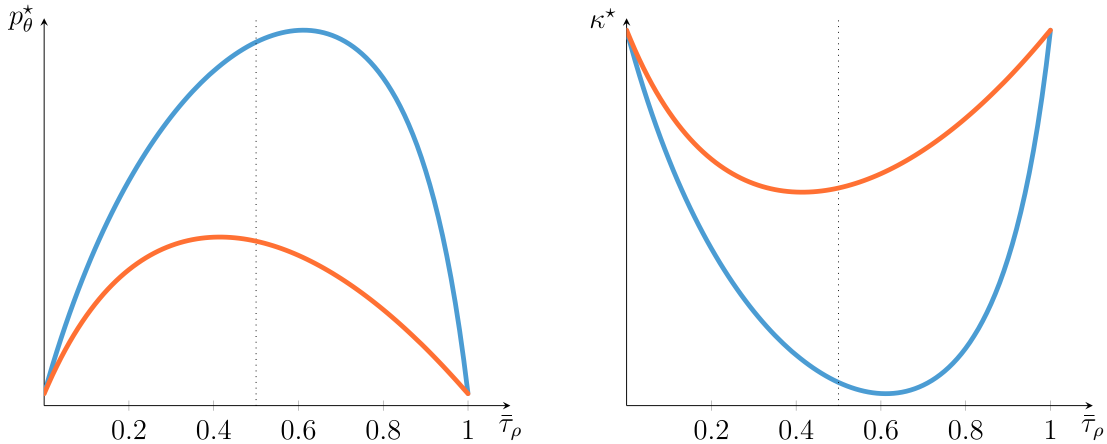

Figure 2: The firm's ability to maximize stage profits while confounding information is maximized when there is an intermediate amount of information regarding common costs ($\overline{\tau}_\rho$ interior). When there is either no or complete information regarding common costs ($\overline{\tau}_\rho \in \{0, 1\}$), private information regarding specific costs cannot be hidden, and cost-misrepresentation incentives are maximized. Where price responsiveness is maximized and information transmission is minimized depends on whether goods are complements (red curve) or substitutes (blue curve). The ability of price to signal specific information ($\kappa^\star$) is inversely proportional to the sensitivity of price to information about specific costs.

When $e > 0$, so that goods are substitutes, firms' first period prices are more sensitive to information on the common cost component than to information on their idiosyncratic cost component. If a firm receives a high signal on the common cost component this often implies the other firm will set a high price, increasing demand and making it optimal to further increase price. When $e < 0$, so that goods are complements, prices are strategic substitutes and will not respond strongly to the common cost signal. When $e = 0$, so that there are no cross-firm demand effects, there is no need to either adjust for the opponent's price and information about each cost component affects first period prices identically. Moreover, in the monopoly case there will be no attempt to conceal information regarding cost. However, in general the information conveyed by first period prices will affect second period profits. Proposition 2 illustrates firms' incentives to not reveal too much information about their specific cost components.

Proposition 2. *The values of $p_\theta^\star$ and $\kappa^\star$ are inversely related: $p_\theta^\star$ increases when $\kappa^\star$ decreases and vice versa. Additionally, $p_\theta^\star$ is decreasing and $\kappa^\star$ is increasing in $\bar{\tau}_\theta$, and there is a $\hat{\tau}$ such that for all $\bar{\tau}_\rho > \hat{\tau}$, $\kappa^\star$ is increasing and $p_\theta^\star$ is decreasing in $\bar{\tau}_\rho$, and for all $\bar{\tau}_\rho < \hat{\tau}$, $\kappa^\star$ is decreasing and $p_\theta^\star$ is increasing in $\bar{\tau}_\rho$. When $e > 0$, $\hat{\tau} > 1/2$ and when $e < 0$, $\hat{\tau} < 1/2$.*

The presence of uncertainty on the common component of cost adds noise to the relationship between first period price and the signal on idiosyncratic cost. When this relationship is more noisy, the price reveals less information about this signal, allowing the firm to use the information in its pricing decision without revealing too much. If the signal about the common cost is relatively imprecise ($\bar{\tau}_\rho \approx 0$) then firms do not learn much information from this signal, and relatively little noise is added to this relationship. Additionally, if the signal is very precise ($\bar{\tau}_\rho \approx 1$) then when firms learn the true value of ρ in the second round, they will learn with little error what signal their opponents received and will be able to disentangle the noise in the pricing strategy. Therefore, for a given value of $\bar{\tau}_\theta$, an intermediate level of precision $\bar{\tau}_\rho$ will maximize $p^\star_\theta$.

In general the incentives to hide idiosyncratic cost information leads firms to be less responsive to their idiosyncratic cost signal than is optimal in a one-stage game (without the informational channels implied by our two-stage model). Firms will increase the sensitivity of the first period price to information on the idiosyncratic cost component when incentives to signal jam are relaxed. Since the second dimension of uncertainty introduces noise into equilibrium pricing decisions, prices will be more responsive to information about specific costs than in a model without a common cost component.

3.3 Sharing industry relevant information

We now consider the effect of the firms sharing information about costs, for example through a trade association. We assume that signals about the common cost component are shared, while those of firms' specific costs are not. Information shared via a trade association is that which is relevant to the production process of all firms, for example common input costs, and firms prefer to maintain private information about specific costs.

When firms share their signals about their common cost component they will have the same expectation about this parameter. This simplifies the two stage competition model to a variation of the single cost component models in Mailath [1989] and Jeitschko et al. [2018]. While there are still two cost components, the informational structure is simplified so that firms only possess private information about their specific cost components; the remaining uncertainty regarding the common cost component is common to both firms. While the optimality conditions look similar in this setting, the equilibrium pricing strategies in the first period fully reveal the private information of each firm. We outline the significant differences from the previous section.

In the second period the information that is available to each firm now includes both common cost signals, $s_\rho \equiv (s_{i\rho}, s_{j\rho})$. The new first order conditions are given in Lemma 5.

Lemma 5. *Firm i's optimal second period price is*

$$p_{i2}^c = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[\left.p_{j2}^c\right| \rho, s_\rho, \mathbf{p_1}\right]\right).$$

Firm i's optimal first period price for a given pricing strategy of firm j is

$$\begin{aligned}\hat{p}_{i1} = &\left(\frac{1}{2b}\right)\mathbb{E}\left[bc_i + a + ep_{j1}\right| s_\rho, s_{i\theta}] \\ &+ e\left(\frac{1}{2b}\right)^2 \mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j2}\right| \rho, s_\rho, \hat{p}_{i1}]\right)\frac{\partial}{\partial p_{i1}}\mathbb{E}\left[\left.p_{j2}^c\right| \rho, s_\rho, \hat{p}_{i1}\right]\middle| s_\rho, s_{i\theta}\right].\end{aligned}$$

In a symmetric linear equilibrium, the first period price given signals $(s_{i\theta}, s_{i\rho})$ is $p_{i,1}^c = p_0^c + p_\theta^c\mathbb{E}[\theta_i|s_{i\theta}] + p_\rho^c\mathbb{E}[\rho|s_\rho]$. Because s_ρ and p_{i1} are publicly observable, in equilibrium the value of $s_{i\theta}$ can be inferred by competing firms. Therefore the expectation of each firm's cost in the second period given publicly available information is $\mathbb{E}[c_i|\rho, s_\rho, p_{i1}] = \rho + \mathbb{E}[\theta_i|s_{i,\theta}]$, where $s_{i\theta}$ can be determined from the first period price. The impact of firm i's first period price on firm j's second period price is

$$\frac{\partial}{\partial p_{i1}}\mathbb{E}\left[\left.p_{j2}^c\right| \rho, p_{i1}^c\right] = \frac{be}{4b^2 - e^2}\kappa^c, \text{ where } \kappa^c \equiv \frac{\partial}{\partial p_{i1}}\mathbb{E}\left[c_i\right| \rho, s_\rho, p_{i1}^c] = \frac{1}{p_\theta^c}.$$

Given the simplified information structure after information sharing, in equilibrium firms have a stronger incentive to misrepresent costs; at the same time, first period prices are more informative about private signals of specific costs. Therefore in the unique linear equilibrium, p_θ^c is strictly less than $p_\theta^\star$, indicating that firms use less specific cost information in their first period price and $\kappa^c \geq \kappa^*$.

Proposition 3. *In the unique equilibrium in linear pricing strategies the coefficient on specific cost information is less than the corresponding coefficient in the equilibrium without information sharing:*

$$p_\theta^c = \frac{1-\beta}{2} \leq p_\theta^\star.$$

Additionally, prices are more informative than in the corresponding equilibrium without information sharing: $\kappa^c \geq \kappa^\star$. Both inequalities are strict when $e \neq 0$.

Once firms have shared common cost information, firms' second period prices are more responsive to the price choices in the first period. In this setting, it is easier to soften future competition and therefore firms have a greater incentive to choose a higher first period price. The increase in expected price imposes a first order negative effect on consumer welfare.

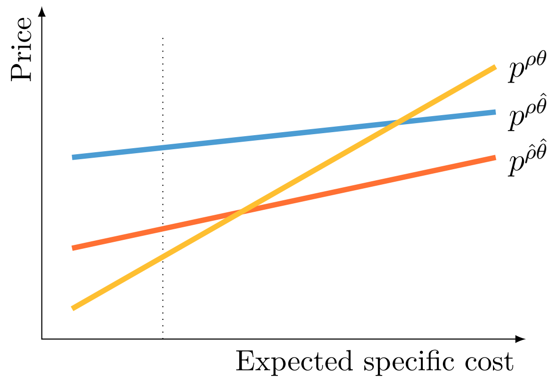

Figure 3: With private information about shared costs, prices ($p^{\hat{\rho}\hat{\theta}}$) are more responsive to information about specific costs than when there is no private information about shared costs ($p^{\rho\hat{\theta}}$), but less responsive to information than when there is no private information at all ($p^{\rho\theta}$). Expected prices (where conditional expected costs equal unconditional expected costs, the dotted line) are higher when firms share information, and $p^{\rho\hat{\theta}}$ will intersect $p^{\rho\theta}$ to the right of the intersection with $p^{\hat{\rho}\hat{\theta}}$.

Theorem 2. *Expected first period prices are higher when firms share signals about common cost information,* $\mathbb{E}[p^{\star}_{i1}] \leq \mathbb{E}[p^{c}_{i1}]$ *where the inequality is strict when* $e \neq 0$. *Moreover, expected second period prices are the same regardless of firms sharing information or not.*

From the first order conditions in Lemmas 3 and 5, there are two differences in determining the optimal price in the first period. The first is the expected price by the competing firm in the second period. The difference in information structure does not change expected prices in the second period as optimal prices are linear in the beliefs about the competing firms costs, and on average, the beliefs must be correct in equilibrium. The second is the rate at which an increase in first period price increases the competitor's second period price. From Proposition 3, the rate of increase is higher when firms share common cost information.

4 Welfare and profits

Section 3 gives equilibrium characterizations when firms share no information, and when they share information about common costs. We now consider the welfare effects of information sharing. The linear demand structure we use is induced by the utility specification in (1),[23]

$$u(\mathbf{q};\mathbf{p}) = \frac{a}{b-e}(q_i+q_j) - \frac{1}{2}\left(\frac{b}{b^2-e^2}\right)(q_i^2+q_j^2) - \left(\frac{e}{b^2-e^2}\right)q_iq_j - (p_iq_i+p_jq_j) \qquad (1)$$

[23]It is assumed that consumer utility is quadratic in consumption and linear in payments.

From this utility specification, algebraic manipulation gives Lemmas 6 and 7 regarding consumer and producer surplus.

Lemma 6. *In a symmetric linear equilibrium, expected consumer surplus in each period of competition is represented by the following utility function.*

$$\mathbb{E}\left[u\left(\mathbf{p}_{\mathbf{t}}^{\star}\right)\right] = \left(-2a + (b-e)\mathbb{E}\left[p_{it}^{\star}\right]\right)\mathbb{E}\left[p_{it}^{\star}\right] + b\,\mathrm{Var}\left(p_{it}^{\star}\right) - e\,\mathrm{Cov}\left(p_{it}^{\star}, p_{jt}^{\star}\right)$$

When expected demand is positive, it is the case that $a > (b-e)\mathbb{E}[p_i]$. Then the expression in Lemma 6 is decreasing in $\mathbb{E}[p_{it}^{\star}]$ and $e\,\mathrm{Cov}(p_{it}^{\star}, p_{jt}^{\star})$ and increasing in $\mathrm{Var}(p_{it}^{\star})$. Higher average prices harm consumers, as does correlation in prices when goods are substitutes ($e > 0$). More volatile prices benefit consumers — expected surplus losses are dominated by expected surplus gains — as does correlation in prices when goods are complements ($e < 0$). That the effect of correlation depends on substitutability follows from the fact that correlation increases variance of the average purchase price of a good within a bundle when goods are complements, and decreases this variance when goods are substitutes.

Lemma 7. *In a symmetric linear equilibrium, expected producer surplus in each period t of competition is given by*

$$\begin{aligned}\mathbb{E}\left[\Pi_t^{\star}\right] = 2\big[&\left(a - (b-e)\,\mathbb{E}\left[p_{it}^{\star}\right]\right)\left(\mathbb{E}\left[p_{it}^{\star}\right] - \mathbb{E}\left[c_i\right]\right) + b\left(\mathrm{Cov}\left(c_i, p_{it}^{\star}\right) - \mathrm{Var}\left(p_{it}^{\star}\right)\right) \\ &-e\left(\mathrm{Cov}\left(c_i, p_{jt}^{\star}\right) - \mathrm{Cov}\left(p_{it}^{\star}, p_{jt}^{\star}\right)\right)\big]\,.\end{aligned} \tag{2}$$

Producer surplus increases with expected price. From Theorem 2, first period prices are higher when firms share industry cost information (and second period prices are unaffected). When demand is relatively inelastic (the parameter a is large relative to b and e) or goods are very substitutable ($e \approx b$), the change in expected price will dominate all other welfare effects from information sharing.

Proposition 4. *When a is large relative to b and e, or when $e \approx b$, sharing common cost information decreases expected consumer surplus. When a is large relative to b and e, sharing common cost information increases expected producer surplus.*

Remark 2. *Although we model Bertrand competition, our analytical approach can be straightforwardly applied to Cournot competition. As is familiar from the literature (see, e.g., Vives [1984]), the basic structure of equilibrium is unaffected by the mode of competition, but the sign of welfare effects is reversed. For example, under Cournot competition, sharing common cost information* increases *expected consumer surplus when demand is inelastic ($a \gg b, e$), or goods are sufficiently substitutable ($e \approx b$).*

4.1 Extension to n firms

Proposition 4 shows that expected producer surplus increases and expected consumer surplus falls when information is shared, provided $a \gg b \geq |e|$. A more general comparison is hampered by the size of the parameter space: comparisons of welfare across regimes will in general depend not only on the demand specification but also on the information structure induced by noisy cost signals. As noted in Proposition 2, equilibrium price coefficients and inference are not monotone in precision $\overline{\tau}_\rho$, making it difficult to directly apply standard methods from comparative statics.

To obtain sharper predictions for surplus we analyze to the n-firm analogue of our basic model. In each period t, and firm i's demand is

$$q_{itn}\left(p_{itn}, p_{-itn}\right) = \frac{1}{n-1}\left(a - bp_{itn} + \frac{e}{n-1}\sum_{j\neq i} p_{jtn}\right). \tag{3}$$

When $n = 2$ the normalization terms $n-1$ are identically 1; this returns our base model, $q_{it2} \equiv q_{it}$. All other assumptions from the base model — for example, conditionally independent signals of ρ — are maintained. This demand function is generated by consumer utility

$$\begin{aligned} u\left(\mathbf{q_t};\mathbf{p_t}\right) = {} & \frac{a}{b-e}\sum_{i=1}^{n} q_{itn} - \frac{n-1}{2}\left(\frac{(n-1)\,b-(n-2)\,e}{((n-1)\,b+e)\,(b-e)}\right)\sum_{i=1}^{n} q_{itn}^2 \\ & - \frac{n-1}{2}\left(\frac{e}{((n-1)\,b+e)\,(b-e)}\right)\sum_{i=1}^{n}\sum_{j\neq i} q_{itn}q_{jtn} - \sum_{i=1}^{n} p_{itn}q_{itn}. \end{aligned} \tag{4}$$

We maintain our focus on symmetric equilibria in linear pricing strategies. Unlike the base model we do not establish uniqueness. We first show that form of equilibrium pricing coefficients depends in a natural way on the number of firms. Then it is natural to view the n-firm extension as providing an approximation to results in our base model. The linear equilibrium analysis of the n-firm extension is not substantially different from that of the base case, and we omit most of the basic calculations; details are found in Appendix B.

Theorem 3. *In the linear equilibrium of the n-firm model,*

$$p_{i1n}^\star\left(s_{i\theta}, s_{i\rho}\right) = p_{0n}^\star + p_{\theta n}^\star \mathbb{E}\left[\theta_i \middle| s_{i\theta}\right] + p_{\rho n}^\star \mathbb{E}\left[\rho \middle| s_{i\rho}\right],$$

where

$$p_{\theta n}^\star = \frac{1}{2+\beta_n\kappa_n^\star}, \quad p_{\rho n}^\star = \frac{b - \left(\frac{b-e}{2b-e}\right)\beta_n\kappa_n^\star}{2b - e\overline{\tau}_\rho - \frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta_n^2\kappa_n^{\star\,2}}, \quad \text{and } \beta_n = \frac{e^2}{(2b-e)\left(2\,(n-1)\,b+e\right)}.$$

An indirect implication of Theorem 3 is that equilibrium inference from prices does not substantively change in the extension to n firms. That is, what firm $j \neq i$ can learn about firm i's specific cost θ_i from its first period price p_{i1n} depends only on $\kappa_n^\star$, which retains the same form as in the base case. It is not the case that $\kappa^\star$ is identical in the base case and the n-firm extension, but $\kappa_n^\star$ depends in the same way on $p_{\theta n}^\star$ and $p_\rho^\star$ regardless of the number of firms. Equivalently, $\kappa^\star$ and $\kappa_n^\star$ are identical functions of different price coefficients, $p_\theta^\star$ and $p_\rho^\star$ versus $p_{\theta n}^\star$ and $p_{\rho n}^\star$, respectively. Intuitively this is straightforward: once ρ is known, firm j can separate inferences about firm i's initial information from inferences about firm k's initial information. Because signals of ρ are conditionally uncorrelated, nothing learned about firm k can affect what is learned about firm i. If the independence assumption were relaxed, or if ρ were not made public in the second period, this would no longer be the case.

The linear equilibrium of the n-firm extension retains the cross-dependency of price coefficients and inference. We now turn to the limiting case where n becomes large to obtain comparative statics on producer and consumer surplus.

4.2 Welfare for n large

Even when the number of firms is large, inference about any particular firm remains relatively stable. How information affects prices when markets are large depends mostly on the incentive of any one firm to hide information from its opponents. As it turns out, in the limit no firm faces any incentive to obfuscate its private information. This is intuitive, as when the market is large any firm is one of many; since individual complementarities (or substitutabilities) $e/(n-1)$ are going to zero as the market becomes large, exposing private information does not dramatically affect opponent pricing incentives. This is true even as aggregate complementarities $\sum_{j\neq i} e/(n-1) = e$ are held constant.[24]

It is straightforward to see that equilibrium inference $\kappa_n^\star$ is bounded (between 0 and 3).[25] Then the effect of firm i's revelation, $\beta_n \kappa_n^\star$, goes to 0 as n becomes large, since

$$\lim_{n \nearrow \infty} \beta_n = \lim_{n \nearrow \infty} \frac{e^2}{(2b-e)\left(2\left(n-1\right)b+e\right)} = 0.$$

This implies a simple analytic form for equilibrium prices with a large number of firms.

[24]In the limit there is the additional question of whether the aggregation of these opponent incentives results in a strictly positive effect on the firm's incentives to hide information. Our results answer this in the negative.

[25]By definition, $\kappa_n^\star \leq 1/p_{\theta n}^\star$. Since $p_{\theta n}$ is bounded away from zero, $\kappa_n^\star$ is bounded above. A full proof is provided in Appendix D.

Theorem 4. *In the linear equilibrium of the large-n extension, equilibrium prices are*

$$p^{\star}_{i1\infty}\left(s_{i\theta}, s_{i\rho}\right) = p^{\star}_{0\infty} + p^{\star}_{\theta\infty}\mathbb{E}\left[\theta_i \middle| s_{i\theta}\right] + p^{\star}_{\rho\infty}\mathbb{E}\left[\rho \middle| s_{i\rho}\right]$$

where $r \equiv e/b$ *and*

$$p^{\star}_{0\infty} = \frac{1}{2-r}\left(\frac{a}{b} + \frac{1}{2}r\mu_\theta + \mu_\rho\right) - \frac{\mu_\rho}{2 - r\overline{\tau}_\rho}, \quad p^{\star}_{\theta\infty} = \frac{1}{2}, \quad \textit{and} \ p^{\star}_{\rho\infty} = \frac{1}{2 - r\overline{\tau}_\rho}.$$

The lack of incentives to hide information is immediate in Theorem 4. $p^{\star}_{\theta\infty} = 1/2$ is exactly the dependence of price on private cost information in the equivalent monopoly problem. The value of $p^{\star}_{\rho\infty}$ is similar to the dependence of price on commonly known costs in a standard duopoly problem. This dependence is adjusted by $\overline{\tau}_\rho$ to account for the fact that firm i's beliefs about firm j's beliefs are a reversion to the mean of firm i's ex ante beliefs. That is, if firm i believes $\mathbb{E}[\rho|s_{i\rho}] = \rho_i < \mu_\rho$, firm i believes that firm j believes $\mathbb{E}[\rho|s_j] \in (\rho_i, \mu_\rho)$.

Additionally, in the linear equilibrium with a large number of firms expected first-period price is independent of $\overline{\tau}_\rho$. While $p^{\star}_{\rho\infty}$ depends on $\overline{\tau}_\rho$, in expectation this is exactly offset by the $\mu_\rho p^{\star}_{\rho\infty}$ term in $p^{\star}_{0\infty}$. Then to the extent that information sharing (an increase in $\overline{\tau}_\rho$) alters producer or consumer surplus, it is through the coefficients $p^{\star}_{0\infty}$, $p^{\star}_{\rho\infty}$ appearing in the variance and covariance of prices.

Proposition 5. *There exist constants* $C_u, C_\pi \in \mathbb{R}$ *such that for any* $\overline{\tau}_\rho$*, first period consumer and producer surplus in a linear equilibrium of the large-n extension are given by*

$$\begin{aligned}
\mathbb{E}\left[u_{1\infty}\right] &\propto (1-r)\operatorname{Var}\left(p^{\star}_{i1\infty}\right) - r\operatorname{Cov}\left(p^{\star}_{i1\infty}, p^{\star}_{j1\infty}\right) + C_u,\\
\mathbb{E}\left[\Pi_{1\infty}\right] &\propto \left(\operatorname{Cov}\left(c_i, p^{\star}_{i1\infty}\right) - \operatorname{Var}\left(p^{\star}_{i1\infty}\right)\right) - r\left(\operatorname{Cov}\left(c_i, p^{\star}_{j1\infty}\right) - \operatorname{Cov}\left(p^{\star}_{i1\infty}, p^{\star}_{j1\infty}\right)\right) + C_\pi.
\end{aligned}$$

where i, j *are any firms such that* $i \neq j$.

The consumer surplus expression in Proposition 5 shows that expected consumer surplus is increasing in the variance of first period prices, and decreasing in covariance when goods are substitutes and increasing in covariance when goods are complements. Producer surplus is impacted by variance and covariance in a similar way. When information is shared in the first period, aggregation of an infinite number of informative signals is equivalent to common knowledge of ρ prior to setting first period prices.[26] Firms will then choose prices as in Theorem 4 where $\mathbb{E}[\rho|s_\rho] = \rho$ and $\bar{\tau}_\rho = 1$. Comparison of surplus in the first period

[26]This is the case when $\overline{\tau}_\rho > 0$. When $\overline{\tau}_\rho = 0$ aggregation yields no additional information, but this problem remains equivalent to profit maximization with only firm specific costs.

depends on how the variance and covariance of prices are affected by these changes in the first period strategy.

Second period strategies, and therefore second period surplus, are unaffected by information sharing. Common costs, ρ, are public knowledge in the second period, allowing potential inference of opponent information from first period prices. However the pricing strategies are symmetric, linear and depend only on expected opponent prices. Then an optimal strategy reduces to a linear strategy on expected opponent costs. With a large number of firms the law of large numbers applies, and the sum of expected opponent costs is equivalent to an average opponent cost. This is independent of whether or not information is shared.

Corollaries 1 and 2 summarize the welfare impact of sharing information when the number of firms is large. Information sharing never harms producer surplus and in almost all cases strictly increases it.

Corollary 1. *Producer surplus is increasing in precision $\overline{\tau}_\rho$, and therefore information sharing strictly improves producer surplus whenever $\overline{\tau}_\rho \in (0,1)$.*

The impact of information sharing on consumer surplus depends on $\bar{\tau}_\rho$ prior to information sharing and substitutability of the firm's products r. Specifically, when $r \leq (\sqrt{33}-5)/2 \approx 0.372$, i.e. products are complements or weakly substitutable, information sharing always improves consumer surplus. For intermediate values of r, surplus may increase when information is relatively dispersed prior to sharing, $\bar{\tau}_\rho \approx 0$. For low initial precision the ability to tie prices more directly to costs outweighs strategic effects, while for high initial precision there is not much information gained when signals are shared and strategic effects dominate. When good are relatively substitutable, $r \geq 1/2$, consumer surplus will be harmed for any initial value of $\bar{\tau}_\rho$.

Corollary 2. *When goods are complements, consumer surplus is increasing in precision $\overline{\tau}_\rho$. When goods are substitutes:*

- *Consumer surplus is increasing in precision $\overline{\tau}_\rho$ when $r \lesssim 0.372$;*
- *Consumer surplus is single-peaked in precision $\overline{\tau}_\rho$ when $0.372 \lesssim r < 1/2$;*
- *Consumer surplus is decreasing in precision $\overline{\tau}_\rho$ when $r \geq 1/2$.*

The proof of Corollary 2 proceeds from the observation that, when n is large, information sharing perfectly reveals the common cost parameter ρ. Determining the effect of information sharing on consumer welfare algebraically reduces to comparing a point on a quadratic (in $\overline{\tau}_\rho$) to 0. The bounds presented in Corollary 2 derive from analysis of this quadratic. When goods are moderately substitutable, $0.372 \lesssim r < 1/2$, information sharing increases consumer surplus when $\overline{\tau}_\rho$ is small and decreases consumer surplus when $\overline{\tau}_\rho$ is large.

5 Generalized information

So far we have taken the precision of information to be exogenous. In reality, firms may exert effort to improve their estimates of cost parameters. Treating the precision as a choice variable, we now consider the value of this information to the firms. Specifically, we examine the effect of an (unobserved) marginal increase in precision of cost information on the firm's profits over the two periods of competition.

The precision of cost information allows firms to make better pricing decisions directly affecting expected profits in the first period. However, marginal deviations from equilibrium levels of acquisition will not affect the linear pricing coefficients of either firm. By the envelope theorem, marginal changes in precision will not change the optimal pricing strategy of that firm. Additionally, the other firm cannot update its pricing strategy based on an unobserved deviation. Therefore, increases in precision will affect profits only through the ex ante variance and covariance of prices.

Lemma 8. *The marginal changes in first period profits with respect to the information precision on each cost component are*

$$\frac{\partial}{\partial \tau_{i\theta}} \mathbb{E}\left[\pi_{i1}^{\star}\right] = \left(\frac{\left(1-p_{i\theta}^{\star}\right) p_{i\theta}^{\star}}{\left(\tau_{i\theta}+\tau_{\theta}\right)^{2}}\right) b, \ \text{and} \ \frac{\partial}{\partial \tau_{i\rho}} \mathbb{E}\left[\pi_{i1}^{\star}\right] = \left(\frac{\left(1-p_{i\rho}^{\star}\right) p_{i\rho}^{\star}}{\left(\tau_{i\rho}+\tau_{\rho}\right)^{2}}\right)\left(b-\bar{\tau}_{j\rho} e\right).$$

First period profits respond to precision of the two components of marginal cost in a similar way with respect to own demand (the terms postmultiplied by b), but the response differs with respect to cross-firm demand (the term postmultiplied by e). To a first approximation, the precision of the informational signals affects the opponent's payoffs only through information on the common cost component; increasing this precision will increase the correlation in first period prices, and decreasing this precision will reduce the correlation in first period prices. Because improved precision will affect best response price coefficients, the aggregate effect of a large change in precision is less clear.

A marginal increase in precision does not impact the expected profit in the second period of competition.[27] Because the firm perfectly observes its cost components after the first period, increased precision about these parameters has no impact on the information the firm has in the second period. Since neither firm changes its pricing strategies, inferences about the opposing firm's cost structure given its first period price does not change. Lastly, because the increase in precision is not observed by the other firm, the inference its makes will also be unaffected. Therefore neither firm's pricing strategy in the second stage is affected

[27] Note that while marginal deviations in investment in precision cannot affect second period profits, different levels of believed investment will typically generate different second period profits.

by a deviation in informational investment.

Finally, which effect is larger — that is, whether it is more advantageous to invest in information regarding common costs or specific costs — depends in an ambiguous way on complementarity and precision. Although Proposition 1 establishes a natural ordering of price sensitivity to information depending on whether goods are complements or substitutes, we show in Appendix D that the natural bounds on price coefficients do not imply an ordering on $(1 - p_\theta)p_\theta \gtreqless (1 - p_\rho)p_\rho$, unless goods are complements ($e < 0$), in which case the postmultiplied term $b - e\overline{\tau}_\rho > b$. Since signal precision appears in the denominator of each marginal value expression, direct comparison will depend on the specific parameters of the model. Nonetheless, in the case of complements a general comparitive static is possible with regard to incentives for specific cost investment.

Corollary 3. *When goods are complements, κ is increasing, p_θ is decreasing, and investment incentives $d\mathbb{E}[\pi^\star_{i1}]/d\tau_{i\theta}$ are decreasing as $|e|$ increases.*

In the case of complements firms can use strategic ignorance as a commitment device. Increasing the degree of complementarity will decrease price responsiveness to specific costs, to avoid communicating private information too precisely. Information regarding specific costs becomes less valuable because it cannot be used, and investment incentives fall. As shown in Proposition 2, p_θ increases as τ_θ falls, partially offsetting the reduction in investment incentives. Each of these effects is at play with regard to the analysis of other investment incentives, but no effect generically dominates.

5.1 Sharing industry relevant information

We now analyze the effect of precision on the firms' profits in the case of information sharing within the industry. Because all information acquired about the common cost component is shared, the two periods of competition will follow as in Section 3.3. In this setting, pricing strategies in either period do not depend on information precision, and changes in precision does not change the information available to firms in the second stage. Therefore an increase in precision of either the common or idiosyncratic cost component will affect expected profits in the first stage only.

Lemma 9. *The marginal changes in first period profits with respect to precision on each cost component when firms are sharing information are*

$$\frac{\partial}{\partial \tau_{i\theta}}\mathbb{E}\left[\pi^c_{i1}\right] = \left(\frac{\left(1 - p^c_{i\theta}\right) p^c_{i\theta}}{\left(\tau_{i\theta} + \tau_\theta\right)^2}\right) b \text{ and } \frac{\partial}{\partial \tau_{i\rho}}\mathbb{E}\left[\pi^c_{i1}\right] = \left(\frac{\left(1 - p^c_{i\rho}\right) p^c_{i\rho}}{\left(\tau_{i\rho} + \tau_{j\rho} + \tau_\rho\right)^2}\right)(b - e).$$

Comparing the marginal benefit of precision on the private cost component between the two information regimes, it is clear that the value of information depends on how much this information is used in the first period of competition. The incentive to signal jam when sharing common cost information reduces the extent to which firms use information about private costs, which in turn reduces the value of acquiring this information.

Proposition 6. *The marginal value of precision in information about firm-specific costs is lower when firms share industry relevant information,*

$$\frac{\partial}{\partial \tau_{i\theta}} \mathbb{E}\left[\pi_{i1}^{\star}\right] \geq \frac{\partial}{\partial \tau_{i\theta}} \mathbb{E}\left[\pi_{i1}^{c}\right].$$

When $e \neq 0$ *this inequality is strict.*

Proof. The result follows directly from Lemmas 8 and 9, and Proposition 3, which implies $p_{i\theta}^{c} \leq p_{i\theta}^{\star} \leq 1/2$, with equality if and only if $e = 0$. □

In a market with a large number of firms, $p_{\theta\infty}^{c} = p_{\theta\infty}^{\star} = 1/2$. Of the terms directly relevant to firm profits (given in Proposition 5), increased precision of private cost information affects only $\mathrm{Var}(\mathrm{p_{i1}})$ and $\mathrm{Cov}(\mathrm{c_i}, \mathrm{p_{i1}})$, just as in the case of two firms. These terms only depend on $p_{\theta\infty}$ and the precision of cost signals. Therefore, the value of increased precision of specific cost information does not depend on information sharing on the common cost component in large markets. Moreover, since information sharing perfectly reveals the value of ρ whenever $\overline{\tau}_{\rho} > 0$, increased precision of common cost information is valuable only when information is not shared.

Corollary 4. *In the symmetric linear equilibrium of the large-n extension, the marginal value of precision of firm-specific cost signals is unaffected by information sharing, and the marginal value of precision of common costs is zero when information is shared,*

$$\frac{\partial}{\partial \tau_{i\theta}} \mathbb{E}\left[\pi_{i1n}^{\star}\right] = \frac{\partial}{\partial \tau_{i\theta}} \mathbb{E}\left[\pi_{i1n}^{c}\right], \;\; \textit{and} \;\; \frac{\partial}{\partial \tau_{i\rho}} \mathbb{E}\left[\pi_{i1n}^{\star}\right] > \frac{\partial}{\partial \tau_{i\rho}} \mathbb{E}\left[\pi_{i1n}^{c}\right] = 0.$$

5.2 Generalized information sharing agreements

In our main results, we consider an all-or-nothing information sharing agreement. Although it is possible to share information about common costs and not about specific costs, firms either share all information about a particular signal, or no information about that signal. In this subsection we generalize the analysis to consider partial information sharing agreements, and firms may share information which contains an arbitrary portion of each of their two signals. As an example, consider the aggregation of information about transportation costs.

This might represent (for example) half of all costs that are common to the firms and 20 percent of costs that are individual to each firm.

Formally, we assume that firm i receives a set of signals regarding cost x, $r_{ixm} = x + \varepsilon_{ixm}$, where $x \in \{\rho, \theta_i\}$, and $m \in \{1, \ldots, M_x\}$. Each error term ε_{ixm} is normally distributed with source-dependent variance, $\varepsilon_{ixm} \sim N(0, \sigma^2_{\varepsilon x})$, and is independent of $\varepsilon_{i'x'm'}$ for $(i, x, m) \neq (i', x', m')$. Then, prior to potential infomation sharing, firm i's signal about cost x is $s_{ix} = x + \sum_{m=1}^{M_x} \varepsilon_{ixm}/M_x$.[28]

Firms abide by an information sharing agreement, determining how many signals r_{ixm} to share with their competitors. Let $\tilde{M}_{i\to jx}$ be the number of firm i's signals about cost x which are shared with firm j.[29] Because the ε_{ixm} terms are independently and identically distributed, the effects of information sharing agreements depend only on the number of signals shared, and not on which signals are shared; without loss of generality, we therefore assume that firm j shares signals s_{jxm} for $m \in \{1, \ldots, \tilde{M}_{j\to ix}\}$. Consistent with the definition of $\tilde{M}_{i\to jx}$, we decorate post-sharing variables with tildes. After information sharing, firm i's signal about cost $x \in \{\rho, \theta_i, \theta_j\}$ is $\tilde{s}_{ix}$, where

$$\tilde{s}_{i\rho} = \rho + \frac{1}{M_\rho + \tilde{M}_{j\to i\rho}} \left(\sum_{m=1}^{M_\rho} \varepsilon_{i\rho m} + \sum_{m=1}^{\tilde{M}_{j\to i\rho}} \varepsilon_{j\rho m} \right) = \frac{1}{M_\rho + \tilde{M}_{j\to i\rho}} \left(\sum_{m=1}^{M_\rho} r_{i\rho m} + \sum_{m=1}^{\tilde{M}_{j\to i\rho}} r_{j\rho m} \right),$$

$$\tilde{s}_{i\theta_i} = \theta_i + \frac{1}{M_\theta} \sum_{m=1}^{M_\theta} \varepsilon_{i\theta m} = \frac{1}{M_\theta} \sum_{m=1}^{M_\theta} r_{i\theta m}, \quad \tilde{s}_{i\theta_j} = \theta_j + \frac{1}{\tilde{M}_{j\to i\theta}} \sum_{m=1}^{\tilde{M}_{j\to i\theta}} \varepsilon_{j\theta m} = \frac{1}{\tilde{M}_{j\to i\theta}} \sum_{m=1}^{\tilde{M}_{j\to i\theta}} r_{j\theta m}.$$

The precision of information on each component after sharing is

$$\tilde{\tau}_{i\rho} = \frac{M_\rho + \tilde{M}_{j\to i\rho}}{\sigma^2_{\varepsilon\rho}}, \quad \tilde{\tau}_{i\theta_i} = \frac{M_\theta}{\sigma^2_{\varepsilon\theta}}, \text{ and } \tilde{\tau}_{i\theta_j} = \frac{\tilde{M}_{j\to i\theta}}{\sigma^2_{\varepsilon\theta}}.$$

Note that, unlike in our base model, firms may directly share information about their specific costs. Then firm i has a signal $\tilde{s}_{i\theta_j}$ regarding firm j's specific costs, and equilibrium prices will typically respond to this information. Firm i's first period price $\tilde{p}_{i1}$ will depend not only on its signal of firm j's specific costs, $\tilde{s}_{i\theta_j}$, but also on what it knows to be firm j's information about its own specific costs, $\tilde{s}_{j\theta_i}$. The linear pricing strategy from our main results must be generalized to account for these new sources of price-relevant information.

Definition 2. *In a linear pricing strategy under generalized information sharing, first-period*

[28] The variance of this signal is $\sigma^2_{\varepsilon x}/M_x$. This can be mapped to our base model by rescaling the variance $\sigma^2_{\varepsilon x}$ by $\sqrt{M_x}$.

[29] This notation is related to that presented in Vives [2001]. In the appendix, we shorten equations by writing $\tilde{M}_{jx} \equiv \tilde{M}_{i\to jx}$, but for clarity of exposition we retain the "giving to" notation here in the main text.

prices are given by

$$\tilde{p}_{i1} = \tilde{p}_0 + \tilde{p}_{i\rho}\mathbb{E}\left[\rho\left|\tilde{s}_{i\rho}\right.\right] + \tilde{p}_{i\theta_i}\mathbb{E}\left[\theta_i\left|\tilde{s}_{i\theta_i}\right.\right] + \tilde{p}_{i\theta_j}\mathbb{E}\left[\theta_j\left|\tilde{s}_{i\theta_j}\right.\right] + \tilde{p}_{i\tilde{s}_{j\theta_i}}\mathbb{E}\left[\theta_i\left|\tilde{s}_{j\theta_i}\right.\right].$$

Theorem 5. *In the linear equilibrium under a generalized information sharing agreement, first-period price coefficients are*

$$\begin{aligned}
\tilde{p}_{i\theta_i} &= \frac{1}{2+\beta\tilde{\kappa}},\\
\tilde{p}_{i\theta_j} &= \frac{1}{2}\left(\frac{r}{2+\beta\tilde{\kappa}} + \frac{r}{2(4-r^2)}\beta\tilde{\kappa}\right),\\
\tilde{p}_{i\rho} &= \frac{1-\left(\frac{1-r}{2-r}\right)\beta\tilde{\kappa}}{2-(1-\eta_{i\rho})\left(r\tilde{\tilde{\tau}}_{j\rho} - \frac{1}{2}\beta^2\tilde{\kappa}^2(1-\tilde{\tilde{\tau}}_{j\rho})\right) - r\eta_{i\rho}},\\
\tilde{p}_{i\tilde{s}_{j\theta_i}} &= \frac{1}{4}\left(\frac{r^2}{2+\beta\kappa} + \frac{r^2}{2(4-r^2)}\beta\tilde{\kappa} + \beta^2\tilde{\kappa}^2\left(1-\frac{\tilde{\kappa}}{2+\beta\tilde{\kappa}}\right)\right),
\end{aligned}$$

where

$$\eta_{i\rho} = \frac{2\tilde{M}_{j\to i\rho}}{M_\rho + \tilde{M}_{j\to i\rho}}, \text{ and}$$

$$\tilde{\kappa} = \frac{\left(1-\frac{\sigma_\theta^2+\left[\sigma_{\varepsilon\theta}^2/M_\theta\right]}{\sigma_\theta^2+\left[\sigma_{\varepsilon\theta}^2/\tilde{M}_{j\to i\theta}\right]}\right)\tilde{p}_{j\theta_j}\tilde{\tilde{\tau}}_{j\theta_j}\sigma_\theta^2}{\left(1-\frac{\sigma_\theta^2+\left[\sigma_{\varepsilon\theta}^2/M_\theta\right]}{\sigma_\theta^2+\left[\sigma_{\varepsilon\theta}^2/\tilde{M}_{j\to i\theta}\right]}\right)\tilde{p}_{j\theta_j}^2\tilde{\tilde{\tau}}_{j\theta_j}\sigma_\theta^2 + \left(1-\frac{\tilde{M}_{i\to j\rho}+\tilde{M}_{j\to i\rho}}{M_\rho+\tilde{M}_{i\to j\rho}}\right)\tilde{p}_{j\rho}^2\tilde{\tilde{\tau}}_{j\rho}\left(1-\tilde{\tilde{\tau}}_{j\rho}\right)\sigma_\rho^2}.$$

Remark 3. *Our base model, with no information sharing, corresponds to* $\tilde{M}_{i\to j\theta} = \tilde{M}_{i\to j\rho} = 0$*. Our all-or-nothing information sharing model corresponds to* $\tilde{M}_{i\to jx} \in \{0, M_x\}$*. In both cases, substituting in for* $\tilde{M}_{i\to jx}$ *in Theorem 5 returns the equilibrium price coefficients from our earlier analyses. Note that if no information is shared about, e.g., specific cost* θ_i*, the conditional expectation* $\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}] = \mu_\theta$ *is independent of the signals* $r_{ix\theta}$*, and the coefficients* $\tilde{p}_{i\theta_j}$ *and* $\tilde{p}_{i\tilde{s}_{j\theta_i}}$ *are subsumed into* p_0*.*

Corollary 5. *Let* $\tilde{M}_{j\to ix} = \tilde{M}_{i\to jx} = \tilde{M}_x$*, and define* $\lambda_x = \tilde{M}_x/M_x$*. Suppose that the variance of error term* ε_{ixm} *is rescaled to match the base model,* $\mathrm{Var}(\varepsilon_{ixm}) = M_x\sigma_{\varepsilon x}^2$*. In the symmetric linear pricing equilibrium with generalized information sharing, the informational parameters are given by*

$$\eta_\rho = \frac{2\lambda_\rho}{1+\lambda_\rho}, \text{ and } \tilde{\kappa} = \frac{\left(1-\frac{\sigma_\theta^2+\sigma_{\varepsilon\theta}^2}{\sigma_\theta^2+\left[\sigma_{\varepsilon\theta}^2/\lambda_\theta\right]}\right)\tilde{p}_{j\theta_j}\tilde{\tilde{\tau}}_{j\theta_j}\sigma_\theta^2}{\left(1-\frac{\sigma_\theta^2+\sigma_{\varepsilon\theta}^2}{\sigma_\theta^2+\left[\sigma_{\varepsilon\theta}^2/\lambda_\theta\right]}\right)\tilde{p}_{j\theta_j}^2\tilde{\tilde{\tau}}_{j\theta_j}\sigma_\theta^2 + (1-\eta_\rho)\,\tilde{p}_{j\rho}^2\tilde{\tilde{\tau}}_{j\rho}\left(1-\tilde{\tilde{\tau}}_{j\rho}\right)\sigma_\rho^2}.$$

Compared to our model with all-or-nothing information sharing, the coefficient on θ_i does not change except through the change in the informativeness of the price, $\kappa \mapsto \tilde{\kappa}$. Therefore this coefficient and κ have an inverse relationship (as in Proposition 2). Corollary 5 makes clear that when no information about specific costs is shared, $\lambda_\theta = 0$, $\tilde{\kappa}$ is a version of our informativeness parameter κ when common-cost information may be fractionally shared; in this case, $\lambda_\rho = 0$ implies $\tilde{\kappa} = \kappa^\star$ and $\lambda_\rho = 1$ implies $\tilde{\kappa} = \kappa^c$.

Relative to the equilibrium in Section 3, the new pricing coefficients $\tilde{p}_{i\theta_j}$ and $\tilde{p}_{i\tilde{s}_{j\theta_i}}$ do not affect the informativeness of first period prices. Firm i knows the signal which is shared with firm j, and therefore any variance in firm j's price due to the shared signal is fully accounted-for by firm i. It follows that these coefficients should not appear in $\tilde{\kappa}$; the effect of information sharing is fully captured in the reduction of variance of the conditional expectation of opponent specific cost.

We can now generalize the results of Section 3 about the impact of information trading agreements on expected prices.

Proposition 7. *For a fixed information structure, an information sharing agreement which increases (decreases) $\tilde{\kappa}$ will cause ex-ante expected first period price to increase (decrease) and will not impact ex-ante expected second period prices.*

Given the level of expected price can be ranked by the equilibrium informativeness of the price, then the impact of an information sharing agreement on expected prices is determined by its impact on $\tilde{\kappa}$. When informativeness increases with sharing, then the agreement leads to higher prices, and vice versa. Specifically, Proposition 8 implies that an information sharing agreement only on common cost information will increase the expected price, while an agreement to share information about idiosyncratic costs will decrease expected prices.

Proposition 8. *For a fixed information structure, sharing more information on the common cost component (larger $\tilde{M}_{j\to i\rho}$ for given M_ρ, or larger $\lambda_{i\rho}$) will increase the equilibrium informativeness of prices while sharing more information on the idiosyncratic cost component (larger $\tilde{M}_{i\to j\theta}$ for given M_θ, or larger $\lambda_{i\theta}$) will decrease the informativeness of prices.*

6 Conclusion

Firms in a given industry typically have heterogeneous costs of production, but these costs may include a common component. When firms have idiosyncratic information about common costs, there may be benefits to sharing information through a trade association. The competitive impact of sharing information about industry-wide costs depends on how information is used and inferred without an information sharing agreement. To this end, we study a dynamic pricing competition model that allows for uncertainty in common cost and specific cost parameters. We characterize the symmetric linear equilibrium of this model and use it to examine how information sharing affects competition and welfare within the industry.

In the setting with two firms, information sharing increases incentives for firms to soften competition, leading to higher average prices and reducing the value of acquiring firm-specific cost information prior to competition. In settings where demand is relatively inelastic or products are close substitutes, information sharing can reduce consumer surplus while increasing producer surplus. As the number of firms in the market increases, the effect of competition softening is reduced; in particular, as the number of firms in the market becomes arbitrarily large no individual firm's price decision conveys useful additional information, and strategic signaling vanishes. In a market with a large number of firms, information sharing no longer has an impact on expected prices and can lead to both higher producer surplus and consumer surplus when products are not close substitutes and industry relevant information is dispersed among the firms.

Because sharing information about industry relevant costs may lead to higher producer surplus, agreements to share this information may not stem from purely collusive motives. In fact, there are cases where an information sharing agreement between many firms can improve both producer and consumer surplus. However, our results suggest increased consumer surplus is less likely for firms that sell products that are close substitutes. In particular, these agreements can be a concern for competition in concentrated markets where the agreements can lead to higher and more coordinated prices even in the absence of an explicit or implicit collusive agreement.

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A Proofs for Section 3

Proof of Lemma 1. This follows directly from firm i's first-order condition with respect to second-period price,

$$(a - bp_{i2}^{\star}) + \int exdF^{j}(x)\,dx - (p_{i2}^{\star} - c_i)\,b = 0 \quad \Longrightarrow \quad p_{i2}^{\star} = \frac{1}{2b}\left(a + bc_i + e\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho, \mathbf{p}_1\right]\right).$$

Substituting in to the firm's profit function yields the expression in Lemma 1. $\square$

Proof of Lemma 2. This follows from Lemmas 17 and 18, derived for the model with n firms. We give a proof for the two-firm case below.

Because firm i's optimal second period price, given in Lemma 1, holds given any information set, it also holds in expectation. That is,

$$\mathbb{E}\left[p_{i2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]=\frac{1}{2b}\mathbb{E}\left[a+bc_i+ep_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right].$$

Then

$$2b\mathbb{E}\left[p_{i2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]-e\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]=a+b\mathbb{E}\left[c_i\middle|\,\rho,\mathbf{p}_1\right].$$

This gives two equations, one each for firm i and firm j, in two unknowns, $\mathbb{E}[p_{i2}^{\star}|\rho,\mathbf{p}_1]$ and $\mathbb{E}[p_{j2}^{\star}|\rho,\mathbf{p}_1]$. Algebraic rearrangement yields the first expression in Lemma 2, and substituting into the firm's profit function yields the second. □

Proof of Lemma 3. This follows from standard monopoly profit maximization and application Lemma 1 to firm i's second-period profits,

$$\begin{aligned}&\max_p\mathbb{E}\left[\pi_{i1}\middle|\,s_{i\rho},s_{i\theta}\right]+\mathbb{E}\left[\pi_{i2}^{\star}\middle|\,s_{i\rho},s_{i\theta}\right]\\&=\max_p\mathbb{E}\left[\left(a-bp+ep_{j1}^{\star}\right)(p-c_i)+\frac{1}{4b}\left(a-bc_i+e\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]\right)^2\middle|\,s_{i\rho},s_{i\theta}\right].\end{aligned}$$

Note that second-period profits depend on p_{i1} only through $\mathbf{p}_1$'s effect on firm j's beliefs. Without substituting in with the expression in Lemma 1 we could have obtained a similar reduction by applying the envelope theorem (firm i's second period price is optimal, conditional on its first period price). Firm i's first order condition is

$$0=\mathbb{E}\left[\left(a+bc_i+ep_{j1}^{\star}\right)-2bp+\frac{e}{2b}\left(a-bc_i+e\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]\right)\frac{\partial}{\partial p_{i1}}\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]\middle|\,s_{i\rho},s_{i\theta}\right].$$

Rearrangement gives the desired result. □

Lemma 10. *Expected costs conditional on second period information are*

$$\mathbb{E}\left[c_i\middle|\,\rho,\mathbf{p}_1\right]=(\mu_\theta+\mu_\rho)+(1-\kappa_i\overline{\tau}_\rho p_{i\rho})(\rho-\mu_\rho)+\kappa_i\left(p_{i1}-(p_{i0}+p_{i\theta}\mu_\theta+p_{i\rho}\mu_\rho)\right),$$

$$\textit{subject to } \kappa_i=\frac{\overline{\tau}_\theta\sigma_\theta^2p_{i\theta}}{\overline{\tau}_\theta\sigma_\theta^2p_{i\theta}^2+(1-\overline{\tau}_\rho)\,\overline{\tau}_\rho\sigma_\rho^2p_{i\rho}^2}.$$

Proof. Note that, conditional on ρ, p_{j1} conveys no information about c_i. Then consider the joint distribution of c_i, p_{i1}, and ρ. Let $\tau_x\equiv 1/\sigma_x^2$ be the precision of the random variable x, and let $\overline{\tau}_x\equiv\tau_{\varepsilon_x}/(\tau_x+\tau_{\varepsilon_x})$ be the relative precision of the signal s_x, $x\in\{\theta_i,\theta_j,\rho\}$. Under a linear pricing strategy,

$$\begin{aligned} p_{i1} &= p_{i0} + p_{i\theta}\mathbb{E}\left[\theta_i \middle| s_{i\theta}\right] + p_{i\rho}\mathbb{E}\left[\rho \middle| s_{i\rho}\right] \\ &= \left(p_{i0} + \left(1-\overline{\tau}_\theta\right)p_{i\theta}\mu_\theta + \left(1-\overline{\tau}_\rho\right)p_{i\rho}\mu_\rho\right) + p_{i\theta}\overline{\tau}_\theta s_{i\theta} + p_{i\rho}\overline{\tau}_\rho s_{i\rho}. \end{aligned}$$

Then c_i, ρ, and p_{i1} are jointly normal,

$$(c_i, \rho, p_{i1})^T \sim N\left(\begin{pmatrix} \mu_\theta + \mu_\rho \\ \mu_\rho \\ \mathbb{E}\left[p_{i1}\right] \end{pmatrix}, \begin{pmatrix} \sigma_\theta^2 + \sigma_\rho^2 & \sigma_\rho^2 & \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta} + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} \\ \sigma_\rho^2 & \sigma_\rho^2 & \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} \\ \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta} + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} & \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} & \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2 \end{pmatrix}\right).$$

Then the conditional expectation of c_i, given ρ and p_{i1}, is

$$\begin{gathered} \mathbb{E}\left[c_i \middle| \rho, p_{i1}\right] = (\mu_\theta + \mu_\rho) + \Sigma_{12}\Sigma_{22}^{-1}\left((\rho, p_{i1})^T - (\mu_\rho, \mathbb{E}\left[p_{i1}\right])^T\right), \\ \Sigma_{12} = \left(\sigma_\rho^2, \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta} + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}\right), \quad \Sigma_{22} = \begin{pmatrix} \sigma_\rho^2 & \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} \\ \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho} & \overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2 \end{pmatrix}. \end{gathered}$$

Write the matrix product as $\Sigma_{12}\Sigma_{22}^{-1} = (m_{i1}, m_{i2})$. Then

$$\begin{aligned} m_{i1} &= \frac{1}{\left(\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2\right)\sigma_\rho^2 - \overline{\tau}_\rho^2\sigma_\rho^4 p_{i\rho}^2}\left(\overline{\tau}_\theta\sigma_\theta^2\sigma_\rho^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^4 p_{i\rho}^2 - \overline{\tau}_\theta\overline{\tau}_\rho\sigma_\theta^2\sigma_\rho^2 p_{i\theta}p_{i\rho} - \overline{\tau}_\rho^2\sigma_\rho^4 p_{i\rho}^2\right) \\ &= \frac{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2 - \overline{\tau}_\theta\overline{\tau}_\rho\sigma_\theta^2 p_{i\theta}p_{i\rho} - \overline{\tau}_\rho^2\sigma_\rho^2 p_{i\rho}^2}{\left(\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2\right) - \overline{\tau}_\rho^2\sigma_\rho^2 p_{i\rho}^2} = 1 - \kappa_i\overline{\tau}_\rho p_{i\rho}; \\ m_{i2} &= \frac{1}{\left(\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2\right)\sigma_\rho^2 - \overline{\tau}_\rho^2\sigma_\rho^4 p_{i\rho}^2}\left(-\overline{\tau}_\rho\sigma_\rho^4 p_{i\rho} + \overline{\tau}_\theta\sigma_\theta^2\sigma_\rho^2 p_{i\theta} + \overline{\tau}_\rho\sigma_\rho^4 p_{i\rho}\right) \\ &= \frac{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}}{\overline{\tau}_\theta\sigma_\theta^2 p_{i\theta}^2 + \left(1-\overline{\tau}_\rho\right)\overline{\tau}_\rho\sigma_\rho^2 p_{i\rho}^2} = \kappa_i. \end{aligned}$$

The result is then immediate. □

Proof of Lemma 4. This follows immediately from Lemma 10. □

Lemma 11. *There is an equilibrium in symmetric linear pricing strategies, where*

$$\begin{aligned} &\kappa = \frac{\sigma_\theta^2\overline{\tau}_\theta p_\theta}{\left(1-\overline{\tau}_\rho\right)\overline{\tau}_\rho\sigma_\rho^2 p_\rho^2 + \overline{\tau}_\theta\sigma_\theta^2 p_\theta}, \\ &\textit{subject to } p_\theta = \frac{1}{2+\beta\kappa} \textit{ and } p_\rho = \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa}{2 - r\overline{\tau}_\rho - \frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2}, \end{aligned}$$

where $r = e/b$ and $\beta = r^2/(4-r^2)$.

Proof. From Lemmas 3 and 4, first period prices are given by

$$4bp_{i1}^{\star} = 2\mathbb{E}\left[bc_i + a + ep_{j1}^{\star}\middle| s_{i\rho}, s_{i\theta}\right] + \mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j2}^{\star}\middle| \rho, p_{i1}^{\star}, p_{j1}^{\star}\right]\right)\beta\kappa_i\middle| s_{i\rho}, s_{i\theta}\right].$$

Lemma 2 gives second period expected prices,

$$\mathbb{E}\left[p_{j2}^{\star}\middle| \rho, \mathbf{p}_1\right] = \frac{1}{4b^2 - e^2}\left((2b+e)\,a + 2b^2\mathbb{E}\left[c_j\middle| \rho, p_{j1}\right] + be\mathbb{E}\left[c_i\middle| \rho, p_{i1}\right]\right).$$

Following Lemma 10,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[c_j\middle| \rho, \mathbf{p}_1\right]\middle| s_{i\theta}, s_{i\rho}\right] &= \mu_\theta + \mathbb{E}\left[\rho\middle| s_{i\rho}\right], \\
\mathbb{E}\left[\mathbb{E}\left[c_i\middle| \rho, \mathbf{p}_1\right]\middle| s_{i\theta}, s_{i\rho}\right] &= \mu_\theta + \mathbb{E}\left[\rho\middle| s_{i\rho}\right] \\
&\qquad + \kappa_i p_{i\theta}\left(\mathbb{E}\left[\theta_i\middle| s_i\right] - \mu_\theta\right) + \kappa_i p_{i\rho}\left(1 - \overline{\tau}_\rho\right)\left(\mathbb{E}\left[\rho\middle| s_{i\rho}\right] - \mu_\rho\right).
\end{aligned}$$

Substituting in gives

$$\begin{aligned}
4bp_{i1}^{\star} = 2\mathbb{E}\left[bc_i + a + ep_{j1}^{\star}\middle| s_{i\rho}, s_{i\theta}\right] + \beta\kappa_i\mathbb{E}\left[\left(a - bc_i + \frac{e}{4b^2 - e^2}\left((2b+e)\,a\right)\right)\middle| s_{i\rho}, s_{i\theta}\right] \\
+ \frac{e\beta\kappa_i}{4b^2 - e^2}\mathbb{E}\left[2b^2\left(\mu_\theta + \rho\right) + be\left(\mu_\theta + \rho + \kappa_i p_{i\theta}\left(\theta_i - \mu_\theta\right) + \kappa_i p_{i\rho}\left(1 - \overline{\tau}_\rho\right)\left(\rho - \mu_\rho\right)\right)\middle| s_{i\rho}, s_{i\theta}\right].
\end{aligned}$$

Recall that $\mathbb{E}[p_{j1}^{\star}|s_{i\rho}, s_{i\theta}] = p_{j0} + p_{j\theta}\mu_\theta + p_{j\rho}\overline{\tau}_\rho\mathbb{E}[\rho|s_{i\rho}] + p_{j\rho}(1-\overline{\tau}_\rho)\mu_\rho$. Matching coefficients gives

$$4bp_{i\theta} = 2b - b\beta\kappa_i + \frac{be^2\beta\kappa_i^2 p_{i\theta}}{4b^2 - e^2}; \tag{5}$$

$$4bp_{i\rho} = 2b + 2e\overline{\tau}_\rho p_{j\rho} - b\beta\kappa_i + \frac{e\beta\kappa_i}{4b^2 - e^2}\left(2b^2 + be\left(1 + \left(1 - \overline{\tau}_\rho\right)\kappa_i p_{i\rho}\right)\right). \tag{6}$$

In a symmetric equilibrium, $p_{i\theta} \equiv p_\theta$, $p_{i\rho} \equiv p_\rho$, and $\kappa_i \equiv \kappa$ for both firms $i \in \{1, 2\}$. Then the coefficients in equations (5) and (6) can be solved,

$$\begin{aligned}
p_\theta &= \frac{2 - \beta\kappa}{4 - \beta^2\kappa^2} = \frac{1}{2 + \beta\kappa}; \\
p_\rho &= \frac{1}{2}\left(1 - \frac{1}{2}\beta\kappa + \frac{1}{r}\beta^2\kappa + \frac{1}{2}\beta^2\kappa\right)\left(1 - \frac{1}{2}r\overline{\tau}_\rho - \frac{1}{4}\left(1 - \overline{\tau}_\rho\right)\beta^2\kappa^2\right)^{-1} \\
&= \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa}{2 - r\overline{\tau}_\rho - \frac{1}{2}\left(1 - \overline{\tau}_\rho\right)\beta^2\kappa^2}.
\end{aligned}$$

The conditional definition of κ follows from Lemma 4. □

Proof of Theorem 1. The expression for linear price coefficients follows from Lemma 11. Substituting price coefficients into κ and letting $\hat{\kappa} \equiv \beta\kappa$ gives

$$\underbrace{(2+\hat{\kappa})^2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)^2(1-\overline{\tau}_\rho)\,\overline{\tau}_\rho\sigma_\rho^2\hat{\kappa}}_{\text{LHS}(\hat{\kappa})}$$
$$=\underbrace{\left(2-r\overline{\tau}_\rho-\frac{1}{2}(1-\overline{\tau}_\rho)\,\hat{\kappa}^2\right)^2((2+\hat{\kappa})\,\beta-\hat{\kappa})\,\overline{\tau}_\theta\sigma_\theta^2}_{\text{RHS}(\hat{\kappa})}.$$

Note that LHS(0) $= 0 <$ RHS(0). Furthermore, we show in Appendix D that $\hat{\kappa} \leq r^2/(2-r^2) \equiv \bar{\kappa}$; then we have LHS$(\bar{\kappa}) > 0 =$ RHS$(\bar{\kappa})$. Since both LHS and RHS are continuous in $\hat{\kappa}$, it follows that there is a $\hat{\kappa} \in [0, \bar{\kappa}]$ that solves LHS$(\hat{\kappa}) =$ RHS$(\hat{\kappa})$.

It is clear that RHS is decreasing in $\hat{\kappa}$, since $(2+\hat{\kappa})\beta + \hat{\kappa} = 2\beta - (1-\beta)\hat{\kappa}$. We now show that LHS is either increasing, or increasing-then-decreasing and concave; in the latter case, we show also that RHS is convex where LHS is decreasing. Since LHS(0) $<$ RHS(0) and LHS$(\bar{\kappa}) >$ RHS$(\bar{\kappa})$, this is sufficient to show that there is a unique $\hat{\kappa}$ such that LHS$(\hat{\kappa}) =$ RHS$(\hat{\kappa})$.

To begin, the derivative of LHS is given by

$$\begin{aligned}\frac{d\,\text{LHS}}{d\hat{\kappa}} &= 2\,(2+\hat{\kappa})\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)^2(1-\overline{\tau}_\rho)\,\overline{\tau}_\rho\sigma_\rho^2\hat{\kappa}\\ &\quad -2\left(\frac{1-r}{2-r}\right)(2+\hat{\kappa})^2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)(1-\overline{\tau})\,\overline{\tau}\sigma_\rho^2\hat{\kappa}\\ &\quad +(2+\hat{\kappa})^2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)^2(1-\overline{\tau})\,\overline{\tau}\sigma_\rho^2\\ &\propto (2+\hat{\kappa})\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\\ &\quad \times\left[2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\hat{\kappa}-2\left(\frac{1-r}{2-r}\right)(2+\hat{\kappa})\,\hat{\kappa}+(2+\hat{\kappa})\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\right].\end{aligned}$$

The leading terms are positive for $\hat{\kappa} \in [0, \bar{\kappa}]$. The trailing term is

$$\begin{aligned}2\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\hat{\kappa}-2\left(\frac{1-r}{2-r}\right)(2+\hat{\kappa})\,\hat{\kappa}+(2+\hat{\kappa})\left(1-\left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)\\ \propto -5\,(1-r)\,\hat{\kappa}^2+3r\hat{\kappa}+2\,(2-r)\,. \qquad (7)\end{aligned}$$

This is a negative quadratic in $\hat{\kappa}$, and is strictly positive when $\hat{\kappa} = 0$; thus LHS is either increasing for $\hat{\kappa} \in [0, \bar{\kappa}]$, or it is increasing-then-decreasing on this range. When goods are

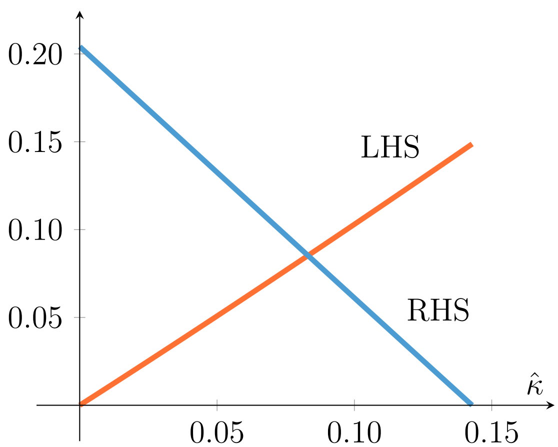

Figure 4: A graphical depiction of the proof of equilibrium existence and uniqueness. The existence of an equilibrium amounts to finding a $\hat{\kappa}$ such that $\text{LHS}(\hat{\kappa}) = \text{RHS}(\hat{\kappa})$. Since $\text{LHS}(0) < \text{RHS}(0)$ and $\text{LHS}(\overline{\kappa}) > \text{RHS}(\overline{\kappa})$ and both functions are continuous, such a $\hat{\kappa}$ is guaranteed to exist. For all parameter specifications RHS is decreasing. We show that either LHS is increasing (left panel) or increasing and then decreasing (right panel). In the former case, it is clear that there is a unique point of intersection and hence a unique equilibrium. In the latter case, we show that LHS is concave where it is decreasing and RHS is convex anywhere LHS is decreasing. Then $\text{LHS} - \text{RHS}$ is concave, ensuring that equilibrium $\hat{\kappa}$ is unique. Plot ranges differ, since the upper bound on $\hat{\kappa}$, $\overline{\kappa} = r^2/(2 - r^2)$ depends on the parameter r. Dashed lines in the right panel appear at $\hat{\kappa} = 1/2$ and $\hat{\kappa} = (\sqrt{249} - 3)/20$, the bounds applied in the proof.

substitutes ($r \geq 0$) this is positive for all relevant $\hat{\kappa}$ and the proof is complete. We then focus on the case where goods are complements ($r < 0$).

Replacing the leading positive terms in LHS gives

$$\frac{d\,\mathrm{LHS}}{d\hat{\kappa}} \propto \left(2\left(2-r\right)+3r\hat{\kappa}-5\left(1-r\right)\hat{\kappa}^2\right)\left(2\left(2-r\right)+r\hat{\kappa}-\left(1-r\right)\hat{\kappa}^2\right).$$

This implies

$$\frac{d^2\,\mathrm{LHS}}{d\hat{\kappa}^2} \propto 8\left(2-r\right)r-6\left(3r^2-12r+8\right)\hat{\kappa}-24\left(1-r\right)r\hat{\kappa}^2+20\left(1-r\right)^2\hat{\kappa}^3.$$

This is negative at $\hat{\kappa}=0$ and $\hat{\kappa}=1\geq\bar{\kappa}$. Moreover,

$$\frac{d^3\,\mathrm{LHS}}{d\hat{\kappa}^3} \propto -\left(18r^2-72r+48\right)-48\left(1-r\right)r\hat{\kappa}+60\left(1-r\right)^2\hat{\kappa}^2.$$

This is a positive quadratic in $\hat{\kappa}$, thus $d^2\,\mathrm{LHS}/d\hat{\kappa}^2$ is either decreasing, decreasing-then-increasing, or increasing for $\hat{\kappa}\in[0,\bar{\kappa}]$. Since $d^2\,\mathrm{LHS}/d\hat{\kappa}^2\leq 0$ for $\hat{\kappa}\in\{0,\bar{\kappa}\}$, it follows that $d^2\,\mathrm{LHS}/d\hat{\kappa}^2\leq 0$ for all $\hat{\kappa}\in[0,\bar{\kappa}]$, and LHS is concave.

If LHS is decreasing, it must be that the quadratic in (7) is negative. The zeros of this quadratic are given by

$$\hat{\kappa}_{\pm}\in\frac{3r}{10-10r}\pm\frac{1}{10-10r}\sqrt{9r^2+8\left(2-r\right)\left(5-5r\right)}=\frac{1}{10-10r}\left(3r+\sqrt{49r^2-120r+80}\right).$$

LHS is decreasing only if goods are complements, $r<0$, so only the "+" solution is valid. Note that

$$\begin{aligned}\frac{d\hat{\kappa}_+}{dr} &\overset{\text{sign}}{=}\left(3+\frac{49r-60}{\sqrt{49r^2-120r+80}}\right)\left(10-10r\right)+10\left(3r+\sqrt{49r^2-120r+80}\right)\\ &\overset{\text{sign}}{=}20-11r+3\sqrt{49r^2-120r+80}>0.\end{aligned}$$

Then $\hat{\kappa}_+$ is minimized when $r=-1$ (since $r\in[-1,1]$ and $d\hat{\kappa}_+/dr<0$ when $r<0$). This gives that if LHS is decreasing at $\hat{\kappa}$,

$$\hat{\kappa}\geq\bar{\kappa}_+=\frac{1}{20}\left(-3+\sqrt{249}\right)\geq\frac{15-3}{20}>\frac{1}{2}.$$

Finally, we compute the second derivative of RHS with respect to $\hat{\kappa}$ to show that RHS is convex,

$$\begin{aligned}\frac{d^2\,\mathrm{RHS}}{d\hat{\kappa}^2} &= \frac{d}{d\hat{\kappa}}\Bigg[-2\left(1-\overline{\tau}_\rho\right)\hat{\kappa}\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\hat{\kappa}^2\right)\left(2\beta-\left(1-\beta\right)\hat{\kappa}\right)\\
&\qquad -\left(1-\beta\right)\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\hat{\kappa}^2\right)^2\Bigg]\\
&= 2\left(1-\overline{\tau}_\rho\right)\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\hat{\kappa}^2\right)\left(4\left(1-\beta\right)\hat{\kappa}-4\beta\right)\\
&\quad +4\left(1-\overline{\tau}_\rho^2\right)\hat{\kappa}^2\left(2\beta-\left(1-\beta\right)\hat{\kappa}\right).\end{aligned}$$

Note that all involved terms are positive for $\hat{\kappa}\in[0,\bar{\kappa}]$, with the potential exception of $4(1-\beta)\hat{\kappa}-\beta$. As shown above, $\hat{\kappa}\geq 1/2$ whenever LHS is decreasing, hence

$$4\left(1-\beta\right)\hat{\kappa}-4\beta\geq 2\left(1-\beta\right)-4\beta=2-6\beta\geq 0. \qquad \text{(since } \beta=r^2/(4-r^2)\leq 1/3\text{)}$$

Then $d^2\,\mathrm{RHS}\,/d\hat{\kappa}^2>0$ when LHS is decreasing. Then where LHS is decreasing it is convex and RHS is concave, implying a unique intersection. □

Proof of Proposition 1. When $e=0$, $\beta=e^2/(4b^2-e^2)=0$. Then $p_\theta=p_\rho=1/2$. Otherwise, we compare

$$\begin{aligned}p_\theta\gtrless p_\rho \iff&\qquad \frac{1}{2+\beta\kappa}\gtrless\frac{1-\left(\frac{b-e}{2b-e}\right)\beta\kappa}{2-\frac{e}{b}\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2}\\
\iff&\qquad 2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2\gtrless\left(2+\beta\kappa\right)\left(1-\left(\frac{1-r}{2-r}\right)\beta\kappa\right)\\
\iff&\qquad -r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2\gtrless\left(\frac{r}{2-r}\right)\beta\kappa-\left(\frac{1-r}{2-r}\right)\beta^2\kappa^2.\end{aligned}\tag{8}$$

When $r>0$ the left-hand side of (8) is maximized when $\overline{\tau}_\rho=0$. This leads to

$$p_\theta<p_\rho \quad\Longleftarrow\quad \left(\frac{1-r}{2-r}-\frac{1}{2}\right)\beta\kappa<\frac{r}{2-r}.$$

The left-hand side is negative and the right-hand side is positive, so $p_\theta<p_\rho$.

When $r<0$ the left-hand side of (8) is minimized when $\overline{\tau}_\rho=0$. This leads to

$$p_\theta>p_\rho \quad\Longleftarrow\quad \left(\frac{1-r}{2-r}-\frac{1}{2}\right)\beta\kappa>\frac{r}{2-r}.$$

The left-hand side is positive and the right-hand side is negative, so $p_\theta>p_\rho$. □

Proof of Proposition 2. The inverse relationship of p_θ and κ follows immediately from the definition $p_\theta = 1/(2+\beta\kappa)$.

The remaining relationships follow from the quintic implicit equation for κ,

$$\underbrace{(2+\beta\kappa)^2\left(1-\left(\frac{1-r}{2-r}\right)\beta\kappa\right)^2\sigma_{s\rho}^2\overline{\tau}_\rho^2\kappa}_{\text{LHS}(\kappa)}$$
$$=\underbrace{\left(2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2\right)^2\left(2-\left(1-\beta\right)\kappa\right)\sigma_\theta^2\overline{\tau}_\theta}_{\text{RHS}(\kappa)}.[30]$$

Note that LHS is constant in $\overline{\tau}_\theta$ and RHS is linearly increasing in $\overline{\tau}_\theta$. All involved functions are continuous and differentiable, hence we check

$$\frac{d}{d\overline{\tau}_\theta}\left[\text{LHS}\left(\kappa\right)-\text{RHS}\left(\kappa\right)\right]=\left(\frac{\partial\text{LHS}}{\partial\kappa}-\frac{\partial\text{RHS}}{\partial\kappa}\right)\frac{\partial\kappa}{\partial\overline{\tau}_\theta}+\left(\frac{\partial\text{LHS}}{\partial\overline{\tau}_\theta}-\frac{\partial\text{RHS}}{\partial\overline{\tau}_\theta}\right).$$

Then

$$\frac{\partial\kappa}{\partial\overline{\tau}_\theta}=\frac{\frac{\partial\text{RHS}}{\partial\overline{\tau}_\theta}}{\frac{\partial\text{LHS}}{\partial\kappa}-\frac{\partial\text{RHS}}{\partial\kappa}}.\tag{9}$$

At the unique κ such that LHS(κ) = RHS(κ) it is the case that $\partial\text{LHS}(\kappa)/\partial\kappa > \partial\text{RHS}(\kappa)/\partial\kappa$, it follows that κ is increasing in $\overline{\tau}_\theta$.

To compute comparative statics with respect to $\overline{\tau}_\rho$, we first check

$$\begin{aligned}\frac{2-r\overline{\tau}_\rho-\frac{1}{2}\left(1-\overline{\tau}_\rho\right)\beta^2\kappa^2}{\overline{\tau}_\rho\sqrt{\sigma_{s\rho}^2}}&=\frac{1}{\sqrt{\sigma_{s\rho}^2}}\left(\frac{1}{\overline{\tau}_\rho}\left(2-\frac{1}{2}\beta^2\kappa^2\right)-\left(r-\frac{1}{2}\beta^2\kappa^2\right)\right)\\&=\left(\left(\frac{\sqrt{\sigma_{s\rho}^2}}{\sigma_\rho^2}+\frac{1}{\sqrt{\sigma_{s\rho}^2}}\right)\left(2-\frac{1}{2}\beta^2\kappa^2\right)-\frac{1}{\sqrt{\sigma_{s\rho}^2}}\left(r-\frac{1}{2}\beta^2\kappa^2\right)\right)\\&=\frac{1}{\sqrt{\sigma_\rho^2}}\left(\sqrt{\frac{\sigma_{s\rho}^2}{\sigma_\rho^2}}\left(2-\frac{1}{2}\beta^2\kappa^2\right)+\sqrt{\frac{\sigma_\rho^2}{\sigma_{s\rho}^2}}\left(2-r\right)\right).\end{aligned}$$

Fixing σ_ρ^2, $\overline{\tau}_\rho$ increases when $\sigma_{s\rho}^2$ decreases (and vice versa). Letting $R_\rho = \sqrt{\sigma_{s\rho}^2/\sigma_\rho^2}$, we

[30]The functions LHS and RHS differ slightly from those used in the proof of Theorem 1. In particular, they are functions of κ and include β terms, while those used in the proof of Theorem 1 are functions of $\hat{\kappa}$ and do not include β terms.

define LHS^R and RHS^R as

$$\underbrace{(2+\beta\kappa)^2\left(1-\left(\frac{1-r}{2-r}\right)\beta\kappa\right)^2\kappa}_{\mathrm{LHS}^R(\kappa)}$$
$$=\underbrace{\frac{1}{\sqrt{\sigma_\rho^2}}\left(\left(2-\frac{1}{2}\beta^2\kappa^2\right)R_\rho+(2-r)\frac{1}{R_\rho}\right)(2-(1-\beta)\kappa)\,\sigma_\theta^2\overline{\tau}_\theta}_{\mathrm{RHS}^R(\kappa)}.$$

Note that LHS^R is constant in R_ρ. Holding κ fixed, the extent to which RHS^R is affected by R_ρ is given by

$$\frac{\partial}{\partial R_\rho}\left[\left(2-\frac{1}{2}\beta^2\kappa^2\right)R_\rho+(2-r)\frac{1}{R_\rho}\right]=\left(2-\frac{1}{2}\beta^2\kappa^2\right)-\frac{2-r}{R_\rho^2}. \tag{10}$$

Since $\beta\kappa\leq 1$ and $r\leq 1$, the above is negative when R_ρ^2 is small and positive when R_ρ^2 is large. Since $R_\rho^2=(1-\overline{\tau}_\rho)/\overline{\tau}_\rho$, the above is negative when $\overline{\tau}_\rho$ is large and positive when $\overline{\tau}_\rho$ is small. An analysis similar to equation (9) implies that κ is decreasing in R_ρ (increasing in $\overline{\tau}_\rho$) when $\overline{\tau}_\rho$ is large and increasing in R_ρ (decreasing in $\overline{\tau}_\rho$) when $\overline{\tau}_\rho$ is small. To see single-peakedness, note that as R_ρ increases, (10) also increases. Starting from a point at which κ is locally constant, a slight increase in R_ρ from a point at which κ is locally constant must cause κ to rise; otherwise, κ is falling, implying that (10) is even more positive, a contradiction.

Finally, note that

$$\left(2-\frac{1}{2}\beta^2\kappa^2\right)-\frac{2-r}{R_\rho^2}=2\left(\frac{R_\rho^2-1}{R_\rho^2}\right)+\left(\frac{r-\frac{1}{2}\beta^2\kappa^2R_\rho^2}{R_\rho^2}\right).$$

When $R_\rho^2=1$, this is simply $r-\beta^2\kappa^2/2\overset{\text{sign}}{=}r$. Then (10) is positive at $R_\rho^2=1$ ($\overline{\tau}_\rho=1/2$) when $r>0$, and negative at $R_\rho^2=1$ when $r<0$. From single-peakedness, it follows that κ is minimized at $\overline{\tau}^\star>1/2$ when $r>0$ and at $\overline{\tau}^\star<1/2$ when $r<0$. $\square$

Proof of Lemma 5. Second period prices p_{i2}^c follow from the same methodology applied in the proof of Lemma 1. First period prices p_{i1}^c follow from the same methodology applied in the proof of Lemma 3. $\square$

Lemma 12. *When common cost information is shared, expected second period prices are*

$$\mathbb{E}\left[p_{i2}^c\middle|\,\rho,s_\rho,\mathbf{p}_1\right]=\frac{1}{4b^2-e^2}\left((2b+e)\,a+2b^2\mathbb{E}\left[c_i\middle|\,\rho,s_\rho,\mathbf{p}_1\right]+be\mathbb{E}\left[c_j\middle|\,\rho,s_\rho,\mathbf{p}_1\right]\right).$$

Proof. Following Lemma 5, we have

$$\mathbb{E}\left[p_{i2}^c\middle|\,\rho,s_\rho,\mathbf{p}_1\right]=\frac{1}{2b}\left(a+b\mathbb{E}\left[c_i\middle|\,\rho,s_\rho,\mathbf{p}_1\right]+e\mathbb{E}\left[p_{j2}^c\middle|\,\rho,s_\rho,\mathbf{p}_1\right]\right).$$

This yields two linear equations in two unknowns. Solving this linear system gives the desired equation. □

Proof of Proposition 3. We begin by computing p_θ^c, then address comparisons to the no-information-sharing regime.

Lemma 5 and the statement that $\partial\mathbb{E}[p_{j2}^c|\rho,s_\rho,\mathbf{p}_1]/dp_{i1}=b\beta/ep_{i\theta}^c$ give that first-period prices are

$$p_{i1}^c=\frac{1}{2b}\mathbb{E}\left[a+bc_i+ep_{j1}^c\middle|\,s_i\right]+\frac{1}{4b}\mathbb{E}\left[\left(a-bc_i+e\mathbb{E}\left[p_{j2}^c\middle|\,\rho,s_\rho,\mathbf{p}_1\right]\right)\frac{\beta}{p_{i\theta}^c}\middle|\,s_i\right].\qquad(11)$$

Following Lemma 12 we have

$$\begin{aligned}&\mathbb{E}\left[-bc_i+e\mathbb{E}\left[p_{j2}^c\middle|\,\rho,s_\rho,\mathbf{p}_1\right]\middle|\,s_i\right]\\&=\mathbb{E}\left[\mathbb{E}\left[-bc_i+ep_{j2}^c\middle|\,\rho,s_\rho,\mathbf{p}_1\right]\middle|\,s_i\right]\\&=\mathbb{E}\left[\mathbb{E}\left[-bc_i+\frac{e}{4b^2-e^2}\left((2b+e)\,a+2b^2c_j+bec_i\right)\middle|\,\rho,s_\rho,\mathbf{p}_1\right]\middle|\,s_i\right]\\&=\frac{1}{4b^2-e^2}\mathbb{E}\left[\mathbb{E}\left[(2b+e)\,ea+2b^2ec_j-2\left(2b^2-e^2\right)bc_i\middle|\,\rho,s_\rho,\mathbf{p}_1\right]\middle|\,s_i\right].\end{aligned}$$

In a linear equilibrium, $s_{i\theta}$ is perfectly revealed by p_{i1}. Then the above is

$$\begin{aligned}&\mathbb{E}\left[-bc_i+e\mathbb{E}\left[p_{j2}^c\middle|\,\rho,s_\rho,\mathbf{p}_1\right]\middle|\,s_i\right]\\&=\frac{1}{4b^2-e^2}\mathbb{E}\left[(2b+e)\,ea-2\left(2b^2-e^2\right)bc_i+2b^2e\mathbb{E}\left[c_j\middle|\,\rho,s_\rho,\mathbf{p}_1\right]\middle|\,s_i\right].\end{aligned}$$

In the linear equilibrium, $p_{i1}=p_{i0}^c+p_{i\theta}^c\mathbb{E}\left[\theta_i\middle|\,s_i\right]+p_{i\rho}^c\mathbb{E}\left[\rho\middle|\,s_i\right]$. Restricting equation (11) to terms which depend on $\mathbb{E}[\theta_i|s_i]$ gives

$$p_{i\theta}^c=\frac{1}{2}-\frac{1}{2}\left(\frac{2b^2-e^2}{4b^2-e^2}\right)\frac{\beta}{p_{i\theta c}}=\frac{1}{2}+\frac{1}{4}\left(\frac{(\beta-1)\,\beta}{p_{i\theta}^c}\right)\quad\Longrightarrow\quad 4\left[p_{i\theta}^c\right]^2-2p_{i\theta}^c-(\beta-1)\,\beta=0.$$

The solutions of this quadratic are

$$\begin{aligned}p_{i\theta}^c&=\frac{1}{8}\left(2\pm\sqrt{4+16\,(\beta-1)\,\beta}\right)\\&=\frac{1}{4}\pm\frac{1}{4}\sqrt{4\beta^2-4\beta+1}=\frac{1}{4}\left(1\pm(2\beta-1)\right)\in\left\{-\frac{1}{2}\beta,\frac{1}{2}\left(1-\beta\right)\right\}.\end{aligned}$$

Since $\beta = e^2/(4b^2 - e^2) \geq 0$, one solution is positive and the other is negative.[31] Then $p^c_{i\theta} = p^c_\theta = (1-\beta)/2$ for both firms.

Recall that $p^\star_\theta = 1/(2+\beta\kappa)$. By its definition in Lemma 4, $\kappa \leq 1/p_\theta = 2 + \beta\kappa$; then $\kappa \leq 2/(1-\beta)$. It follows that

$$p^\star_\theta \geq \frac{1}{2 + \frac{2\beta}{1-\beta}} = \frac{1-\beta}{2} = p^c_\theta.$$

The κ inequality is strict whenever $\beta > 0$ and $\sigma^2_\rho(1-\overline{\tau}_\rho)\tau_\rho p^2_\rho > 0$; since we have assumed signals are informative, this is true whenever $e \neq 0$.

The inequality $p^c_\theta \leq p^\star_\theta$ implies

$$\beta\kappa^\star = \frac{1-2p^\star_\theta}{p^\star_\theta} \leq \frac{1-2p^c_\theta}{p^c_\theta} = \frac{\beta}{p^c_\theta} = \beta\kappa^c \quad \Longleftrightarrow \quad \kappa^c \geq \kappa^\star.$$

□

Proof of Theorem 2. Following from Lemma 2 ex-ante expected second period prices are

$$\begin{aligned}
\mathbb{E}[p^\star_{j2}] &= \frac{1}{4b^2-e^2}\Big((2b+e)\,a + 2b^2\mathbb{E}\left[\mathbb{E}\left[c_j|\,\rho, p_{j1}\right]\right] + be\mathbb{E}\left[\mathbb{E}\left[c_i|\,\rho, p_{i1}\right]\right]\Big) \\
&= \frac{1}{4b^2-e^2}\Big(\mathbb{E}\left[(2b+e)a + (2b^2+be)\left(\mu_\theta + \mathbb{E}\left[\rho|s_{i,\rho}\right]\right)\right] \\
&\quad + be\mathbb{E}\left[\kappa\left(\mathbb{E}[p_{i,}|s] - p_0 + p_\theta\mu_\theta + p_\rho\mathbb{E}[\rho|s]\right)\right]\Big)
\end{aligned}$$

The latter term equals zero, $\mathbb{E}[\mathbb{E}[p_{i1}|s_i] - p_0 + p_\theta\mu_\theta + p_\rho\mathbb{E}[\rho|s_i])] = 0$. It follows that in equilibrium

$$\mathbb{E}[p^\star_{j2}] = \frac{a + b(\mu_\rho + \mu_\theta)}{2b-e}.$$

Similarly, with information sharing

$$\begin{aligned}
\mathbb{E}[p^c_{j2}] &= \frac{1}{4b^2-e^2}\Big((2b+e)\,a + 2b^2\mathbb{E}\left[\mathbb{E}\left[c_j|\,\rho, s_\rho, p_{j1}\right]\right] + be\mathbb{E}\left[\mathbb{E}\left[c_i|\,\rho, s_\rho, p_{i1}\right]\right]\Big) \\
&= \frac{1}{4b^2-e^2}\Big((2b+e)\,a + 2b^2\mathbb{E}\left[\rho + \mathbb{E}[\theta_j|s_{j\theta}]\right] + be\mathbb{E}\left[\rho + \mathbb{E}[\theta_i|s_{i\theta}]\right]\Big) \\
&= \frac{a + b(\mu_\rho + \mu_\theta)}{2b-e}.
\end{aligned}$$

In equilibrium the ex-ante expected price for each firm in the second period are the same with and without information sharing.

[31] When $e = 0$, $-\beta/2 = 0$. This solution can be ruled out by second order conditions, but we omit this exercise: if $p^c_{i\theta} = 0$, prices do not depend on private cost information, which is not possible in our equilibrium.

From Lemma 3 the first period price in the symmetric equilibrium is

$$\begin{aligned}\mathbb{E}[p_{i1}^{\star}] &= \frac{1}{2b-e}\mathbb{E}\left[\mathbb{E}\left[bc_i + a + \frac{e}{2b}\left(a - bc_i + e\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho, s_{i\rho}, p_{j1}\right]\right)\frac{be}{4b^2-e^2}\kappa\middle|\, s_{i\rho}, s_{i\theta}\right]\right] \\ &= \frac{1}{2b-e}\left(a + b(\mu_\rho+\mu_\theta) + \frac{e^2\kappa}{2(4b^2-e^2)}\mathbb{E}\left[\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho, s_{i,\rho}, p_{j1}\right]\right)\middle|\, s_{i\rho}, s_{i\theta}\right]\right]\right).\end{aligned}$$

Similarly, from Lemma 5 when firms are sharing common cost information expected first period prices are

$$\begin{aligned}\mathbb{E}[p_{i1}^{c}] &= \frac{1}{2b-e}\mathbb{E}\left[bc_i + a + \frac{e}{2b}\left(a - bc_i + e\mathbb{E}\left[p_{j2}^{c}\middle|\,\rho, s_{\rho}, p_{j1}\right]\right)\frac{be}{4b^2-e^2}\kappa^c\middle|\, s_{\rho}, s_{i\theta}\right] \\ &= \frac{1}{2b-e}\left(a + b(\mu_\rho+\mu_\theta) + \frac{e^2\kappa^c}{2(4b^2-e^2)}\mathbb{E}\left[\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j2}^{c}\middle|\,\rho, s_{\rho}, p_{j1}\right]\right)\middle|\, s_{\rho}, s_{i\theta}\right]\right]\right).\end{aligned}$$

All terms are identical except κ and κ^c. From Proposition 3, $\kappa^c \geq \kappa$, where the inequality is strict when $e \neq 0$. Therefore $\mathbb{E}[p_{i1}^c] \geq \mathbb{E}[p_{i1}^{\star}]$, where the inequality is strict when $e \neq 0$. □

B Proofs for Section 4

Lemma 13. *The utility specification in* (4) *generates demand specification* (3)*. In particular, when* $n = 2$ *the utility specification in* (1) *generates demand specification in Section 2.*

Proof. Fix a price vector $\mathbf{p}$. Given utility as in (4), the consumer's first-order condition with respect to quantity q_i is

$$\frac{a}{b-e} - (n-1)\left(\frac{(n-1)\,b-(n-2)\,e}{((n-1)\,b+e)\,(b-e)}\right)q_i - (n-1)\left(\frac{e}{((n-1)\,b+e)\,(b-e)}\right)\sum_{j\neq i} q_j = p_i.$$

Let $Q = \sum_{i=1}^n q_i$. Summing up both sides of the equation over all firms i gives

$$\frac{na}{b-e} - (n-1)\left(\frac{(n-1)\,b-(n-2)\,e}{((n-1)\,b+e)\,(b-e)}\right)Q - (n-1)^2\left(\frac{e}{((n-1)\,b+e)\,(b-e)}\right)Q = \sum_{i=1}^n p_i.$$

Algebraic rearrangement yields

$$(n-1)\,Q = na - (b-e)\sum_{i=1}^n p_i.$$

Note that $\sum_{j\neq i} q_j = Q - q_i$. Then the consumer's first-order condition with respect to q_i

can be written

$$\frac{a}{b-e} - (n-1)^2\left(\frac{1}{(n-1)\,b+e}\right) q_i - (n-1)\left(\frac{e}{((n-1)\,b+e)\,(b-e)}\right) Q = p_i.$$

This implies

$$\begin{aligned} q_i &= \left(\frac{1}{n-1}\right)^2 \left(\left(\frac{(n-1)\,b+e}{b-e}\right) a - \frac{e}{b-e}\left(na - (b-e)\sum_{j=1}^{n} p_j\right) - ((n-1)\,b+e)\,p_i\right) \\ &= \left(\frac{1}{n-1}\right)^2 \left((n-1)\,a - (n-1)\,bp_i + e\sum_{j\neq i} p_j\right). \end{aligned}$$

This is the demand form given in (3). □

Proof of Lemma 6. From Lemma 13, utility is given by

$$\mathbb{E}\left[u\left(\mathbf{q};\mathbf{p}\right)\right] = \frac{a}{b-e}(q_i+q_j) - \frac{1}{2}\left(\frac{b}{b^2-e^2}\right)(q_i^2+q_j^2) - \left(\frac{e}{b^2-e^2}\right) q_i q_j - (p_i q_i + p_j q_j).$$

Substituting in demand and applying equilibrium symmetry,

$$\begin{aligned} \mathbb{E}\left[u\left(\mathbf{q}_t;\mathbf{p}_t\right)\right] &= \frac{2a}{b-e}\mathbb{E}\left[q_{it}\right] - \left(\frac{b}{b^2-e^2}\right)\mathbb{E}\left[q_{it}^2\right] - \left(\frac{e}{b^2-e^2}\right)\mathbb{E}\left[q_{it}q_{jt}\right] - 2\mathbb{E}\left[p_{it}q_{it}\right] \\ &= -2a\mathbb{E}\left[p_{it}\right] + b\mathbb{E}\left[p_{it}^2\right] - e\mathbb{E}\left[p_{it}p_{jt}\right]. \end{aligned}$$

Expressing in terms of the expectation, covariance, and variance of prices,

$$\mathbb{E}\left[u\left(\mathbf{p}_t\right)\right] = \left(-2a + (b-e)\,\mathbb{E}\left[p_{it}\right]\right)\mathbb{E}\left[p_{it}\right] + b\,\mathrm{Var}\left(p_{it}\right) - e\,\mathrm{Cov}\left(p_{it},p_{jt}\right).$$

□

Proof of Lemma 7. In period t firm i's expected profits are

$$\mathbb{E}\left[\pi_{it}\right] = \mathbb{E}\left[\left(a - bp_{it} + ep_{jt}\right)\left(p_{it} - c_i\right)\right].$$

Note that, when considering ex ante expected profits, it is not necessary to condition on learned information, which disappears by the law of iterated expectations. Then we see

$$\begin{aligned}
\mathbb{E}\left[\pi_{it}\right] &= a\mathbb{E}\left[p_{it}-c_i\right] - b\mathbb{E}\left[p_{it}^2 - p_{it}c_i\right] + e\mathbb{E}\left[p_{jt}p_{it} - p_{jt}c_i\right] \\
&= a\mathbb{E}\left[p_{it}-c_i\right] - b\operatorname{Var}\left(p_{it}\right) - b\mathbb{E}\left[p_{it}\right]^2 + b\operatorname{Cov}\left(p_{it},c_i\right) + b\mathbb{E}\left[p_{it}\right]\mathbb{E}\left[c_i\right] \\
&\quad + e\operatorname{Cov}\left(p_{it},p_{jt}\right) + e\mathbb{E}\left[p_{it}\right]\mathbb{E}\left[p_{jt}\right] - e\operatorname{Cov}\left(p_{jt},c_i\right) - e\mathbb{E}\left[p_{jt}\right]\mathbb{E}\left[c_i\right] \\
&= \left(a - b\mathbb{E}\left[p_{it}\right] + e\mathbb{E}\left[p_{jt}\right]\right)\left(\mathbb{E}\left[p_{it}\right] - \mathbb{E}\left[c_i\right]\right) \\
&\quad - b\left(\operatorname{Var}\left(p_{it}\right) - \operatorname{Cov}\left(p_{it},c_i\right)\right) + e\left(\operatorname{Cov}\left(p_{it},p_{jt}\right) - \operatorname{Cov}\left(p_{jt},c_i\right)\right) \\
&= \left(a - (b-e)\mathbb{E}\left[p_{it}\right]\right)\left(\mathbb{E}\left[p_{it}\right] - \mathbb{E}\left[c_i\right]\right) \\
&\quad - b\left(\operatorname{Var}\left(p_{it}\right) - \operatorname{Cov}\left(p_{it},c_i\right)\right) + e\left(\operatorname{Cov}\left(p_{it},p_{jt}\right) - \operatorname{Cov}\left(p_{jt},c_i\right)\right).
\end{aligned}$$

The final line follows by equilibrium symmetry. Since there are two firms, symmetry further implies that expected producer surplus is twice this quantity. □

Lemma 14. *For given values of b and e, consumer surplus decreases with information sharing when a is sufficiently large.*

Proof. From Lemma 6 consumer surplus is

$$\mathbb{E}\left[u\left(\mathbf{p}\right)\right] = \left(-2a + (b-e)\mathbb{E}[p_i]\right)\mathbb{E}[p_i] + b\operatorname{Var}(p_i) - e\operatorname{Cov}(p_i,p_j)$$

Without information sharing, the expected price, variance of price, and covariance of prices in the first period are given by:

$$\begin{aligned}
\mathbb{E}\left[p_{i1}^\star\right] &= \frac{1}{2b-e}\left(a + b(\mu_\rho+\mu_\theta) + \frac{e^2\kappa^\star}{2(4b^2-e^2)}\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j2}^\star \middle| \rho, s_{i\rho}, p_{j1}\right]\right)\right]\right); \\
\operatorname{Var}\left(p_{i1}^\star\right) &= \left[p_\theta^\star\right]^2\operatorname{Var}\left(\mathbb{E}\left[\theta_i \middle| s_{i\theta}\right]\right) + \left[p_\rho^\star\right]^2\operatorname{Var}\left(\mathbb{E}\left[\rho \middle| s_{i\rho}\right]\right) = \left[p_\theta^\star\right]^2\frac{\overline{\tau}_{i\theta}}{\tau_\theta} + \left[p_\rho^\star\right]^2\frac{\overline{\tau}_{i\rho}}{\tau_\rho}; \\
\operatorname{Cov}\left(p_{i1}^\star, p_{j1}^\star\right) &= \left[p_\rho^\star\right]^2\operatorname{Cov}\left(\mathbb{E}\left[\rho \middle| s_{i\rho}\right], \mathbb{E}\left[\rho \middle| s_{j\rho}\right]\right) = \left[p_\rho^\star\right]^2\frac{\overline{\tau}_{i\rho}^2}{\tau_\rho}.
\end{aligned}$$

With information sharing these become

$$\begin{aligned}
\mathbb{E}\left[p_{i1}^c\right] &= \frac{1}{2b-e}\left(a + b(\mu_\rho+\mu_\theta) + \frac{e^2\kappa^c}{2(4b^2-e^2)}\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[p_{j2}^c \middle| \rho, s_\rho, p_{j1}\right]\right)\right]\right); \\
\operatorname{Var}\left(p_{i1}^c\right) &= \left[p_\theta^c\right]^2\operatorname{Var}\left(\mathbb{E}\left[\theta_i \middle| s_{i\theta}\right]\right) + \left[p_\rho^c\right]^2\operatorname{Var}\left(\mathbb{E}\left[\rho \middle| s_\rho\right]\right) = \left[p_\theta^c\right]^2\frac{\overline{\tau}_{i\theta}}{\tau_\theta} + \left[p_\rho^c\right]^2\frac{2\tau_{i\rho}}{\tau_\rho\left(\tau_\rho + 2\tau_{i\rho}\right)}; \\
\operatorname{Cov}\left(p_{i1}^c, p_{j1}^c\right) &= \left[p_\rho^c\right]^2\operatorname{Var}\left(\mathbb{E}\left[\rho \middle| s_\rho\right]\right) = \left[p_\rho^c\right]^2\frac{2\tau_{i\rho}}{\tau_\rho\left(\tau_\rho + 2\tau_{i\rho}\right)}.
\end{aligned}$$

Given an information sharing agreement the differences between the values with and without information sharing are

$$\begin{aligned}
\Delta\mathbb{E}\left[p_{i1}\right] &= \left(\frac{1}{2b-e}\right)^2\left(\frac{e^2}{2(4b^2-e^2)}\right)\left(\kappa^c-\kappa^\star\right)\left(2a-2\left(b-e\right)\left(\mu_\rho+\mu_\theta\right)\right)b;\\
\Delta\,\mathrm{Var}\left(p_{i1}\right) &= \frac{\overline{\tau}_{i\theta}}{\tau_\theta}\left([p^c_\theta]^2-[p^\star_\theta]^2\right)+\left[p^c_\rho\right]^2\frac{2\tau_{i\rho}}{\tau_\rho\left(\tau_\rho+2\tau_{s\rho i}\right)}-\left[p^\star_\rho\right]^2\frac{\overline{\tau}_{i\rho}}{\tau_\rho};\\
\Delta\,\mathrm{Cov}\left(p_{i1},p_{j1}\right) &= \left[p^c_\rho\right]^2\frac{2\tau_{i\rho}}{\tau_\rho\left(\tau_\rho+2\tau_{i\rho}\right)}-\left[p^\star_\rho\right]^2\frac{\overline{\tau}^2_{i\rho}}{\tau_\rho}.
\end{aligned}$$

From Proposition 3 we have $\kappa^c>\kappa^\star$ for $e\neq 0$, so $\Delta\mathbb{E}[p_{i1}]$ is increasing in a; when expected demand is positive, so that $a>(b-e)\mathbb{E}[p]$, $\Delta\mathbb{E}[p_{i1}]>0$. Equilibrium price coefficients (other than p_0) and price informativeness do not depend on the value of a, therefore $\Delta\,\mathrm{Var}(p_{i1})$ and $\Delta\,\mathrm{Cov}(p_{i1},p_{j1})$ are constant for all a. Then the difference in consumer surplus across informational regimes depends only on the leading term in the expression for consumer surplus given in Lemma 6. The effect of a on this term is given by

$$\begin{aligned}
&\frac{\partial}{\partial a}\left[\left(-2a+(b-e)\,\mathbb{E}\left[p^c_{i1}\right]\right)\mathbb{E}\left[p^c_{i1}\right]-\left(-2a+(b-e)\,\mathbb{E}\left[p^\star_{i1}\right]\right)\mathbb{E}\left[p^\star_{i1}\right]\right]\\
&=\frac{\partial}{\partial a}\left[-2a\Delta\mathbb{E}\left[p_{i1}\right]+(b-e)\,\Delta\mathbb{E}\left[p_{i1}\right]\left(\mathbb{E}\left[p^c_{i1}\right]+\mathbb{E}\left[p^\star_{i1}\right]\right)\right]\\
&=-2\Delta\mathbb{E}\left[p_{i1}\right]-2a\frac{\partial\Delta\mathbb{E}\left[p_{i1}\right]}{\partial a}\\
&\quad+(b-e)\,\frac{\partial\Delta\mathbb{E}\left[p_{i1}\right]}{\partial a}\left(\mathbb{E}\left[p^c_{i1}\right]+\mathbb{E}\left[p^\star_{i1}\right]\right)+(b-e)\,\Delta\mathbb{E}\left[p_{i1}\right]\left(\frac{\partial\mathbb{E}\left[p^c_{i1}\right]}{\partial a}+\frac{\partial\mathbb{E}\left[p^\star_{i1}\right]}{\partial a}\right).
\end{aligned}\tag{12}$$

Now, notice that since $e\neq 0$ implies $\kappa^c>\kappa^\star$, it follows that

$$\begin{aligned}
\frac{\partial\mathbb{E}\left[p^c_{i1}\right]}{\partial a} &= \left(\frac{1}{2b-e}\right)\left(1+\left(\frac{e}{2b-e}\right)\frac{be}{4b^2-e^2}\kappa^c\right)\\
&> \left(\frac{1}{2b-e}\right)\left(1+\left(\frac{e}{2b-e}\right)\frac{be}{4b^2-e^2}\kappa^\star\right)=\frac{\partial\mathbb{E}\left[p^\star_{i1}\right]}{\partial a}.
\end{aligned}$$

Then $\Delta\mathbb{E}[p_{i1}]>0$ allows (12) to be bounded above by

$$\begin{aligned}
&\frac{\partial}{\partial a}\left[\left(-2a+(b-e)\,\mathbb{E}\left[p^c_{i1}\right]\right)\mathbb{E}\left[p^c_{i1}\right]-\left(-2a+(b-e)\,\mathbb{E}\left[p^\star_{i1}\right]\right)\mathbb{E}\left[p^\star_{i1}\right]\right]\\
&<\left(-2+2\,(b-e)\,\frac{\partial\mathbb{E}\left[p^c_{i1}\right]}{\partial a}\right)\Delta\mathbb{E}\left[p_{i1}\right]+\left(-2a+2\,(b-e)\,\mathbb{E}\left[p^c_{i1}\right]\right)\frac{\partial\Delta\mathbb{E}\left[p_{i1}\right]}{\partial a}.
\end{aligned}$$

Straightforward algebraic rearrangment gives $-2+2(b-e)\partial\mathbb{E}[p^c_{i1}]/\partial a<0$, and by assumption $-2a+2(b-e)\mathbb{E}[p^c_{i1}]<0$. Since $\Delta\mathbb{E}[p_{i1}]$ is linear in a and $\partial\Delta\mathbb{E}[p_{i1}]/\partial a$ is constant in a, it follows that for any $\lambda > 0$ there is $\underline{a}$ such that for all $a > \underline{a}$,

$$\frac{\partial}{\partial a}\left[\left(-2a + (b-e)\,\mathbb{E}\left[p_{i1}^c\right]\right)\mathbb{E}\left[p_{i1}^c\right] - \left(-2a + (b-e)\,\mathbb{E}\left[p_{i1}^\star\right]\right)\mathbb{E}\left[p_{i1}^\star\right]\right] < -\lambda.$$

Therefore we can choose an a such that

$$\begin{aligned}&\left(-2a + (b-e)\,\mathbb{E}\left[p_{i1}^c\right]\right)\mathbb{E}\left[p_{i1}^c\right] - \left(-2a + (b-e)\,\mathbb{E}\left[p_{i1}^\star\right]\right)\mathbb{E}\left[p_{i1}^\star\right]\\ &\quad < b\Delta\,\mathrm{Var}\left(p_{i1}\right) - \Delta\,\mathrm{Cov}\left(p_{i1}, p_{j1}\right).\end{aligned} \tag{13}$$

For all such a, the consumer surplus in the first period decreases under information sharing.

In the second period, the expectation, covariance, and variance of prices do not change under information sharing. In the proof of Theorem 2 it is shown that expected prices do not change. The variance and covariance of second period prices are

$$\begin{aligned}\mathrm{Var}(p_{i2}^\star) &= \frac{1}{4}\mathrm{Var}(c_i) + \frac{e^2}{4b^2}\mathrm{Var}\left(\mathbb{E}[p_{j2}^\star|\rho, p_{i1}, p_{j1}]\right) - \frac{e}{4b}\mathrm{Cov}\left(c_i, \mathbb{E}[p_{j2}^\star|\rho, p_{i1}, p_{j1}]\right)\\ &= C(b,e)\sigma_\rho^2 + \frac{1}{4}\sigma_\theta^2 - \frac{e^2}{4(4b^2-e^2)}\kappa p_\theta^\star \mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right)\\ &\quad + \frac{4b^2e^2+e^4}{4(4b^2-e^2)^2}\left(\kappa^2\bar{\tau}_{i\rho}(1-\bar{\tau}_{i\rho})\sigma_\rho^2 p_\rho^{\star 2} + \kappa^2 p_\theta^{\star 2}\mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right)\right);\\ \mathrm{Cov}(p_{i2}^\star, p_{j2}^\star) &= \frac{1}{4}\mathrm{Cov}(c_i, c_j) + \frac{e}{2b}\mathrm{Cov}\left(c_i, \mathbb{E}[p_{j2}^\star|\rho, p_{i1}, p_{j1}]\right)\\ &\quad + \frac{e^2}{4b^2}\mathrm{Cov}\left(\mathbb{E}[p_{i2}^\star|\rho, p_{i1}, p_{j1}], \mathbb{E}[p_{j2}^\star|\rho, p_{i1}, p_{j1}]\right)\\ &= \frac{(2b+e)^2 + 4b^2e^2 + e^4 + 4be^3}{4(4b^2-e^2)^2}\sigma_\rho^2 + \frac{e^2}{2(4b^2-e^2)}\kappa p_\theta^\star \mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right)\\ &\quad + \frac{be^3}{(4b^2-e^2)^2}\left(\kappa^2\bar{\tau}_{i\rho}(1-\bar{\tau}_{i\rho})\sigma_\rho^2 p_\rho^{\star 2} + \kappa^2 p_\theta^{\star 2}\mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right)\right).\end{aligned}$$

Under information sharing the variance and covariance are

$$\begin{aligned}\mathrm{Var}(p_{i2}^c) &= \frac{1}{4}\mathrm{Var}(c_i) + \frac{e^2}{4b^2}\mathrm{Var}\left(\mathbb{E}[p_{j2}^c|\rho, p_{i1}, p_{j1}]\right) - \frac{e}{4b}\mathrm{Cov}\left(c_i, \mathbb{E}[p_{j2}^c|\rho, p_{i1}, p_{j1}]\right)\\ &= C(b,e)\sigma_\rho^2 + \frac{1}{4}\sigma_\theta^2 + \frac{e^4}{2(4b^2-e^2)^2}\mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right);\\ \mathrm{Cov}(p_{i2}^c, p_{j2}^c) &= \frac{1}{4}\mathrm{Cov}(c_i, c_j) + \frac{e}{2b}\mathrm{Cov}\left(c_i, \mathbb{E}[p_{j2}^c|\rho, p_{i1}, p_{j1}]\right)\\ &\quad + \frac{e^2}{4b^2}\mathrm{Cov}\left(\mathbb{E}[p_{i2}^c|\rho, p_{i1}, p_{j1}], \mathbb{E}[p_{j2}^c|\rho, p_{i1}, p_{j1}]\right)\\ &= \frac{(2b+e)^2 + 4b^2e^2 + e^4 + 4be^3}{4(4b^2-e^2)^2}\sigma_\rho^2 + \frac{4b^2e^2 - e^4 + 2be^3}{2(4b^2-e^2)^2}\mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right).\end{aligned}$$

The desired result follows from the fact that

$$\kappa = \frac{p_\theta^\star \mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right)}{\bar{\tau}_{i\rho}(1-\bar{\tau}_{i\rho})\sigma_\rho^2 p_\rho^{\star 2} + p_\theta^{\star 2}\mathrm{Var}\left(\mathbb{E}[\theta_i|s_{i\theta}]\right)}.$$

Therefore two-period consumer surplus decreases for all a that satisfy equation (13). $\square$

Lemma 15. *When $e \approx b$, consumer surplus decreases with information sharing.*

Proof. We show that consumer surplus decreases with information sharing when $e = b$. Since all equilibrium expressions are continuous in the demand parameters a, b, and e, it follows that the same is true when $e \lesssim b$.

When $e = b$, Lemma 6 gives the difference in consumer surplus between informational regimes as

$$\begin{aligned}\Delta\mathbb{E}\left[u\right] &= -2a\Delta\mathbb{E}\left[p_{i1}\right] + b\Delta\,\mathrm{Var}\left(p_{i1}\right) - e\Delta\,\mathrm{Cov}\left(p_{i1}, p_{j1}\right) \\ &= -2a\Delta\mathbb{E}\left[p_{i1}\right] + \left(\Delta\,\mathrm{Var}\left(p_{i1}\right) - \Delta\,\mathrm{Cov}\left(p_{i1}, p_{j1}\right)\right) b.\end{aligned}$$

From Theorem 2, $\Delta\mathbb{E}\left[p_{i1}\right] > 0$. The expressions in the proof of Lemma 14 imply that $\Delta\,\mathrm{Var}\left(p_{i1}\right) < \Delta\,\mathrm{Cov}(p_{i1}, p_{j1})$. The result follows immediately. $\square$

Lemma 16. *For given values of b and e, producer surplus increases with information sharing when a is sufficiently large.*

Proof. From Lemma 7 producer surplus in each period is

$$\begin{aligned}\mathbb{E}\left[\pi_{it}\right] = &\left(a - (b-e)\,\mathbb{E}\left[p_{it}\right]\right)\left(\mathbb{E}\left[p_{it}\right] - \mathbb{E}\left[c_i\right]\right) \\ &+ b\left(\mathrm{Cov}\left(p_{it}, c_i\right) - \mathrm{Var}\left(p_{it}\right)\right) - e\left(\mathrm{Cov}\left(p_{jt}, c_i\right) - \mathrm{Cov}\left(p_{it}, p_{jt}\right)\right).\end{aligned}$$

Similar to the case with consumer surplus, none of the variance or covariance terms depend on the value of a both with and without information sharing. Therefore the following term does not depend on a,

$$b\left(\Delta\,\mathrm{Cov}\left(p_{it}, c_i\right) - \Delta\,\mathrm{Var}\left(p_{it}\right)\right) - e\left(\Delta\,\mathrm{Cov}\left(p_{jt}, c_i\right) - \Delta\,\mathrm{Cov}\left(p_{it}, p_{jt}\right)\right).$$

Then expected producer surplus is given by

$$\begin{aligned}\mathbb{E}\left[\Pi_1\right] &= 2\left(\left(a - (b-e)\,\mathbb{E}\left[p_{i1}\right]\right)\left(\mathbb{E}\left[p_{i1}\right] - \mathbb{E}\left[c_i\right]\right) - b\,\mathrm{Var}\left(p_{i1}\right) + e\,\mathrm{Cov}\left(p_{i1}, p_{j1}\right)\right. \\ &\quad \left. + b\,\mathrm{Cov}\left(c_i, p_{i1}\right) - e\,\mathrm{Cov}\left(c_i, p_{j1}\right)\right) \\ &= 2\left(a - (b-e)\,\mathbb{E}\left[p_{i1}\right]\right)\left(\mathbb{E}\left[p_{i1}\right] - (\mu_\rho + \mu_\theta)\right) + C_\Pi.\end{aligned}$$

As shown in the proof of Lemma 14 (effect of information sharing on consumer surplus, when a is large), second period price statistics — expectation, variance, and covariance — do not change between informational regimes. However there is a change associated with the covariance of prices with costs. This change, which also does not depend on the value of a, is

$$\Delta\mathbb{E}\left[\Pi_2\right] = \mathbb{E}\left[\Pi_2^c\right] - \mathbb{E}\left[\Pi_2^\star\right] = -\frac{be^2}{2\left(4b^2-e^2\right)} \frac{\overline{\tau}_{i\theta}}{\tau_\theta} \left(1-\kappa^\star p_\theta^\star\right).$$

The total change in profits with information sharing is then

$$\begin{aligned}\Delta\mathbb{E}\left[\Pi_1+\Pi_2\right] = {} & 2\left(a-(b-e)\,\mathbb{E}\left[p_{i1}^c\right]\right)\left(\mathbb{E}\left[p_{i1}^c\right]-\mathbb{E}\left[c_i\right]\right) \\ & - \left(2\left(a-(b-e)\,\mathbb{E}\left[p_{i1}^\star\right]\right)\left(\mathbb{E}\left[p_{i1}^\star\right]-\mathbb{E}\left[c_i\right]\right)\right) + C'_\Pi,\end{aligned}$$

where C'_Π does not depend on a. Taking the derivative with respect to a gives

$$\begin{aligned}
&\frac{\partial\Delta\mathbb{E}\left[\Pi_1+\Pi_2\right]}{\partial a} \\
&= 2\left(\mathbb{E}\left[p_{i1}^c\right]-\mathbb{E}\left[p_{i1}^\star\right]\right)\left(1-2\left(b-e\right)\frac{\partial\mathbb{E}\left[p_{i1}^c\right]}{\partial a}\right) \\
&\qquad + 2\left(a+(b-e)\,\mathbb{E}\left[c_i\right]\right)\left(\frac{\partial\mathbb{E}\left[p_{i1}^c\right]}{\partial a}-\frac{\partial\mathbb{E}\left[p_{i1}^\star\right]}{\partial a}\right) \\
&\geq 2\left(a+(b-e)\,\mathbb{E}\left[c_i\right]\right)\left(\frac{1}{2b-e}\right)^2\left(\frac{be^2}{4b^2-e^2}\right)\left(\kappa^c-\kappa^\star\right) \\
&\qquad - 2\left(\frac{1}{2b-e}\right)^2\left(\frac{e^2}{2\left(4b^2-e^2\right)}\right)\left(\kappa^c-\kappa^\star\right)\left(2ab-2\left(b-e\right)\mathbb{E}\left[c_i\right]\right) \\
&= 2\left(\frac{1}{2b-e}\right)^2\left(\frac{e^2}{2\left(4b^2-e^2\right)}\right)\left(\kappa^c-\kappa^\star\right)\left(ab+b\left(b-e\right)\mathbb{E}\left[c_i\right]-ab+\left(b-e\right)\mathbb{E}\left[c_i\right]\right) > \varepsilon.
\end{aligned}$$

The first inequality comes from

$$\left(1-\frac{2\left(b-e\right)}{2b-e}\left(\frac{1}{2b-e}\left(1+\frac{e^2b\kappa^c}{4b^2-e^2}\right)\right)\right) \geq -1$$

Therefore we can choose a large enough so that change in total profits is positive. □

Proof of Proposition 4. The follows immediately from Lemmas 14, 15, and 16. □

Lemma 17. *In the n-firm extension expected second-period prices are given by*

$$\mathbb{E}\left[p_{i2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]=\frac{a}{2b-e}+(n-1)\left(\frac{1}{2\,(n-1)\,b+e}\right)\mathbb{E}\left[bc_i\middle|\,\rho,\mathbf{p}_1\right]$$
$$+\frac{e}{(2\,(n-1)\,b+e)\,(2b-e)}\sum_{j=1}^{n}\mathbb{E}\left[bc_j\middle|\,\rho,\mathbf{p}_1\right].$$

Proof. Firm i's second-period objective is

$$\max_{p}\mathbb{E}\left[\frac{1}{n-1}\left(a-bp+\frac{e}{n-1}\sum_{j\neq i}p_{j2}^{\star}\right)(p-c_i)\middle|\,c_i,\mathbf{p}_1\right].$$

At the optimal price $p_{i2}^{\star}$, firm i's second-period first-order condition is

$$0=a-2bp_{i2}^{\star}+bc_i+\frac{e}{n-1}\sum_{j\neq i}\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right].$$

This equation holds given firm i's second-period information; therefore it holds in expectation, conditional on ρ and $\mathbf{p}_1$. This gives

$$0=a-2b\mathbb{E}\left[p_{i2}^{\star}\middle|\,\rho,\mathbf{p}_1\right]+b\mathbb{E}\left[c_i\middle|\,\rho,\mathbf{p}_1\right]+\frac{e}{n-1}\sum_{j\neq i}\mathbb{E}\left[p_{j2}^{\star}\middle|\,\rho,\mathbf{p}_1\right].$$

Taken over all firms i this is a linear system, $A\mathbb{E}[\mathbf{p}_2^{\star}|\rho,\mathbf{p}_1]=a+b\mathbb{E}[\mathbf{c}|\rho,\mathbf{p}_1]$, where

$$A_{ii}=2b,\quad A_{ij}=-\frac{e}{n-1}\quad(j\neq i).$$

The matrix A is invertible, with

$$A_{ii}^{-1}=\frac{2\,(n-1)\,b-(n-2)\,e}{(2b-e)\,(2\,(n-1)\,b+e)},\quad A_{ij}^{-1}=\frac{e}{(2b-e)\,(2\,(n-1)\,b+e)}\quad(j\neq i).$$

This implies the stated result. □

Corollary 6. *In the n-firm extension second-period prices are given by*

$$p_{i2}^{\star}=\frac{a}{2b-e}+\frac{1}{2}\left(c_i+\frac{e^2}{(2\,(n-1)\,b+e)\,(2b-e)}\mathbb{E}\left[c_i\middle|\,\rho,\mathbf{p}_1\right]\right)$$
$$+\sum_{j\neq i}\frac{be}{(2\,(n-1)\,b+e)\,(2b-e)}\mathbb{E}\left[c_j\middle|\,\rho,\mathbf{p}_1\right].$$

Proof. This follows immediately from the proof of Lemma 17. □

Lemma 18. *In the n-firm extension expected second-period profits are given by*

$$\mathbb{E}\left[\pi_{i2}^{\star}\middle| c_i, \mathbf{p}_1\right] = \frac{1}{4b\left(n-1\right)}\left(a - bc_i + \frac{e}{n-1}\sum_{j\neq i}\mathbb{E}\left[p_{j2}^{\star}\middle|\rho, \mathbf{p}_1\right]\right)^2$$

Proof. This is a standard profit-maximization problem, and follows immediately from the optimization in Lemma 17. □

Proof of Theorem 3. Firm i's first period objective is

$$\max_p \mathbb{E}\left[\frac{1}{n-1}\left(a - bp + \frac{e}{n-1}\sum_{j\neq i}p_{j1}^{\star}\right)(p - c_i) + \pi_{i2}^{\star}(p)\middle| s_i\right].$$

Following Lemma 18, the firm's first-order condition is

$$\begin{aligned}2bp_{i1}^{\star} = {} & \mathbb{E}\left[a + bc_i + \frac{e}{n-1}\sum_{j\neq i}p_{j1}^{\star}\middle| s_i\right] \\ & + \frac{e}{2b}\mathbb{E}\left[\left(a - bc_i + \frac{e}{n-1}\sum_{j\neq i}p_{j2}^{\star}\right)\left(\frac{1}{n-1}\sum_{j\neq i}\frac{dp_{j2}^{\star}}{dp_{i1}}\right)\middle| s_i\right].\end{aligned} \tag{14}$$

Conditional on ρ, firm i's price does not affect firm j's beliefs about firm k's price. Then following Corollary 6,

$$\frac{d}{dp_{i1}}\mathbb{E}\left[p_{j2}^{\star}\middle|\rho, \mathbf{p}_1\right] = \frac{be}{\left(2\left(n-1\right)b + e\right)\left(2b - e\right)}\left(\frac{d}{dp_{i1}}\mathbb{E}\left[c_i\middle|\rho, \mathbf{p}_1\right]\right).$$

Conditional on ρ, the effect of firm i's first period price on firm j's beliefs about i's costs is completely determined by the relative importance of private and public costs in setting first period prices. That is,

$$\frac{d}{dp_{i1}}\mathbb{E}\left[c_i\middle|\rho, \mathbf{p}_1\right] = \kappa \equiv \frac{\sigma_\theta^2\overline{\tau}_\theta p_\theta}{\sigma_\rho^2\left(1 - \overline{\tau}_\rho\right)\overline{\tau}_\rho p_\rho^2 + \sigma_\theta^2\overline{\tau}_\theta p_\theta^2}.$$

Define $\beta_n = e^2/(2(n-1)b+e)(2b-e)$; note that $\beta_2 = \beta$ as defined in the base two-firm case. Then equation (14) becomes, for any $j \neq i$,

$$2bp_{i1}^{\star} = \left(1 + \frac{e}{2b}\right)a + \left(1 - \frac{1}{2}\beta_n\kappa\right)\mathbb{E}\left[bc_i\middle| s_i\right] + e\mathbb{E}\left[p_{j1}^{\star}\middle| s_i\right] + \frac{1}{2}e\beta_n\kappa\mathbb{E}\left[p_{j2}^{\star}\middle| s_i\right].$$

Corollary 6 implies[32]

$$\begin{aligned}
\mathbb{E}\left[\left. p_{j2}^{\star}\right| s_i\right] &= \frac{a}{2b-e} + \frac{1}{2}\left(1+\beta_n\right)\mathbb{E}\left[\left. c_j\right| s_i\right] + \sum_{k\neq i,j}\frac{b}{e}\beta_n\mathbb{E}\left[\left. c_k\right| s_i\right] + \frac{b}{e}\beta_n\mathbb{E}\left[\left.\mathbb{E}\left[\left. c_i\right| \rho,\mathbf{p}_1\right]\right| s_i\right] \\
&= \frac{a}{2b-e} + \left(\frac{1}{2}\left(1+\beta_n\right) + \left(n-2\right)\frac{b}{e}\beta_n\right)\mathbb{E}\left[\left. c_j\right| s_i\right] + \frac{b}{e}\beta_n\mathbb{E}\left[\left.\mathbb{E}\left[\left. c_i\right| \rho,\mathbf{p}_1\right]\right| s_i\right].
\end{aligned}$$

In the linear pricing equilibrium,

$$p_{i1}^{\star} = p_{0n} + p_{\theta n}\mathbb{E}\left[\left.\theta_i\right| s_i\right] + p_{\rho n}\mathbb{E}\left[\left.\rho\right| s_i\right].$$

Matching coefficients gives

$$\begin{aligned}
2bp_{\theta n} &= b + \frac{e}{2b}\left(-b + b\beta_n\kappa p_{\theta n}\right)\frac{b}{e}\beta_n\kappa \\
&= \left(b - \frac{1}{2}b\beta_n\kappa\right) + \frac{1}{2}b\beta_n^2\kappa^2 p_{\theta n}; \\
2bp_{\rho n} &= b + e\overline{\tau}_\rho p_{\rho n} \\
&\quad + \left(\frac{e}{2b}\right)\left(-b + \left(\frac{1}{2}\left(1+\beta_n\right) + \left(n-2\right)\frac{b}{e}\beta_n + \frac{b}{e}\beta_n + \frac{b}{e}\beta_n\kappa p_{\rho n}\left(1-\overline{\tau}_\rho\right)\right)e\right)\frac{b}{e}\beta_n\kappa \\
&= b + e\overline{\tau}_\rho p_{\rho n} + \frac{1}{2}\left(-b + \frac{1}{2}\left(1+\beta_n\right)e + \left(n-1\right)b\beta_n + b\beta_n\kappa p_{\rho n}\left(1-\overline{\tau}_\rho\right)\right).
\end{aligned}$$

The stated equalities are immediate. □

Proof of Theorem 4. The expressions for $p_{\theta\infty}$ and $p_{\rho\infty}$ follow immediately from Theorem 3.

As mentioned in the main text, $\lim_{n\nearrow\infty}\beta_n = 0$. In the n-large limit information about firm i does not affect any firm j's second period pricing strategy. Then firm i's first period first order conditions (equation (14) in the proof of Theorem 3 above) reduce to[33]

$$2bp_{i1}^{\star} = a + b\mathbb{E}\left[\left. c_i\right| s_i\right] + e\mathbb{E}\left[\left. p_{j1}^{\star}\right| s_i\right] \quad \text{for any } j\neq i.$$

In expectation this is

$$2b\mathbb{E}\left[p_{i1}^{\star}\right] = a + b\mathbb{E}\left[\mathbb{E}\left[\left. c_i\right| s_i\right]\right] + e\mathbb{E}\left[\mathbb{E}\left[\left. p_{j1}^{\star}\right| s_i\right]\right] = a + b\mathbb{E}\left[c_i\right] + e\mathbb{E}\left[p_{j1}^{\star}\right].$$

[32]Note that $b\beta_n/e = be/(2(n-1)b+e)(2b-e)$, so divding by zero is not a relevant concern.

[33]Recall that $\mathbb{E}[\sum_{k\neq i} p_{k1}^{\star}|s_i]/(n-1) = \mathbb{E}[p_{j1}^{\star}|s_i]$ for any $j\neq i$.

In the linear equilibrium this implies

$$2b\left(p_{0\infty}+p_{\theta\infty}\mu_\theta+p_{\rho\infty}\mu_\rho\right)=a+b\left(\mu_\theta+\mu_\rho\right)+e\left(p_{0\infty}+p_{\theta\infty}\mu_\theta+p_{\rho\infty}\mu_\rho\right).$$

Algebraic rearrangement gives

$$\left(2b-e\right)p_{0\infty}=a+b\left(\mu_\theta+\mu_\rho\right)-\left(2b-e\right)p_{\theta\infty}\mu_\theta-\left(2b-e\right)p_{\rho\infty}\mu_\rho.$$

Substituting in for $p_{\theta\infty}$ and $p_{\rho\infty}$ yields the stated equation for $p_{0\infty}$. □

Lemma 19. *There exists a constant $C_u \in \mathbb{R}$ such that for any $\overline{\tau}_\rho$, the linear equilibrium with a large number of firms yields expected first-period consumer surplus*

$$\mathbb{E}\left[u_{1\infty}\right]\propto\left(1-r\right)\operatorname{Var}\left(p_{i1}^\star\right)-r\operatorname{Cov}\left(p_{i1}^\star,p_{j1}^\star\right)+C_u.$$

Proof. For any finite number of firms n, the linear pricing equilibrium yields the first-stage expected consumer surplus given in equation (4). Let d_{in} be scaled demand with n firms,

$$d_{in}=\left(n-1\right)q_i=a-bp_{i1}^\star+\frac{e}{n-1}\sum_{j\neq i}p_{j1}^\star.$$

Applying symmetry of the linear pricing equilibrium and the linearity of expectation gives

$$\mathbb{E}\left[u_{1\infty}\right]=\lim_{n\nearrow\infty}\mathbb{E}\left[u_{1n}\right]\propto ab\mathbb{E}\left[d_{i\infty}\right]-\frac{1}{2}\left(b-e\right)\mathbb{E}\left[d_{i\infty}^2\right]-\frac{1}{2}e\mathbb{E}\left[d_{i\infty}d_{j\infty}\right]-\left(b-e\right)b\mathbb{E}\left[p_{i1}^\star d_{i\infty}\right].$$

In the limit with a large number of firms, $\mathbb{E}[p_{i1}^\star]$ does not depend on $\overline{\tau}_\rho$. Then $\mathbb{E}[d_{i\infty}]$ does not depend on $\overline{\tau}_\rho$. Define $d_{in}^a=d_{in}-a$. Then there is a constant $C_{\pi1}$ such that the above equation can be written

$$\mathbb{E}\left[u_{1\infty}\right]\propto-\left(b-e\right)\mathbb{E}\left[\left(d_{i\infty}^a+2bp_{i1}^\star\right)d_{i\infty}^a\right]-e\mathbb{E}\left[d_{i\infty}^a d_{j\infty}^a\right]+C_{\pi1}.$$

We compute each piece in turn.

$$
\begin{aligned}
\mathbb{E}\left[\left(d_{i\infty}^a + 2bp_{i1}^\star\right) d_{i\infty}^a\right] &= \lim_{n\nearrow\infty} \mathbb{E}\left[\left(d_{in}^a + 2bp_{i1,n}^\star\right) d_{in}^a\right] \\
&= \lim_{n\nearrow\infty} \mathbb{E}\left[-b^2 p_{i1,n}^{\star 2} + \left(\frac{e}{n-1}\right)^2 \left(\sum_{j\neq i} p_{j1,n}^\star\right)^2\right] \\
&= -b^2 \mathbb{E}\left[p_{i1}^{\star 2}\right] + \lim_{n\nearrow\infty} \left(\frac{e}{n-1}\right)^2 \mathbb{E}\left[\sum_{j\neq i} p_{j1,n}^{\star 2} + 2\sum_{j\neq i}\sum_{k\neq i,j} p_{j1,n}^\star p_{k1,n}^\star\right] \\
&= -b^2 \mathbb{E}\left[p_{i1}^{\star 2}\right] + 2e^2 \mathbb{E}\left[p_{i1}^\star p_{j1}^\star\right] = -b^2 \operatorname{Var}\left(p_{i1}^\star\right) + 2e^2 \operatorname{Cov}\left(p_{i1}^\star, p_{j1}^\star\right) + C_{\pi 2}; \\
\mathbb{E}\left[d_{i\infty}^a d_{j\infty}^a\right] &= \lim_{n\nearrow\infty} \mathbb{E}\left[d_{in}^a d_{jn}^a\right] \\
&= \lim_{n\nearrow\infty} \mathbb{E}\left[\left(-bp_{i1,n}^\star + \frac{e}{n-1}\sum_{k\neq i} p_{k1,n}^\star\right)\left(-bp_{j1,n}^\star + \frac{e}{n-1}\sum_{k\neq j} p_{k1,n}^\star\right)\right] \\
&= b^2 \mathbb{E}\left[p_{i1}^\star p_{j1}^\star\right] - 2be\mathbb{E}\left[p_{i1}^\star p_{j1}^\star\right] + \lim_{n\nearrow\infty} \left(\frac{e}{n-1}\right)^2 \mathbb{E}\left[\left(\sum_{k\neq i,j} p_{k1,n}^\star\right)^2\right] \\
&= \left(b^2 - 2be\right) \mathbb{E}\left[p_{i1}^\star p_{j1}^\star\right] + 2e^2 \mathbb{E}\left[p_{i1}^\star p_{j1}^\star\right] = \left((b-e)^2 + e^2\right) \operatorname{Cov}\left(p_{i1}^\star, p_{j1}^\star\right) + C_{\pi 3}.
\end{aligned}
$$

Putting these pieces together leaves

$$
\begin{aligned}
\mathbb{E}\left[u_{1\infty}\right] &\propto -(b-e)\left(-b^2 \operatorname{Var}\left(p_{i1}^\star\right) + 2e^2 \operatorname{Cov}\left(p_{i1}^\star, p_{j1}^\star\right)\right) - e\left((b-e)^2 + e^2\right) \operatorname{Cov}\left(p_{i1}^\star, p_{j1}^\star\right) + C_{\pi 4} \\
&= (b-e)\, b^2 \operatorname{Var}\left(p_{i1}^\star\right) - \left(2e^2 (b-e) + e(b-e)^2 + e^3\right) \operatorname{Cov}\left(p_{i1}^\star, p_{j1}^\star\right) + C_{\pi 4} \\
&\propto (1-r) \operatorname{Var}\left(p_{i1}^\star\right) - r \operatorname{Cov}\left(p_{i1}^\star, p_{j1}^\star\right) + C_{\pi 5}.
\end{aligned}
$$

□

Lemma 20. *There exists a constant $C_\pi \in \mathbb{R}$ such that for any $\overline{\tau}_\rho$, the linear equilibrium with a large number of firms yields expected first-period producer surplus*

$$
\mathbb{E}\left[\Pi_{1\infty}\right] \propto \left(\operatorname{Cov}\left(c_i, p_{i1}^\star\right) - \operatorname{Var}\left(p_{i1}^\star\right)\right) + \left(\operatorname{Cov}\left(p_{i1}^\star, p_{j1}^\star\right) - \operatorname{Cov}\left(c_i, p_{j1}^\star\right)\right) r + C_\pi.
$$

Proof. For any finite number of firms n, the linear pricing equilibrium yields first-stage expected producer surplus of

$$\begin{aligned}\mathbb{E}\left[\Pi_{1n}\right] &= \mathbb{E}\left[\sum_{i=1}^{n} \frac{1}{n-1}\left(a - bp_{i1}^\star + \frac{e}{n-1}\sum_{j\neq i} p_{j1}^\star\right)\left(p_{i1}^\star - c_i\right)\right] \\ &= \mathbb{E}\left[\left(a - bp_{i1}^\star + \frac{e}{n-1}\sum_{j\neq i} p_{j1}^\star\right)\left(p_{i1}^\star - c_i\right)\right].\end{aligned}$$

Symmetry in the linear pricing equilibrium implies the second equality; this expression holds for any firm i. With the exception of sensitivity to opponent prices all terms are (first-order) independent of the number of firms n; prices themselves will depend on the number of the firms in the market. Linearity of expectations and symmetry of pricing strategies imply

$$\mathbb{E}\left[\Pi_{1\infty}\right] = \lim_{n\nearrow\infty} \mathbb{E}\left[\Pi_{1n}\right] = \mathbb{E}\left[\left(a - bp_{i1}^\star\right)\left(p_{i1}^\star - c_i\right)\right] + e\mathbb{E}\left[\left(p_{i1}^\star - c_i\right)p_{j1}^\star\right] \qquad (j \neq i).$$

Recall that when n is large, expected prices do not depend on $\overline{\tau}_\rho$. Then there are constants $C_{\pi k}$ such that

$$\begin{aligned}\mathbb{E}\left[\Pi_{1\infty}\right] &= e\mathbb{E}\left[\left(p_{i1}^\star - c_i\right)p_{j1}^\star\right] - b\mathbb{E}\left[\left(p_{i1}^\star - c_i\right)p_{i1}^\star\right] + C_{\pi 1} \\ &= e\mathbb{E}\left[p_{i1}^\star p_{j1}^\star\right] - e\mathbb{E}\left[c_i p_{j1}^\star\right] - b\mathbb{E}\left[p_{i1}^{\star 2}\right] + b\mathbb{E}\left[c_i p_{i1}^\star\right] + C_{\pi 1} \\ &= e\,\mathrm{Cov}\left(p_{i1}^\star, p_{j1}^\star\right) - e\,\mathrm{Cov}\left(c_i, p_{j1}^\star\right) - b\,\mathrm{Var}\left(p_{i1}^\star\right) + b\,\mathrm{Cov}\left(c_i, p_{i1}^\star\right) + C_{\pi 2} \\ &\propto \left(\mathrm{Cov}\left(c_i, p_{i1}^\star\right) - \mathrm{Var}\left(p_{i1}^\star\right)\right) + \left(\mathrm{Cov}\left(p_{i1}^\star, p_{j1}^\star\right) - \mathrm{Cov}\left(c_i, p_{i1}^\star\right)\right) r + C_{\pi 3}.\end{aligned}$$

This establishes the stated result. ☐

Proof of Proposition 5. This follows immediately from Lemmas 19 and 20. ☐

Proof of Corollary 1. Following Proposition 5, the extent to which producer surplus depends on $\overline{\tau}_\rho$ is given by

$$\mathbb{E}\left[\Pi_{1\infty}\right] \simeq \left(\mathrm{Cov}\left(c_i, p_{i1}^\star\right) - \mathrm{Var}\left(p_{i1}^\star\right)\right) + \left(\mathrm{Cov}\left(p_{i1}^\star, p_{j1}^\star\right) - \mathrm{Cov}\left(c_i, p_{j1}^\star\right)\right) r.$$

To simplify notation we use $\simeq$ to denote an equivalence of all terms that depend directly on $\overline{\tau}_\rho$; that is, $f(\cdot) \simeq g(\cdot)$ if there is $C \in \mathbb{R}$ such that for any $\overline{\tau}_\rho$, $g(\overline{\tau}_\rho) - f(\overline{\tau}_\rho) = C$. We compute in turn:

$$\begin{aligned}
\operatorname{Cov}\left(c_i, p_{i1}^{\star}\right) &\simeq p_{\rho\infty}\overline{\tau}_\rho\sigma_\rho^2;\\
\operatorname{Var}\left(p_{i1}^{\star}\right) &\simeq p_{\rho\infty}^2\overline{\tau}_\rho^2\left(\sigma_\rho^2+\sigma_{\varepsilon\rho}^2\right) = p_{\rho\infty}^2\overline{\tau}_\rho\sigma_\rho^2;\\
\operatorname{Cov}\left(p_{i1}^{\star}, p_{j1}^{\star}\right) &\simeq p_{\rho\infty}^2\overline{\tau}_\rho^2\sigma_\rho^2;\\
\operatorname{Cov}\left(c_i, p_{j1}^{\star}\right) &\simeq p_{\rho\infty}\overline{\tau}_\rho\sigma_\rho^2.
\end{aligned}$$

Then

$$\begin{aligned}
\mathbb{E}\left[\Pi_{1\infty}\right] &\simeq \left(p_{\rho\infty}\overline{\tau}_\rho\sigma_\rho^2 - p_{\rho\infty}^2\overline{\tau}_\rho\sigma_\rho^2\right) + \left(p_{\rho\infty}^2\overline{\tau}_\rho^2\sigma_\rho^2 - p_{\rho\infty}\overline{\tau}_\rho\sigma_\rho^2\right) r\\
&\propto (1-r)\, p_{\rho\infty}\overline{\tau}_\rho - (1-r\overline{\tau}_\rho)\, p_{\rho\infty}^2\overline{\tau}_\rho\\
&= \frac{(1-r)\,\overline{\tau}_\rho}{2-r\overline{\tau}_\rho} - \frac{(1-r\overline{\tau}_\rho)\,\overline{\tau}_\rho}{\left(2-r\overline{\tau}_\rho\right)^2} = \frac{(1-2r)\,\overline{\tau}_\rho + r^2\overline{\tau}_\rho^2}{\left(2-r\overline{\tau}_\rho\right)^2}.
\end{aligned}$$

The derivative of this expression with respect to $\overline{\tau}_\rho$ is[34]

$$\frac{d}{d\overline{\tau}_\rho}\mathbb{E}\left[\Pi_{1\infty}\right] = \frac{2+(2r^2-4r)\,\overline{\tau}_\rho}{\left(2-r\overline{\tau}_\rho\right)^3}.$$

Since $r\overline{\tau}_\rho \leq 1$, the denominator is always positive. The numerator is linear in $\overline{\tau}_\rho$ and thus will obtain extrema at $\overline{\tau}_\rho = 0$ and $\overline{\tau}_\rho = 1$. At $\overline{\tau}_\rho = 0$ the numerator is $2 > 0$; at $\overline{\tau}_\rho = 1$ the numerator is $2(r-1)^2 \geq 0$. Then producer surplus is strictly increasing in $\overline{\tau}_\rho$ for all $\overline{\tau}_\rho < 1$. It follows that producer surplus is strictly greater when information is shared, provided $\overline{\tau}_\rho \in (0,1)$. $\square$

Proof of Corollary 2. Following Proposition 5, the extent to which consumer surplus depends on $\overline{\tau}_\rho$ is given by

$$\mathbb{E}\left[u_{1\infty}|\,\overline{\tau}_\rho\right] \simeq (1-r)\operatorname{Var}\left(p_{i1}^{\star}\right) - r\operatorname{Cov}\left(p_{i1}^{\star}, p_{j1}^{\star}\right).$$

We compute in turn:

$$\begin{aligned}
\operatorname{Var}\left(p_{i1}^{\star}\right) &\simeq p_{\rho\infty}^2\overline{\tau}_\rho\sigma_\rho^2;\\
\operatorname{Cov}\left(p_{i1}^{\star}, p_{j1}^{\star}\right) &\simeq p_{\rho\infty}^2\overline{\tau}_\rho^2\sigma_\rho^2.
\end{aligned}$$

[34]Recall that $\simeq$ is a relationship removing terms constant in $\overline{\tau}_\rho$. Then the derivative of $\mathbb{E}[\Pi_{1\infty}]$ with respect to $\overline{\tau}_\rho$ depends only on the terms on the right-hand side of $\simeq$ whether or not the constant terms are ignored, and can be written as $=$.

Then

$$\mathbb{E}\left[u_{1\infty}|\,\overline{\tau}_\rho\right] \simeq (1-r)\,p_{\rho\infty}^2\overline{\tau}_\rho\sigma_\rho^2 - rp_{\rho\infty}^2\overline{\tau}_\rho^2\sigma_\rho^2 \propto \frac{(1-r)\,\overline{\tau}_\rho - r\overline{\tau}_\rho^2}{(2-r\overline{\tau}_\rho)^2}.$$

To determine the effects of information sharing, we compare expected consumer surplus with precision $\overline{\tau}_\rho$ against precision $\overline{\tau}'_\rho = 1$,

$$\begin{aligned} & \mathbb{E}\left[u_{1\infty}|\,\overline{\tau}_\rho\right] \gtrless \mathbb{E}\left[u_{1\infty}|\,\overline{\tau}'_\rho = 1\right] \\ \iff \quad & \frac{(1-r)\,\overline{\tau}_\rho - r\overline{\tau}_\rho^2}{(2-r\overline{\tau}_\rho)^2} \gtrless \frac{1-2r}{(2-r)^2} \\ \iff \quad & \left(r^3+3r^2-4r\right)\overline{\tau}_\rho^2 + \left(4-4r-r^3-3r^2\right)\overline{\tau}_\rho + (8r-4) \gtrless 0. \end{aligned} \tag{15}$$

When $\overline{\tau}_\rho = 0$ the left-hand side is $8r-4$, and the value of information sharing will depend on whether $r \gtrless 1/2$. When $\overline{\tau}_\rho = 1$, the left-hand side is 0. The left-hand side of the above inequality is a quadratic in $\overline{\tau}_\rho$, so properties of the parabola will determine the effect of information sharing on consumer surplus. In particular, it is sufficient to analyze the slope of the parabola at $\overline{\tau}_\rho = 1$. If this slope is positive the left-hand side of (15) is negative and information sharing improves expected consumer welfare; if this slope is negative information sharing may lower expected consumer welfare, depending on initial informational precision.

The derivative of the left-hand side of (15) with respect to $\overline{\tau}_\rho$, evaluated at $\overline{\tau}_\rho = 1$, is

$$2\left(r^3+3r^2-4r\right) + \left(4-4r-r^3-3r^2\right) = r^3+3r^2-12r+4 = -\,(2-r)\left(r^2+5r-2\right).$$

Then the slope of the left-hand term will depend on $-(r^2+5r-2) \gtrless 0$. Solving the quadratic gives

$$r^\perp = -\frac{5}{2} + \frac{1}{2}\sqrt{33} \approx 0.372.$$

When $r \lesssim 0.372$ the slope of the left-hand term is positive at $\overline{\tau}_\rho = 1$, and when $r \gtrsim 0.372$ the slope of the left-hand term is negative at $\overline{\tau}_\rho = 1$. Then when $r \lesssim 0.372$ information sharing strictly improves expected consumer surplus, and when $r \gtrsim 0.372$ information sharing may harm expected consumer surplus, depending on the initial level of precision $\overline{\tau}_\rho$.

Finally, the derivative of the left-hand side of (15) is $(4-6r-3r^2)(\overline{\tau}_\rho - \overline{\tau}_\rho^2) + 8 > 0$. Then the negative effect of information sharing on consumer welfare is increasing in substitutability. When $r = 1/2$ the left-hand side of (15) is $9(\overline{\tau}_\rho - \overline{\tau}_\rho^2)/8 \geq 0$, and hence for all $r \geq 1/2$ the left-hand side of (15) is weakly positive. Then for all $r \geq 1/2$ information sharing decreases consumer welfare, strictly so when either $\overline{\tau}_\rho \in (0,1)$ or $r > 1/2$.

□

C Proofs for Section 5

Proof of Lemma 8. From the proof of Lemma 7, firm i's first-period profits are

$$\begin{aligned}\mathbb{E}\left[\pi_{i1}^{\star}\right] &= \left(a-(b-e)\,\mathbb{E}\left[p_{i1}\right]\right)\left(\mathbb{E}\left[p_{i1}\right]-\mathbb{E}\left[c_i\right]\right) \\ &\quad - b\left(\operatorname{Var}\left(p_{i1}\right)-\operatorname{Cov}\left(p_{i1},c_i\right)\right)+e\left(\operatorname{Cov}\left(p_{i1},p_{j1}\right)-\operatorname{Cov}\left(p_{j1},c_i\right)\right).\end{aligned}$$

Expected cost and first period pricing strategies are not effected by marginal increases in precision so the first term can be treated as a constant $C_i \in \mathbb{R}$.

Writing the variance and covariance terms in terms of the precision parameters we first note that for $x=\theta,\rho$

$$\operatorname{Var}\left(\mathbb{E}\left[x\right|s_{ix}]\right)=\frac{\tau_{ix}}{\left(\tau_{ix}+\tau_x\right)\tau_x},$$

and for ρ,

$$\operatorname{Cov}\left(\mathbb{E}\left[\rho\right|s_{i\rho}],\mathbb{E}\left[\rho\right|s_{j\rho}]\right)=\frac{\tau_{i\rho}\tau_{j\rho}}{\left(\tau_{i\rho}+\tau_\rho\right)\left(\tau_{j\rho}+\tau_\rho\right)\tau_\rho}.$$

Then the terms become

$$\begin{aligned}\operatorname{Var}\left(p_{i1}^{\star}\right) &= \frac{\tau_{i\theta}\left[p_{i\theta}^{\star}\right]^2}{\left(\tau_{i\theta}+\tau_\theta\right)\tau_\theta}+\frac{\tau_{i\rho}\left[p_{i\rho}^{\star}\right]^2}{\left(\tau_{i\rho}+\tau_\rho\right)\tau_\rho}, \\ \operatorname{Cov}\left(p_{i1}^{\star},c_i\right) &= \frac{\tau_{i\theta}p_{i\theta}^{\star}}{\left(\tau_{i\theta}+\tau_\theta\right)\tau_\theta}+\frac{\tau_{i\rho}p_{i\rho}^{\star}}{\left(\tau_{i\rho}+\tau_\rho\right)\tau_\rho}, \\ \operatorname{Cov}\left(p_{i1}^{\star},p_{j1}^{\star}\right) &= \frac{\tau_{i\rho}\tau_{j\rho}p_{i\rho}^{\star}p_{j\rho}^{\star}}{\left(\tau_{i\rho}+\tau_\rho\right)\left(\tau_{j\rho}+\tau_\rho\right)\tau_\rho}, \\ \operatorname{Cov}\left(p_{j1}^{\star},c_i\right) &= \frac{\tau_{i\rho}\tau_{j\rho}p_{j\rho}^{\star}}{\left(\tau_{i\rho}+\tau_\rho\right)\left(\tau_{j\rho}+\tau_\rho\right)\tau_\rho}.\end{aligned}$$

Putting everything together,

$$\mathbb{E}\left[\pi_{i,1}\right]=\left(\frac{\left(1-p_{i\theta}^{\star}\right)p_{i\theta}^{\star}\tau_{i\theta}}{\left(\tau_{i\theta}+\tau_\theta\right)\tau_\theta}\right)b+\left(\frac{\left(1-p_{i\rho}^{\star}\right)p_{i\rho}^{\star}\tau_{i\rho}}{\left(\tau_{i\rho}+\tau_\rho\right)\tau_\rho}\right)b-\left(\frac{\left(1-p_{i\rho}^{\star}\right)p_{j\rho}^{\star}\tau_{i\rho}\tau_{j\rho}}{\left(\tau_{i\rho}+\tau_\rho\right)\left(\tau_{j\rho}+\tau_\rho\right)\tau_\rho}\right)e+C_i.$$

The result follows from taking the derivative with respect to each precision parameter. □

Proof of Lemma 9. Firm i's first period profits are

$$\begin{aligned}\mathbb{E}\left[\pi_{i1}^{c}\right] &= \left(a-(b-e)\,\mathbb{E}\left[p_{i1}^{c}\right]\right)\left(\mathbb{E}\left[p_{i1}^{c}\right]-\mathbb{E}\left[c_i\right]\right) \\ &\quad - b\left(\operatorname{Var}\left(p_{i1}^{c}\right)-\operatorname{Cov}\left(p_{i1}^{c},c_i\right)\right)+e\left(\operatorname{Cov}\left(p_{i1}^{c},p_{j1}^{c}\right)-\operatorname{Cov}\left(p_{j1}^{c},c_i\right)\right).\end{aligned}$$

For ρ the pertinent variance term is

$$\operatorname{Var}\left(\mathbb{E}\left[\rho \middle| s_{i\rho}, s_{j\rho}\right]\right) = \frac{\tau_{i\rho} + \tau_{j\rho}}{\left(\tau_{i\rho} + \tau_{j\rho} + \tau_{\rho}\right)\tau_{\rho}}.$$

Then the variance and covariance terms are

$$\begin{aligned}
\operatorname{Var}\left(p_{i1}^{c}\right) &= \frac{\tau_{i\theta}\left[p_{i\theta}^{c}\right]^{2}}{\left(\tau_{i\theta} + \tau_{\theta}\right)\tau_{\theta}} + \frac{\left(\tau_{i\rho} + \tau_{j\rho}\right)\left[p_{i\rho}^{c}\right]^{2}}{\left(\tau_{i\rho} + \tau_{j\rho} + \tau_{\rho}\right)\tau_{\rho}}, \\
\operatorname{Cov}\left(p_{i1}^{c}, c_{i}\right) &= \frac{\tau_{i\theta}p_{i\theta}^{c}}{\left(\tau_{i\theta} + \tau_{\theta}\right)\tau_{\theta}} + \frac{(\tau_{i\rho} + \tau_{j\rho})p_{i\rho}^{c}}{\left(\tau_{i\rho} + \tau_{j\rho} + \tau_{\rho}\right)\tau_{\rho}}, \\
\operatorname{Cov}\left(p_{i1}^{c}, p_{j1}^{c}\right) &= \frac{\left(\tau_{i\rho} + \tau_{j\rho}\right)p_{i\rho}^{c}p_{j\rho}^{c}}{\left(\tau_{i\rho} + \tau_{j\rho} + \tau_{\rho}\right)\tau_{\rho}}, \\
\operatorname{Cov}\left(p_{j1}^{c}, c_{i}\right) &= \frac{\left(\tau_{i\rho} + \tau_{j\rho}\right)p_{j\rho}^{c}}{\left(\tau_{i\rho} + \tau_{j\rho} + \tau_{\rho}\right)\tau_{\rho}}.
\end{aligned}$$

Putting everything together,

$$\begin{aligned}
&\mathbb{E}\left[\pi_{i1}^{c}\right] \\
&= \left(\frac{\left(1 - p_{i\theta}^{c}\right)p_{i\theta}^{c}\tau_{i\theta}}{\left(\tau_{i\theta} + \tau_{\theta}\right)\tau_{\theta}}\right)b + \left(\frac{\left(1 - p_{i\rho}^{c}\right)p_{i\rho}^{c}(\tau_{i\rho} + \tau_{j\rho})}{\left(\tau_{i\rho} + \tau_{j\rho} + \tau_{\rho}\right)\tau_{\rho}}\right)b - \left(\frac{\left(1 - p_{i\rho}^{c}\right)p_{j\rho}^{c}(\tau_{i\rho} + \tau_{j\rho})}{\left(\tau_{i\rho} + \tau_{j\rho} + \tau_{\rho}\right)\tau_{\rho}}\right)e + C_{i}.
\end{aligned}$$

The result follows from taking the derivative with respect to each precision parameter. □

Proof of Proposition 6. This proof is included in the main text. □

Proof of Theorem 5. The public information that is relevant to firm j's marginal cost includes, ρ, p_{j1}, shared information on ρ: $\tilde{s}_{\rho} = \rho + \frac{1}{\tilde{M}_{j\rho}+\tilde{M}_{i\rho}}\left(\sum_{m=1}^{\tilde{M}_{j\rho}} u_{i\rho m} + \sum_{m=1}^{\tilde{M}_{i\rho}} u_{j\rho m}\right)$, and the information shared with firm i about θ_j: $\tilde{s}_{i\theta_j}$. Additionally, because ρ is observed, then the remainder of the first period public signal on common costs can be identified: $\tilde{\varepsilon}_{\rho} \equiv \tilde{s}_{\rho} - \rho$. We are interested the expected cost given this public information, $\mathbb{E}[c_j|\rho, \hat{p}_{ij}, \tilde{\varepsilon}_{\rho}, \tilde{s}_{i\theta_j}]$, and how it changes as $\hat{p}_{j1}$ changes. The five variables $(\theta_j, \rho, \hat{p}_{j1}, \tilde{\varepsilon}_{\rho}, \tilde{s}_{i\theta_j})^T$ are distributed joint normally with means $(\mu_{\theta}, \mu_{\rho}, \mathbb{E}\left[\hat{p}_{j1}\right], 0, \mu_{\theta})^T$ and covariance matrix

$$\begin{pmatrix}
\sigma_{\theta}^{2} & 0 & \tilde{p}_{j\theta_j}\tilde{\bar{\tau}}_{j\theta_j}\sigma_{\theta}^{2} & 0 & \sigma_{\theta}^{2} \\
0 & \sigma_{\rho}^{2} & \tilde{p}_{j\rho}\tilde{\bar{\tau}}_{j\rho}\sigma_{\rho}^{2} & 0 & 0 \\
\tilde{p}_{j\theta}\tilde{\bar{\tau}}_{j\theta_j}\sigma_{\theta}^{2} & \tilde{p}_{j\rho}\tilde{\bar{\tau}}_{j\rho}\sigma_{\rho}^{2} & \tilde{p}_{j\theta_j}^{2}\tilde{\bar{\tau}}_{j\theta_j}\sigma_{\theta}^{2} + \tilde{p}_{j\rho}^{2}\tilde{\bar{\tau}}_{j\rho}\sigma_{\rho}^{2} & p_{j\rho}\tilde{\bar{\tau}}_{j\rho}\left(\frac{\sigma_{u\rho}^{2}}{M_{j\rho}+\tilde{M}_{j\rho}}\right) & \tilde{p}_{j\theta_j}\tilde{\bar{\tau}}_{j\theta_j}\left(\sigma_{\theta}^{2} + \frac{\sigma_{u\theta_j}}{M_{j\theta_j}}\right) \\
0 & 0 & p_{j\rho}\tilde{\bar{\tau}}_{j\rho}\left(\frac{\sigma_{u\rho}^{2}}{M_{j\rho}+\tilde{M}_{j\rho}}\right) & \frac{\sigma_{u\rho}^{2}}{\tilde{M}_{j\rho}+\tilde{M}_{i\rho}} & 0 \\
\sigma_{\theta}^{2} & 0 & \tilde{p}_{j\theta_j}\tilde{\bar{\tau}}_{j\theta_j}\left(\sigma_{\theta}^{2} + \frac{\sigma_{u\theta_j}}{M_{j\theta_j}}\right) & 0 & \sigma_{\theta}^{2} + \frac{\sigma_{u\theta_j}^{2}}{\tilde{M}_{i\theta_j}}
\end{pmatrix}.$$

Then the conditional expectation of c_j, given ρ, $\hat{p}_{j1}$, $\tilde{\varepsilon}_\rho$, and $\tilde{s}_{i\theta_j}$ is

$$\mathbb{E}\left[\theta_j|\,\rho,\hat{p}_{j1},\tilde{\varepsilon}_\rho,\tilde{s}_{i\theta_j}\right]=\mu_\theta+\Sigma_{12}\Sigma_{22}^{-1}\left((\rho,\hat{p}_{j1},\tilde{\varepsilon}_\rho,\tilde{s}_{i\theta_j})^T-(\mu_\rho,\mathbb{E}\left[\hat{p}_{j1}\right],0,\mu_\theta)^T\right),\text{ with}$$

$$\Sigma_{12}=\left(0,\tilde{p}_{j\theta_j}\tilde{\tilde{\tau}}_{j\theta_j}\sigma_\theta^2,0,\sigma_\theta^2\right),$$

$$\Sigma_{22}=\begin{pmatrix}\sigma_\rho^2 & \tilde{p}_{j\rho}\tilde{\tilde{\tau}}_{j\rho}\sigma_\rho^2 & 0 & 0\\ \tilde{p}_{j\rho}\tilde{\tilde{\tau}}_{j\rho}\sigma_\rho^2 & \tilde{p}_{j\theta_j}^2\tilde{\tilde{\tau}}_{j\theta_j}\sigma_\theta^2+\tilde{p}_{j\rho}^2\tilde{\tilde{\tau}}_{j\rho}\sigma_\rho^2 & p_{j\rho}\tilde{\tilde{\tau}}_{j\rho}\left(\frac{\sigma_{u\rho}^2}{M_{j\rho}+\tilde{M}_{j\rho}}\right) & \tilde{p}_{j\theta_j}\tilde{\tilde{\tau}}_{j\theta_j}\left(\sigma_\theta^2+\frac{\sigma_{u\theta_j}}{M_{j\theta_j}}\right)\\ 0 & p_{j\rho}\tilde{\tilde{\tau}}_{j\rho}\left(\frac{\sigma_{u\rho}^2}{M_{j\rho}+\tilde{M}_{j\rho}}\right) & \frac{\sigma_{u\rho}^2}{\tilde{M}_{j\rho}+\tilde{M}_{i\rho}} & 0\\ 0 & \tilde{p}_{j\theta_j}\tilde{\tilde{\tau}}_{j\theta_j}\left(\sigma_\theta^2+\frac{\sigma_{u\theta_j}}{M_{j\theta_j}}\right) & 0 & \sigma_\theta^2+\frac{\sigma_{u\theta_j}^2}{\tilde{M}_{i\theta_j}}\end{pmatrix}.$$

The conditional expectation matrix is

$$\Sigma_{21}\Sigma_{22}^{-1}=\left(-\tilde{p}_{j\rho}\tilde{\tilde{\tau}}_{j\rho}\tilde{\kappa},\ \tilde{\kappa},\ -\tilde{p}_{j\rho}\tilde{\tilde{\tau}}_{j\rho}\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{M_{j\rho}+\tilde{M}_{j\rho}}\tilde{\kappa},\ \tilde{\tilde{\tau}}_{i\theta_j}(1-\tilde{\kappa}\tilde{p}_{j\theta_j})\right),$$

where

$$\tilde{\kappa}=\frac{\tilde{p}_{j\theta_j}\tilde{\tilde{\tau}}_{j\theta_j}\sigma_\theta^2\left(1-\frac{\tilde{M}_{i\theta_j}}{M_{j\theta_j}}\frac{M_{j\theta_j}\sigma_\theta^2+\sigma_{u\theta_j}^2}{\tilde{M}_{i\theta_j}\sigma_\theta^2+\sigma_{u\theta_j}^2}\right)}{\tilde{p}_{j\theta_j}^2\tilde{\tilde{\tau}}_{j\theta_j}\sigma_\theta^2\left(1-\frac{\tilde{M}_{i\theta_j}}{M_{j\theta_j}}\frac{M_{j\theta_j}\sigma_\theta^2+\sigma_{u\theta_j}^2}{\tilde{M}_{i\theta_j}\sigma_\theta^2+\sigma_{u\theta_j}^2}\right)+\tilde{p}_{j\rho}^2\tilde{\tilde{\tau}}_{j\rho}\left(1-\tilde{\tilde{\tau}}_{j\rho}\right)\sigma_\rho^2\left(1-\frac{\tilde{M}_{j\rho}+\tilde{M}_{i\rho}}{M_{j\rho}+\tilde{M}_{j\rho}}\right)}.$$

Then the expected costs as a function of the public information in the second period are given by

$$\begin{aligned}\mathbb{E}[c_j|\rho,\hat{p}_{j1},\tilde{\varepsilon}_\rho,\tilde{s}_{i\theta_j}]&=\rho+\mu_\theta-(\rho-\mu_\rho)\tilde{p}_{j\rho}\tilde{\tilde{\tau}}_{j\rho}\tilde{\kappa}+\left(\hat{p}_{j1}-\mathbb{E}[\hat{p}_{j1}]\right)\tilde{\kappa}\\&\quad-\tilde{\varepsilon}_\rho\tilde{p}_{j\rho}\tilde{\tilde{\tau}}_{j\rho}\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{M_{j\rho}+\tilde{M}_{j\rho}}\tilde{\kappa}+(1-\tilde{\kappa}\tilde{p}_{j\theta_j})\tilde{\tilde{\tau}}_{i\theta_j}(\tilde{s}_{i\theta_j}-\mu_\theta)\end{aligned}$$

Firm i's expectation of these expected costs given the signals it has in the first period are

$$\begin{aligned}\mathbb{E}\left[\mathbb{E}\left[c_j|\rho,p_{j1},\tilde{\varepsilon}_\rho,\tilde{s}_{i\theta_j}\right]|\tilde{s}_{i\rho},\tilde{s}_{i\theta_j},\tilde{s}_{i\theta_i}\right]&=\mathbb{E}[\rho|\tilde{s}_{i\rho}]+\mathbb{E}[\theta_j|\tilde{s}_{i\theta_j}]\\ \mathbb{E}\left[\mathbb{E}\left[c_i|\rho,p_{j1},\tilde{\varepsilon}_\rho,\tilde{s}_{i\theta_j}\right]|\tilde{s}_{i\rho},\tilde{s}_{i\theta_j},\tilde{s}_{i\theta_i}\right]&=\mathbb{E}[\rho|\tilde{s}_{i\rho}]+\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}]+\tilde{\kappa}\tilde{p}_{i\theta_i}\left(\mathbb{E}[\theta_i|\tilde{s}_{i\theta_i}]-\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}]\right)\\&\quad+\tilde{\kappa}\tilde{p}_{i\rho}(1-\tilde{\tilde{\tau}}_{i\rho})\left(1-\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{\tilde{M}_{i\rho}+M_{i\rho}}\right)\left(\mathbb{E}[\rho|\tilde{s}_{i\rho}]-\mu_\rho\right)\end{aligned}$$

Expected first period price of the firm j given the signals of firm i.

$$\begin{aligned}
\mathbb{E}[\tilde{p}_{j1}|\tilde{s}_{i\rho},\tilde{s}_{i\theta_i},\tilde{s}_{i\theta_j}] &= \mathbb{E}\left[\tilde{p}_{j0}+\mathbb{E}[\theta_j|\tilde{s}_{j\theta_j}]\tilde{p}_{j\theta_j}+\mathbb{E}[\rho|\tilde{s}_{i\rho}]\tilde{p}_{j\rho}+\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}]\tilde{p}_{j\theta_i}]|\tilde{s}_{i\rho},\tilde{s}_{i\theta_i},\tilde{s}_{i\theta_j}\right] \\
&= \tilde{p}_{j0}+\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\mathbb{E}[\rho|\tilde{s}_{i\rho}]\tilde{p}_{j\rho}+\mathbb{E}[\theta_j|\tilde{s}_{i\theta_j}]\tilde{p}_{j\theta_j}+\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}]\tilde{p}_{j\theta_i} \\
&\quad+\left(1-\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)\left(\tilde{\overline{\tau}}_{j\rho}\mathbb{E}[\rho|\tilde{s}_{i\rho}]+(1-\tilde{\overline{\tau}}_{j\rho})\mu_\rho\right)\tilde{p}_{j\rho}
\end{aligned}$$

Plugging these into the first order condition from Lemma 3

$$\begin{aligned}
4b\tilde{p}_{i1} &= 2\mathbb{E}[bc_i+a+e\tilde{p}_{j1}|\tilde{s}_{i\rho},\tilde{s}_{i\theta_i},\tilde{s}_{i\theta_j}]+\beta\tilde{\kappa}_i\mathbb{E}\left[\left(a-bc_i+\frac{e}{4b^2-e^2}((2b+e)a)\right)\middle|\tilde{s}_{i\rho},\tilde{s}_{i\theta_i},\tilde{s}_{j\theta_i}\right] \\
&\quad+\frac{e\beta\tilde{\kappa}_i}{4b^2-e^2}\left[2b^2\left(\mathbb{E}[\rho|\tilde{s}_{i\rho}]+\mathbb{E}[\theta_j|\tilde{s}_{i\theta_j}]\right)+be\left(\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}]+\tilde{\kappa}_i\tilde{p}_{i\theta_i}\left(\mathbb{E}[\theta_i|\tilde{s}_{i\theta_i}]-\mathbb{E}[\theta_i|\tilde{s}_{j\theta_i}]\right)\right)\right] \\
&\quad+\frac{e\beta\tilde{\kappa}_i}{4b^2-e^2}\left[be\left(\mathbb{E}[\rho|\tilde{s}_{i\rho}]+\tilde{\kappa}_i\tilde{p}_{i\rho}(1-\tilde{\overline{\tau}}_{i\rho})\left(1-\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{\tilde{M}_{i\rho}+M_{i\rho}}\right)\left(\mathbb{E}[\rho|\tilde{s}_{i\rho}]-\mu_\rho\right)\right)\right]
\end{aligned}$$

Matching coefficients we get the following system

$$\begin{aligned}
4bp_{i\theta_i} &= 2b-b\beta\kappa_i+\frac{be^2\beta\kappa_i^2p_{i\theta}}{4b^2-e^2};\quad 4bp_{i\theta_j}=2e\tilde{p}_{j\theta_j}+\frac{2b^2e}{4b^2-e^2}\beta\tilde{\kappa}_i; \\
4bp_{i\rho} &= 2b+2e\left(\tilde{\overline{\tau}}_\rho p_{j\rho}+(1-\tilde{\overline{\tau}}_\rho p_{j\rho})\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)-b\beta\kappa_i \\
&\quad+\frac{e\beta\kappa_i}{4b^2-e^2}\left(2b^2+be\left(1+(1-\overline{\tau}_\rho)\kappa_ip_{i\rho}\left(1-\frac{\tilde{M}_{i\rho}+\tilde{M}_{j\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)\right)\right); \\
4bp_{i\tilde{s}_{j\theta_i}} &= 2e\tilde{p}_{j\theta_i}+\frac{be^2}{4b^2-e^2}\beta\tilde{\kappa}_i(1-\tilde{\kappa}_i\tilde{p}_{i\theta_i}).
\end{aligned}$$

Imposing symmetry (in both strategies and shared information ($\tilde{M}_{i\rho}=\tilde{M}_{j\rho}$ and $\tilde{M}_{i\theta_j}=\tilde{M}_{j\theta_i}$)) and rearranging the coefficients we get the desired result. □

Proof of Proposition 7. Following the proof of Theorem 2 we calculate the ex-ante expected price in period 1 and 2.

Ex-ante expected first period prices are

$$\mathbb{E}[\tilde{p}_{i1}] = \frac{1}{2b-e}\mathbb{E}\left[\mathbb{E}\left[bc_i + a + \frac{e}{2b}\left(a - bc_i + e\mathbb{E}\left[\tilde{p}_{j2}\right|\rho, \hat{p}_{j1}, \tilde{\varepsilon}_\rho, \tilde{s}_{i\theta_j}\right]\right)\frac{be}{4b^2-e^2}\kappa\right|\tilde{s}_{i\rho}, \tilde{s}_{i\theta_i}, \tilde{s}_{i\theta_j}, \tilde{s}_{j\theta_i}\right]\right]$$

$$= \frac{1}{2b-e}\Bigg(a + b(\mu_\rho + \mu_\theta)$$

$$+ \frac{e^2\kappa}{2(4b^2-e^2)}\mathbb{E}\left[\mathbb{E}\left[\left(a - bc_i + e\mathbb{E}\left[\tilde{p}_{j2}\right|\rho, \hat{p}_{j1}, \tilde{\varepsilon}_\rho, \tilde{s}_{i\theta_j}\right]\right)\right|\tilde{s}_{i\rho}, \tilde{s}_{i\theta_i}, \tilde{s}_{i\theta_j}, \tilde{s}_{j\theta_i}\right]\right]\Bigg).$$

Where ex-ante expected second period prices do not depend on the information sharing agreement:

$$\mathbb{E}[\tilde{p}_{j2}] = \frac{1}{4b^2-e^2}\left((2b+e)\,a + 2b^2\mathbb{E}\left[\mathbb{E}\left[c_j\right|\rho, \hat{p}_{j1}, \tilde{\varepsilon}_\rho, \tilde{s}_{i\theta_j}\right]\right] + be\mathbb{E}\left[\mathbb{E}\left[c_i\right|\rho, \hat{p}_{i1}, \tilde{\varepsilon}_\rho, \tilde{s}_{j\theta_i}\right]\right]\right)$$

$$= \frac{1}{4b^2-e^2}\left(\mathbb{E}\left[(2b+e)a + 2b^2\left(\mathbb{E}\left[\theta_j|\tilde{s}_{i\theta_j}\right] + \mathbb{E}\left[\rho|\tilde{s}_{i\rho}\right]\right) + be\left(\mathbb{E}\left[\theta_i|\tilde{s}_{j\theta_i}\right] + \mathbb{E}\left[\rho|\tilde{s}_{i\rho}\right]\right)\right]\right)$$

$$= \frac{a + b(\mu_\theta + \mu_\rho)}{2b-e}$$

Therefore ex-ante expected prices in the first period become

$$\mathbb{E}[\tilde{p}_{i1}] = \frac{a + b(\mu_\rho + \mu_\theta)}{2b-e} + \frac{e^2\tilde{\kappa}}{2(4b^2-e^2)}\left(\frac{a + b(\mu_\theta + \mu_\rho)}{2b-e}\right) = \frac{a + b(\mu_\rho + \mu_\theta)}{2b-e}\left(1 + \frac{1}{2}\beta\tilde{\kappa}\right).$$

□

Proof of Proposition 8. We combine the equation for the informativeness of price with the coefficients $\tilde{p}_{i\theta_i}$ and $\tilde{p}_{i\rho}$ to get the following expression for the equilibrium value of $\hat{\kappa} = \beta\tilde{\kappa}$

$$\hat{\kappa}\left(1 - \left(\frac{1-r}{2-r}\right)\hat{\kappa}\right)^2 (2+\hat{\kappa})^2\tilde{\tilde{\tau}}_{i\rho}(1-\tilde{\tilde{\tau}}_{i\rho})\sigma_\rho^2$$

$$= (\beta(2+\hat{\kappa}) - \hat{\kappa})\,\tilde{\tilde{\tau}}_{i\theta_i}\sigma_\theta^2\left(1 - \frac{\tilde{M}_{i\theta_j}}{M_{j\theta_j}}\frac{M_{j\theta_j}\sigma_\theta^2 + \sigma_{u\theta_j}^2}{\tilde{M}_{i\theta_j}\sigma_\theta^2 + \sigma_{u\theta_j}^2}\right)$$

$$\times\frac{\left(2 - \left(1 - \frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)\left(r\tilde{\tilde{\tau}}_{j\rho} - \frac{1}{2}\hat{\kappa}^2(1-\tilde{\tilde{\tau}}_{j\rho})\right) - r\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)^2}{\left(1 - \frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)}$$

Relative to no information sharing, the LHS is the same. The right hand side is still decreasing in $\hat{\kappa}$ therefore the basic argument of existence (and uniqueness?) will go through in the more general setting.

An increase in $\tilde{M}_{i\theta_j}$ decreases the RHS for all $\hat{\kappa}$, lowering the value of $\hat{\kappa}$ where the two lines intersect. An increase in $\tilde{M}_{i\rho}$ will increase the RHS for all $\hat{\kappa}$ increasing the equilibrium value of $\hat{\kappa}$. To see this second point we take the derivative of the RHS with respect to $\frac{2\tilde{M}_{i\rho}}{\tilde{M}_{i\rho}+M_{i\rho}}$ and get something proportional to

$$\left(1-\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)\left(-2r+2\left(r\tilde{\bar{\tau}}_{i\rho}+\frac{1}{2}\hat{\kappa}(1-\tilde{\bar{\tau}}_{i\rho})\right)\right)$$
$$+2-\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}r-\left(1-\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}\right)\left(r\tilde{\bar{\tau}}_{i\rho}+\frac{1}{2}\hat{\kappa}(1-\tilde{\bar{\tau}}_{i\rho})\right).$$

This is equal to

$$-2r+2r\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}+2-r\frac{2\tilde{M}_{i\rho}}{M_{i\rho}+\tilde{M}_{i\rho}}+(2-1)\left(r\tilde{\bar{\tau}}_{i\rho}+\frac{1}{2}\hat{\kappa}(1-\tilde{\bar{\tau}}_{i\rho})\right)>0.$$

□

D Bounds on values (for online publication)

The following inequalities are used throughout the paper.

$$\beta \in \left[0,\frac{1}{3}\right] \tag{16}$$

$$\frac{b-e}{2b-e} \in \left[0,\frac{2}{3}\right] \tag{17}$$

$$\left(\frac{b-e}{2b-e}\right)\beta \in \left[0,\frac{2}{9}\right] \tag{18}$$

$$p_\theta \in \left[\frac{1}{3},\frac{1}{2}\right] \tag{19}$$

$$\kappa \in \left[0,\frac{2}{1-\beta}\right] \subseteq [0,3] \tag{20}$$

$$\beta\kappa \in \left[0,\frac{r^2}{2-r^2}\right] \subseteq \left[0,r^2\right] \subseteq [0,1] \tag{21}$$

$$\left|\frac{be}{4b^2-e^2}\right|\kappa \in \left[0,\frac{r}{2-r^2}\right] \subseteq [0,r] \subseteq [0,1] \tag{22}$$

$$p_\rho \in \left[\frac{1}{9},\frac{1}{2}\right] \text{ (when } e<0) \tag{23}$$

$$p_\rho \in [0.46,1] \text{ (when } e>0) \tag{24}$$

D.1 Proofs of bounds

Proof of inequality (16). Since $|e| \leq b$, $\beta = e^2/(4b^2 - e^2) \geq 0$. To establish the upper bound, note that the numerator is increasing in e^2 and the denominator is decreasing in e^2, so the maximum value of β will be attained when e^2 is at its maximum. Since $e^2 \leq b^2$, it follows that $\beta \leq 3$. $\square$

Proof of inequality (17). Since $|e| \leq b$, it is immediate that $(b-e)/(2b-e) \geq 0$. To establish the upper bound we examine the first derivative of the expression with respect to e,[35]

$$\frac{-(2b-e)+(b-e)}{(2b-e)^2} = -\frac{b}{(2b-e)^2} < 0.$$

Then the derivative is negative everywhere, and the expression is maximized when e is at its minimum, $e = -b$. This gives

$$\frac{b-(-b)}{2b-(-b)} = \frac{2}{3}.$$

$\square$

Proof of inequality (18). This follows directly from inequalities (16) and (17). $\square$

Proof of inequalities (19) *and* (20). Since $\beta\kappa \geq 0$ and $p_\theta = 1/(2+\beta\kappa)$, it must be that $p_\theta \leq 1/2$. Further, p_θ will be minimized when $\beta\kappa$ is maximized. Looking at κ in isolation,

$$\kappa = \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\sigma_{s,\rho}^2 \bar{\tau}_{s,\rho}^2 p_\rho^2 + \left(\sigma_\theta^2 + \sigma_{s,\theta}^2\right) \bar{\tau}_{s,\theta}^2 p_\theta^2}.$$

All involved terms are positive, so κ can be bounded above by assuming that $\bar{\tau}_{s,\rho} = 0$. This gives

$$\kappa \leq \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\left(\sigma_\theta^2 + \sigma_{s,\theta}^2\right) \bar{\tau}_{s,\theta}^2 p_\theta^2} = \frac{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta}{\sigma_\theta^2 \bar{\tau}_{s,\theta} p_\theta^2} = \frac{1}{p_\theta}.$$

Let $\underline{p}_\theta$ be the minimum feasible value of p_θ and $\overline{\beta} = 1/3$ be the maximum feasible value of β; then $\kappa \leq 1/\underline{p}_\theta$. It follows that

$$p_\theta \geq \frac{1}{2 + \frac{\beta}{\underline{p}_\theta}} \quad \Longrightarrow \quad \underline{p}_\theta \geq \frac{1}{2 + \frac{\overline{\beta}}{\underline{p}_\theta}}.$$

This gives

$$2\underline{p}_\theta + \overline{\beta} \geq 1 \quad \Longrightarrow \quad \underline{p}_\theta \geq \frac{1}{3}.$$

[35]Basic intuition about fractions is sufficient for this maximization. We find that straightforward calculus is simpler to analyze.

Then $p_\theta \geq 1/3$. It follows that $\kappa \leq 3$. Since $|e| \leq b$, $be/(4b^2 - e^2) \leq 1/3$, hence

$$\left(\frac{be}{4b^2 - e^2}\right) \kappa \leq \left(\frac{1}{3}\right) 3 = 1.$$

From $\kappa \leq \frac{1}{p_\theta} = 2 + \beta\kappa$ we can bound

$$\kappa \leq \frac{2}{1-\beta} = \frac{4-r^2}{2-r^2}.$$

$\square$

Proof of inequalities (21) *and* (22). This follows directly from (20),

$$0 \leq \beta\kappa \leq \beta\left(\frac{2}{1-\beta}\right) = \frac{2r^2}{4-2r^2} = \frac{r^2}{2-r^2}.$$

Then

$$0 \leq \left|\frac{be}{4b^2-e^2}\right| \kappa = \left|\frac{b}{c}\right| \beta\kappa = \left|\frac{1}{r}\right| \beta\kappa \leq \frac{r}{2-r^2}.$$

$\square$

Proof of inequalities (23) *and* (24). Recall the equilibrium equation for p_ρ,

$$p_\rho = \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa}{(2 - r\bar{\tau}_{s,\rho}) - \frac{1}{2}\left(1-\bar{\tau}_{s,\rho}\right)\beta^2\kappa^2}, \text{ where } r = \frac{e}{b}.$$

By inequality (21), $\beta\kappa \leq |r|$, so the bound on the denominator will depend on the sign of r.

When $r < 0$, the denominator is bounded below by $2 - \beta^2\kappa^2/2$ and above by $2 - r$. The numerator is bounded above by $1 - \beta\kappa/2$. This gives

$$\begin{aligned} p_\rho &\leq \frac{1 - \frac{1}{2}\beta\kappa}{2 - \frac{1}{2}\beta^2\kappa^2} & \qquad p_\rho &\geq \frac{1 - \left(\frac{1-r}{2-r}\right)\beta\kappa}{2-r} \\ &= \frac{2-\beta\kappa}{4-\beta^2\kappa^2} & &\geq \frac{(2-r)-(1-r)}{(2-r)^2} \\ &= \frac{1}{2+\beta\kappa} \leq \frac{1}{2}; & &= \frac{1}{(2-r)^2} \geq \frac{1}{9}. \end{aligned}$$

When $r > 0$, the denominator is bounded below by $2 - r$ and above by $2 - \beta^2\kappa^2/2$. The numerator is bounded below by $1 - \beta\kappa/2$. This gives[36]

$$p_\rho \leq \frac{1}{2-r} \leq 1; \qquad p_\rho \geq \frac{1 - \left(\frac{1-r}{2-r}\right)\left(\frac{r^2}{2-r^2}\right)}{2 - \frac{1}{2}\left(\frac{r^2}{4-r^2}\right)^2 \kappa^2}$$
$$\geq \frac{1}{2} - \left(\frac{1-r}{2-r}\right)\left(\frac{r^2}{4-2r^2}\right) \gtrsim 0.46.$$

□

[36]While not obvious from the approach here, simple cases and numerical investigation show that these bounds are tight.