Unawareness and Framing

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Abstract

We introduce a model of unawareness founded on preferences over descriptions of acts. The model highlights the effects of how the contingencies of an act or contract are framed by distinguishing different descriptions of the same act. A more detailed description immediately confers a higher level of awareness to the decision maker. The primitive is a family of preferences, indexed by partitions of the state space. Each partition corresponds to an enumeration or frame of the state space. We axiomatically characterize the following partition-dependent expected utility representation: the decision maker has a (nonadditive) set function over contingencies which she adapts and normalizes to her level of awareness; she then computes expected utility with respect to her partition-dependent belief. Unawareness can then be expressed through betting preference and subjective likelihood rather than through knowledge. Absolute and relative notions of unawareness and response to unawareness are presented.

1 Introduction

Consider a newly hired worker comparing available health insurance plans during open enrollment. While she understands some broad possible contingencies, like requiring a surgery or becoming pregnant, she is unaware of more specific contingencies, like requiring a laminotomy.[1] Despite this partial awareness of the environment, she still has to decide to enroll in some health plan before the end of the month. How much is this employee willing to pay for the different insurance options? Can an outside observer, who knows what kind of coverage she will purchase, distinguish the treatments of which the employee is aware from those of which she has never heard? Can the observer distinguish those of which the employee has never heard from those that she believes are impossible? Can the observer place predictive restrictions on the employee's behavior as she becomes aware of more contingencies?

This paper introduces a novel methodology for answering these sorts of questions. It uses different framing of acts to measure the decision maker's response to different levels of awareness. For

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[1]Incidentally, a laminotomy is a surgery which removes a thin bony layer covering the spinal canal.

example, consider the following contract, which associates deductibles on the left with contingencies on the right:

$$\begin{pmatrix} \$500 & \text{surgery} \\ \$100 & \text{prenatal care} \\ \vdots & \vdots \end{pmatrix}.$$

Compare this to the following contract, which includes some redundancies:

$$\begin{pmatrix} \$500 & \text{laminotomy} \\ \$500 & \text{other surgeries} \\ \$100 & \text{prenatal care} \\ \vdots & \vdots \end{pmatrix}.$$

These contracts represent the same levels of coverage, but the decision maker might evaluate these lists differently, because the second formulation makes her aware of laminotomies. If the employee is willing to pay more for the second contract, this disparity might reflect her unawareness of laminotomies when she was presented the first contract. Her unawareness while evaluating the first contract reveals itself when she is willing to pay more for the second contract, perhaps reflecting an updated and increased personal belief of the likelihood of surgery. The broader conceptual point is that the presentation and expression of an act immediately confers some information to the reader about the structure of the state space. Specifically, the decision maker must at least understand the coarsest partition required to express the structure of the contact. Moreover, increasingly refined expressions of the contract must confer correspondingly more awareness. To our knowledge, our model is the first axiomatic attempt to connect the measurability of an act and the decision maker's awareness. It attempts to provide a unified treatment of awareness and framing.

The following example perhaps illustrates the relationship between awareness and framing more sharply. The mathematician Jean d'Alembert argued that "the probability of observing at least one head in two tosses of a fair coin is 2/3 rather than 3/4. Heads, as he said, might appear on the first toss, or, failing that, it might appear on the second, or, finally, might not appear on either. D'Alembert considered the three possibilities equally likely (Savage 1954, p. 65)." D'Alembert's fundamental mistake was in his framing of the states; he failed to split the first event into its two atoms: heads then tails, and heads then heads. His view of the world was as three events: $\{HH, HT\}$, $\{TH\}$, and $\{TT\}$. Had he been aware of these contingencies and framed the possible tosses appropriately, he may have avoided the error.

Such framing effects are precluded in the standard models of decision making under uncertainty introduced by Savage (1954) and by Anscombe and Aumann (1963). These models do not distinguish between different presentations of the same act, implicitly assuming that the framing of the state space is inconsequential. We introduce a richer set of primitives which treats the different frames for an act as distinct choice objects.

In particular, our model treats lists of contingencies and outcomes as the primitive objects of

Office visit	\$20
Hospital visit	no charge
Preventive physical exam	\$20
Maternity outpatient care	\$20
Maternity inpatient care	\$250

Table 1: Blue Cross health insurance plan

choice. The following list

$$\begin{pmatrix} x_1 & E_1 \\ x_2 & E_2 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix},$$

denotes an act which delivers the outcome x_i if the state of the world is in E_i. In paper, acts are often denoted by such lists for ease of exposition. We take these expressions literally. For example, if $E_1' \cup E_1'' = E_1$, then the following list

$$\begin{pmatrix} x_1 & E_1' \\ x_1 & E_1'' \\ x_2 & E_2 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix},$$

denotes the same act, but is modeled as a distinct object. The decision maker might have different attitudes about the two presentations, because the second has made her aware of the more specific contingencies E_1' and E_2''. This discrimination between lists is the primary methodological innovation of the paper.

This also provides a natural framework to understanding how the decision maker updates her decision making as she becomes aware of possibilities she did not previously understand. For example, what would d'Alembert have done if he had realized that HH is distinct from HT? Can we make any predictive restrictions on his behavior after this realization from his preferences before the realization? One can view the different lists as reflective of more or less awareness. The response to new awareness is central to our approach. In fact, given that the decision maker uses expected utility to aggregate uncertainty at a fixed enumeration of states, the comparison across enumerations is the only way to identify unawareness from preference.

Aside from theoretical concerns, many real contracts are presented as such lists. Insurance plans are often described by a table of contingencies and coverage amounts. Table 1 is a partial verbatim copy of a Blue Cross medical plan available to University of California employees expressed as procedures and deductibles.

We study a decision maker who acts as if she places a weight $\nu(E)$ on each event E. When presented the enumeration $E_1, \ldots, E_n$, she judges the probability of E_i to be $\nu(E_i)/\sum_j \nu(E_j)$. Since the weighting function ν is not necessarily additive, her probability of E_1 can depend on whether it is expressed as E_1 or expressed as $E_1' \cup E_1''$. Her utility for a list is her expected utility aggregating the cardinal utility of each consequence by the weight on its accompanying event. Then the nonadditivity of ν can be used to measure and compare the awareness of E_1' when she is only told of E_1. So, although the primitives are richer, our proposed utility representation maintains the essential notions of expected utility and subjective probability from the standard model.

The articulation of awareness in decision theoretic terms of betting and likelihood provides several benefits. First, it provides a language to discuss partial awareness of unawareness. While having nothing directly to say about interactive epistemology, we hope our work complements a sizable literature studying semantic models which poses awareness and unawareness as epistemic operators on events.[2] Often, it either begins with or arrives at the position that the decision maker should have no awareness of her unawareness. For example, Dekel, Lipman, and Rustichini (1998, p. 161) argue that "an agent who is unaware of a possibility should have *no* positive knowledge of it at all." This claim was cast in the context of possibility correspondences in epistemic models, where the dichotomous nature of the awareness operator hinders the expression of partial awareness in the required terms. Either a decision maker is aware of an event or she is not. Then she is either aware of her unawareness or she is not. In this sense, the severity is at least partially an artifact of modeling choices.

While concurring that one should never be completely aware of her unawareness, our intuition departs from the severe conclusion that she should have no awareness at all. For example, a consumer may be partially aware that she has a less than complete medical understanding of her health or an investor may be partially aware that he cannot consider the universe of all mutual funds. What they cannot do is express the diseases or the mutual funds they don't know exactly. By expressing awareness in terms of betting and likelihood, rather than knowledge, we hope to introduce some notion of partial awareness, hence also a notion of partial awareness of unawareness. In our model, partial awareness is not a part of the description of the state space, but identified through comparisons of belief between descriptions. For example, we compare the consumer's willingness to pay for insurance contracts with varying levels of descriptive detail.

Some recent work expands the basic semantic structure to include different projections or partitions of state spaces to reflect different levels of awareness (Heifetz, Meier, and Schipper forthcoming, Li 2006). But, at a fixed state space, the austere epistemic model still provides no channel for the decision maker to express a partial conception of the more refined semantics to which he does not have immediate access. These different state spaces might also be viewed as different frames of a single space. Under this interpretation, one contribution of this paper is to introduce an axiomatic notion of choice into these enriched structures.

However, this interpretation is strained by the following major difference between these papers and ours. Heifetz, Meier, and Schipper (forthcoming) and Li (2006) treat each state space as the

[2]Feinberg (2004) and Heifetz, Meier, and Schipper (2005) provide syntactic analyses of interactive unawareness.

fixed awareness of the decision maker; she does not have any more awareness. On the other hand, we treat each frame as the *minimal* awareness that the analyst can attribute to the decision maker, based on the expression of the acts. We then let the choice behavior provide some insight on her awareness beyond this minimal level. Because the representation involves likelihoods, it provides another channel to express partial awareness. Specifically, as alluded to in the beginning of this introduction, the disparity in subjective beliefs between different frames gauges the decision maker's partial awareness. For example, if there is no change in these beliefs, then the decision maker acts as if she is completely of the more refined state space.

We also hope this model complements the existing decision theoretic work on unforeseen contingencies. Kreps (1979) introduced an axiomatic model of preference for flexibility by considering menus of objects as the primitives of choice. Demand for flexibility is interpreted as a response to unforeseen contingencies, which are captured in the proposed representation as subjective taste uncertainties. Dekel, Lipman, and Rustichini (2001), henceforth DLR, extend this approach to menus of lotteries, where the linear structure essentially identifies the subjective state space.[3] Then the analyst can remarkably determine the space of uncertainty as a theoretical artifact of preference, rather than assume a state space a priori.

The DLR methodology provides a powerfully unified treatment of states, beliefs, and utilities. On the other hand, because it depends on the decision maker's preferences to elicit the states, recovering unawareness is difficult, encountering the basic conundrum that the decision maker cannot reveal something of which she is totally unaware. In fact, in DLR's main representations, the decision maker acts as if she has complete awareness of some state space.

Epstein and Marinacci (2005) propose a generalization of DLR to address this conundrum. A form of maxmin expected utility, similar to that used by Gilboa and Schmeidler (1989) to model ambiguity, over the likelihood of the subjective states is suggested as a response to the decision maker's partial awareness that his understanding of the world is coarse. This resonates with work by Ghirardato (2001), Mukerji (1996), and Nehring (1999) who capture the decision maker's partial awareness of unforeseen contingencies through Choquet integration of belief functions or capacities, which is also used to model ambiguity by Schmeidler (1989). One point of this paper is to demonstrate another method of detecting unawareness without invoking ambiguity aversion. In fact, our representation satisfies the standard expected utility axioms for each enumeration of states.

One way to distinguish DLR's approach and ours is that DLR relaxes Savage's assumption that the analyst has a complete understanding of the state space and studies a model where the states of the world are revealed through preference. On the other hand, then unawareness can only be elicited through violations of expected utility. We equip the model, hence the analyst, with the comprehensive view. This means that it is not a part of the representation in the model.

This assumption is a strong one, but we hope that it is justified by its conceptual dividends. It also seems more palatable when imagining applications. For example, one party often has more

[3]Dekel, Lipman, Rustichini, and Sarver (2005) report a technical corrigendum to the original paper.

awareness than another. Then the completely specified state space only has to be the more specified space in situations of asymmetric awareness, where perhaps one party has to decide how much awareness information it wishes to reveal by presenting a contract. It also seems difficult to verify and enforce contracts which depend on subjective states, which are purely theoretical constructions. Insofar as unawareness and unforeseen contingencies bear on contracting, assuming some sort of objective state space seems less heroic. One of the motivations of the model is to accommodate unawareness in a Savage setting without invoking menus or multi-valued consequences, so the primitives of the model bear as close a resemblance as possible to the way that actual contracts, like insurance policies or warranties, are presented.

Finally, we believe that the framing interpretation of the model is of interest, independently of its application to awareness. While economists appreciate the framing effects of consequences, especially as formalized by prospect theory (Kahneman and Tversky 1979), the framing effects of states seems to be relatively obscure. This is despite a large psychological literature which questions extensionality, the psychological term for the invariance of the judged probability of an event to its particular expression. For example, Fischoff, Slovic, and Lichtenstein (1978) found that car mechanics assign higher value to the conditional probability that a car fails to start because of something other than the battery, fuel system, or engine when this complement is expressed as a union of more specific causes. It is difficult to attribute this distortion to unawareness on the part of the mechanics. Tversky and Koehler (1994) propose a theory of judgement, which they coin support theory, with many similarities to the theory of decision forwarded here. One contribution of the paper is to provide an axiomatic foundation for a generalized version of support theory. While the connections are discussed throughout the sequel, we should note now that many of the behavioral intuitions of the model should be credited to this psychological literature in general and to Tversky and Koehler (1994) in particular. We hope this paper bring more attention by economists to violations of extensionality, which we think is an important psychological factor that might have very large economic consequences.

In the next section, we introduce the primitives of our theory. We then propose a utility representation for the model and provide an axiomatic characterization. Finally, we suggest methods for detecting correction for unawareness and comparing this correction across individuals.

2 A model of decision making with unawareness and framing

We introduce our formal model. Let S denote an arbitrary state space. Let X denote the finite set of consequences, and ΔX denote the lotteries on consequences. Let Π denote the collection of all finite partitions of S.[4] For any $\pi \in \Pi$, let $\sigma(\pi)$ denote the algebra induced by π. Let $\mathcal{F}$ denote the simple Anscombe–Aumann acts, $\{f \in (\Delta X)^S : |f(S)| < \infty\}$. We slightly abuse notation and let $p \in \Delta X$ denote the obvious constant act. Let $\mathcal{F}_\pi = \{f \in \mathcal{F} : f \text{ is } \pi\text{-measurable}\}$ denote the acts which respect the partition π. We consider a *family* of preferences $\{\succsim_\pi\}_{\pi \in \Pi}$ indexed by π, where

[4]The restriction to finite partitions is mainly for technical ease.

each preference $\succsim_\pi \subset \mathcal{F}_\pi \times \mathcal{F}_\pi$ is on π-measurable acts. The strict and symmetric components $\succ_\pi$ and $\sim_\pi$ carry their standard meanings. Given a partition $\pi = \{E_1, \ldots, E_n\}$ and acts $f_1, \ldots, f_n \in \mathcal{F}$ define a new act by:

$$\begin{pmatrix} f_1 & E_1 \\ \vdots & \vdots \\ f_n & E_n \end{pmatrix}(s) = \begin{cases} f_1(s) & \text{if } s \in E_1 \\ \vdots & \vdots \\ f_n(s) & \text{if } s \in E_n \end{cases}.$$

The following is a generalization of the concept of null events for our setting.

Definition 1. Given $\pi \in \Pi$, an event $E \in \sigma(\pi)$ is **π-null** if

$$\begin{pmatrix} p & E \\ f & E^{\complement} \end{pmatrix} \sim_\pi \begin{pmatrix} q & E \\ f & E^{\complement} \end{pmatrix},$$

for all $f \in \mathcal{F}_\pi$ and $p, q \in \Delta X$. $E \in \sigma(\pi)$ is **π-nonnull** if it is not π-null. The event E is **null** if $E = \emptyset$ or if E is π-null for any π such that $E \in \pi$. E is **nonnull** if it is not null.

This family of preferences might not immediately appear to be related to our original motivation of studying lists. In fact, this model provides a parsimonious primitive which is isomorphic to a model which begins with preferences over lists. Suppose we started with a list

$$\begin{pmatrix} x_1 & E_1 \\ \vdots & \vdots \\ x_n & E_n \end{pmatrix}$$

which is a presentation of the act f. This could be more compactly represented by a pair (f, π), where $\pi = \{E_1, \ldots E_n\}$ denotes the enumeration of contingencies on the right hand side of the list. This enumeration π must be at least rich enough to describe the act f, so we can assume $f \in \mathcal{F}_\pi$. Now suppose the decision maker is deciding between two lists, which are represented as (f, π_1) and (g, π_2). Since she has read both enumerations, she is now aware of both π_1 and π_2; alternatively, she must be aware of both π_1 and π_2 to compare the lists. Then her minimal level of awareness is the coarsest common refinement of π_1 and π_2, their join $\pi = \pi_1 \vee \pi_2$. Then (f, π_1) is preferred to (g, π_2) if and only if (f, π) is preferred to (g, π). So, we can restrict attention to the preference restricted to pairs (f, π) and (g, π) where $f, g \in \mathcal{F}_\pi$. Moving the partition from being carried by the acts to being carried by the preference lightens the notation and results exactly in the model being studied here. We stress that the model is really of a decision maker deciding between lists. This notation is economical and aids in the understanding of the axioms.

For example, suppose the decision maker is deciding between the Blue Cross health plan described on Table 1 and the healthy plan available from Kaiser Permanente and depicted in Table 2. The partitioning of the Kaiser Permanente plan differs from the partitioning of the Blue Cross plan. A newly hired and naive assistant professor, in the process of comparing health insurance options, becomes aware of the possibility of maternity outpatient care, from the Blue Cross plan, and of

Primary and specialty care visits	\$50
Well-child visits to age two	\$15
Family planning visits	\$50
Scheduled prenatal care and first postmartum visit	\$50
Maternity inpatient care	\$250

Table 2: Kaiser Permanente health insurance plan

family planning visits, from the Kaiser Permanente plan. This level of awareness is a consequence of simply reading the lists and merely reflects her exposure to both policies.

By assuming $f, g \in \mathcal{F}_\pi$, we assure that π is at least as fine as $\pi(f, g)$. This interpretation highlights an important interpretive difference between standard theories of Bayesian updating and our theory of awareness. In the former, functions must be restricted to respect the information, or lack thereof, embodied in an algebra on the state space. In our theory, it is the algebra that must be expanded to reflect the awareness implicit in the description of an act.

Partitions or subalgebras are often used to model the arrival of information about the actual state of the world, where each cell of a partition represents an updated restriction on the truth. Our interpretation is quite different. We take each partition π as a frame of awareness or a description of the entire state space. Each cell represents an event that a decision maker understands and of which she is aware. For example, she may be aware that her car may break down, yet be unaware that one of the ways it might break down is a sudden disintegration of the tires. In our model, she does *not* learn at some ex interim stage which particular cell actually obtains, i.e. whether her car will actually break down in the future because of tire disintegration or for some other reason.

One feature of our model is that the decision maker's minimal awareness is formalized with respect to partitions. We feel that focusing on the awareness an entire partition of the state space is superior to discussing the awareness of particular states or events. A similar view is articulated in semantic models with lattices or partitions of state spaces by Heifetz, Meier, and Schipper (forthcoming) and Li (2006) and in the interpretation of DLR's subjective state space by Epstein and Marinacci (2005), who suggest the term "coarse contingencies" is more evocative than "unforeseen contingencies." For example, an investor may have been unaware of the possibility of domestic terrorism before September 11, 2001. Afterwards, she updates her awareness. The investor does not become aware of a new state of the world, because a terrorist action does not constitute a full description of relevant uncertainty. She is still concerned with the prime interest rate, the price of oil, and all the other variables that priced her investments before she was aware of terrorism. Rather than becoming aware of a single state, she becomes aware that each cell she had previously considered a full description of the relevant uncertainty had actually been incomplete.

Graphically, suppose the investor's view of the world before September 11 was:

s_1	s_2	s_3

If we model her awareness of terrorism as a new state t, then her new view would look like:

s_1	s_2	s_3	t

If we model her awareness as a new *partition* which filtrates terrorism (t) and no terrorism (n), her new view has six cells and would look like:

s_1, t	s_2, t	s_3, t
s_1, n	s_2, n	s_3, n

We believe the latter view is much more aligned with the general process of updating awareness.

A model which considers only awareness of states also has difficulties accommodating a decision maker who is aware of an event, but unaware of its components. For example, it seems reasonable that a consumer might be aware of the fact that her car might break down, and even have a well-formed probability of this event, yet have only a vague idea of the different components in her car that might fail. In the standard epistemic expressions of unawareness, awareness of an event implies awareness of its subevents. By using partitions, our models separates awareness from set inclusion, since a partition can obviously be fine enough to include an event but be too coarse to include its subevents.

3 Partition-dependent expected utility

We propose the following utility representation for every $\succsim_\pi$. The decision maker has a nonnegative set function $\nu : 2^S \to \mathbb{R}_+$ over all events. When she is presented with an enumerated description $\pi = \{E_1, E_2, \ldots, E_n\}$ of the state space, she places a weight $\nu(E_k)$ on each event. Normalizing these weights by their sum, $\mu_\pi(E_k) = \nu(E)/\sum_i \nu(E_i)$ defines a probability measure μ_π over $\sigma(\pi)$, the algebra induced by π. Then, her utility for the act

$$f = \begin{pmatrix} p_1 & E_1 \\ p_2 & E_2 \\ \vdots & \vdots \\ p_n & E_n \end{pmatrix},$$

is simply $\sum_{i=1}^n u(p_i)\mu_\pi(E_i)$, where $u : \Delta X \to \mathbb{R}$ is an affine von Neumann–Morgenstern utility function on objective lotteries over consequences.

The following restriction is needed to avoid dividing by zero when normalizing the set function.

Definition 2. A set function $\nu : 2^S \to \mathbb{R}$ is *nondegenerate* if $\sum_{E\in\pi} \nu(E) > 0$ for all $\pi \in \Pi$.

We can now formally define our desired representation.

Definition 3. $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a **partition-dependent expected utility** representation if there exist a nondegenerate and positive set function $\nu : 2^S \to \mathbb{R}_+$ and a nonconstant affine vNM utility function $u : \Delta X \to \mathbb{R}$ such that for all $\pi \in \Pi$ and $f, g \in \mathcal{F}_\pi$:

$$f \succsim_\pi g \iff \int_S u \circ f \, d\mu_\pi \geq \int_S u \circ g \, d\mu_\pi,$$

where μ_π is the unique probability measure on $(S, \sigma(\pi))$ such that, for all $E \in \pi$:

$$\mu_\pi(E) = \frac{\nu(E)}{\sum_{F \in \pi} \nu(F)}. \tag{1}$$

When such a pair (u, ν) exists, we will call it a partition-dependent expected utility representation.

The set function ν in this definition is not necessarily additive, and its nonadditivity provides a channel for detecting how aware the decision maker is of more specified states. While the representation involves nonadditive set functions, it is only very superficially similar to Choquet expected utility (Schmeidler 1989). In fact, the decision maker acts as if she maximizes an affine expected utility function at each partition π. The structure of ν is quite general: ν does not have to be monotone nor convex-ranged. It can also be strictly bounded away from zero for nonempty events, in which case there are no null events, even if the state space is uncountably rich.

In the special case that the set function ν is additive, the probabilities of events do not depend on their expressions. Then the decision maker is indistinguishable from someone who has full awareness of the state space.

Definition 4. $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a **partition-independent expected utility** representation if there exist a finitely additive probability measure $\mu : 2^S \to \mathbb{R}_+$ and a nonconstant affine vNM utility function $u : \Delta X \to \mathbb{R}$ such that for all $\pi \in \Pi$ and $f, g \in \mathcal{F}_\pi$:

$$f \succsim_\pi g \iff \int_S u \circ f \, d\mu \geq \int_S u \circ g \, d\mu.$$

Our representation provides the following guidelines for the decision maker's response to unawareness. Each event $E \subsetneq S$ carries a value $\nu(E)$, which corresponds to its relative weight in frames where the decision maker must be aware of E but not necessarily of its subevents. The nonadditivity of ν captures the effects of framing or unawareness: A and B can be disjoint yet $\nu(A) + \nu(B) \neq \nu(A \cup B)$. If π_E is a partition of E, then the difference $\sum_{F \in \pi_E} \nu(F) - \nu(E)$ captures the unawareness of π_E relative to E. Of course, if ν is additive, then the decision maker acts as if she is totally aware of the state space and her behavior corresponds to standard Bayesian updating. Moreover, she may have complete awareness over part of the state space, i.e. if ν is additive over all the subevents of E, without awareness over the entire state space, i.e. if ν is nonadditive over subevents of E^{C}. In the representation, unawareness departs from Bayesian decision theory only in the *response* to new awareness or information, but not in the choices within a fixed mode of awareness, since each $\succsim_\pi$ conforms to expected utility. From the analyst's perspective, the detection or elicitation of unawareness therefore hinges on this response, on the dynamics between partitions. If she accepts our axioms, the analyst can predict the decision maker's behavior in frame π from her behavior in other frames.

Notice that the inequality $\sum_{F \in \pi_E} \nu(F) < \nu(E)$ is not precluded. This is because ν does not capture unawareness in isolation, but also reflects the decision maker's correction for her unawareness. She may be partially aware that her conception of the event E is incomplete, and try to incorporate what "she believes she does not know" into her odds. If she overcompensates for her unawareness, the resulting $\nu(E)$ might be larger than the sum of its components. For example, a car owner might understand that there are myriad ways for her car to break down, but can name only a few. One plausible response might be to purchase more insurance than would be optimal had she possessed a full mechanical understanding of her car.

Tversky and Koehler (1994) introduced a related nonextensional theory of judgement called support theory. Its primitives are different descriptions of events, called hypotheses. It analyzes binary comparisons of likelihood between two hypotheses, which they call evaluation frames, which consist of a focal hypothesis and an alternative hypothesis. The probability judgment of the focal hypothesis A relative to the alternative B in the evaluation frame (A, B) is proposed to be $P(A, B) = s(A)/[s(A) + s(B)]$, where $s(A)$ is a assignment of support for each hypothesis which is based on the strength of its evidence. They offer a characterization of such judgments based on functional equations, but this characterization is primarily technical and not rooted in decision making (Tversky and Koehler 1994, Theorem 1). Our theory translates support theory from judgment to decision making, extends its scope beyond binary evaluation frames, and provides an axiomatic foundation from preference. The motivation is also quite different, since Tversky and Koehler attribute violations of extensionality to heuristic devices like availability, where the decision maker judges probabilities by her ability to recall typical cases, rather than to unawareness. They also present extensive experimental evidence illustrating the sensitivity of judgments of probability to the description and framing of the possibilities.

The following special cases provide some particular intuition for partition-dependent expected utility.

Example 1 (Probability weighting)**.** Suppose $\mu : 2^S \to \mathbb{R}$ is a finitely additive probability measure and $w : [0, 1] \to \mathbb{R}_+$ is a weakly increasing transformation. Now suppose $\nu(E) = w(\mu(E))$. Transformations like w are sometimes called probability weighting functions and featured in the literature on non-expected utility over lotteries, for example in prospect theory (Kahneman and Tversky 1979) and anticapted utility theory (Quiggin 1982), where they are applied in a different manner. The application to our model is closest to the subjectively weighted utility theory of Karmarkar (1978). Quiggin (1982) points out, because it is independent of the consequences tied to the lottery, that the weighting function w in subjectively weighted utility must be linear if the preference satisfies stochastic dominance. Here, because the manner in which w is applied depends on the framing of the act, we avoid this trivial reduction.

This example illuminates a framing dependence of objective theories which depend on weighting functions. Instead of working with the space of probability distributions or lotteries, suppose the objects of choice were lists of outcomes and odds, analogous to the lists of outcomes and events considered here. If a list included redundancies, for example if the probability p of x was broken into $p_1 + p_2 = p$, then a nonlinear weighting function would aggregate the redundant expression differently from the minimal expression.

Example 2 (Principle of insufficient reason)**.** Suppose ν is a constant function, for example $\nu(E) = 1$ for each every nonempty E. Then the decision maker puts equal probability on all the events of which she is cognizant. Such a criterion for cases of extreme ignorance or unawareness was advocated by Laplace as the principle of insufficient reason. This principle is sensitive to the framing of the states. Consider the error of d'Alemebert mentioned in the introduction, who attributed a probability of 2/3 to seeing at least one head among two tosses of a fair coin. The fundamental error was his framing of the state space as H-$, TH, TT$, or partitioned as $\{\{HT, HH\}; \{TH\}; \{TT\}\}$. If ν is constant, d'Alembert would have realized his error had he been presented a bet which pays only on a head followed by a tail, HT. On the other hand, he would have made a similar error had he reasoned that there can be either 0, 1, or 2 heads, partitioning the states into $\{\{TT\}; \{HT, TH\}; \{HH\}\}$. The principle of insufficient reason is often derided for its sensitivity to the framing of events and states. This criticism is difficult to even formalize in a standard decision model; ours is specifically designed to capture such framing effects.

A more tempered resolution of unawareness is a convex combination of a probability measure and the ignorance prior: $\nu(E) = \alpha(\mu) + (1 - \alpha)$. Fox and Rottenstreich (2003) report experimental evidence which suggests that judgement is partially biased towards the ignorance prior.

4 Axioms and characterizations

We now provide axiomatic characterizations of both partition-dependent and partition-independent expected utility. We also discuss the somewhat subtle uniqueness of ν, which requires an additional condition.

4.1 Representations

The first five axioms on preference essentially apply the standard Anscombe–Aumann axioms to each $\succsim_\pi$. We will refer to Axioms 1 to 5 collectively as the Anscombe–Aumann axioms.

Axiom 1 (Preference)**.** $\succsim_\pi$ is complete and transitive for all $\pi \in \Pi$.

Axiom 2 (Independence)**.** For all $f, g, h \in \mathcal{F}_\pi$ and $\alpha \in (0, 1)$: if $f \succ_\pi g$, then $\alpha f + (1 - \alpha)h \succ_\pi \alpha g + (1 - \alpha)h$.

Axiom 3 (Archimedean Continuity)**.** For all $f, g, h \in \mathcal{F}_\pi$: if $f \succ_\pi g \succ_\pi h$, then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1 - \alpha)h \succ_\pi g \succ_\pi \beta f + (1 - \beta)h$.

Axiom 4 (Nondegeneracy)**.** For all $\pi \in \Pi$, there exist $f, g \in \mathcal{F}_\pi$ such that $f \succ_\pi g$.

Axiom 5 (State Independence)**.** For all $\pi \in \Pi$, π-nonnull $E \in \sigma(\pi)$, $p, q \in \Delta X$, and $f \in \mathcal{F}_\pi$:

$$p \succsim_{\{S\}} q \Longleftrightarrow \begin{pmatrix} p & E \\ f & E^{\complement} \end{pmatrix} \succsim_\pi \begin{pmatrix} q & E \\ f & E^{\complement} \end{pmatrix}.$$

State Independence has some additional content in our model. Not only is cardinal utility of a consequence invariant to the state in which it obtains, but it is also invariant to the decision maker's minimal level of awareness.

These familiar axioms guarantee an Anscombe–Aumann expected utility representation for each $\succsim_\pi$: there exist a probability measure $\mu_\pi : \sigma(\pi) \to [0, 1]$ and an affine function $u : \Delta X \to \mathbb{R}$ such that $U_\pi(f) = \int_S u \circ f \, d\mu_\pi$ represents $\succsim_\pi$. Hence, given a fixed partition π, the decision maker's preferences $\succsim_\pi$ are completely standard: she is probabilistically sophisticated on $\sigma(\pi)$ and evaluates lotteries linearly. Probabilistically sophisticated expected utility for a fixed level of awareness is not at odds with our model. The model's interest derives from the relationship between preferences across partitions, i.e. in how the decision maker responds to updated awareness. The following axioms consider this relationship.

To consider an act f, the decision maker must be aware of the events which are necessary for its description, namely those in $\sigma(\pi)$ where π is the coarsest partition such that $f \in \mathcal{F}_\pi$. If she was ignorant of π, reading any description of f would immediately refine her understanding of the states. Similarly, when comparing two acts f and g, she must have the minimal awareness required to describe both f and g. This motivates the following binary relation $\succsim$ on $\mathcal{F}$.

Definition 5. For all $f, g \in \mathcal{F}$ define $f \succsim g$ if $f \succsim_{\pi(f,g)} g$, where $\pi(f, g)$ is the coarsest partition such that $f, g \in \mathcal{F}_\pi$.

In words, $\succsim$ reflects the decision maker's preference when presented with the coarsest possible descriptions of the two acts. The remaining axioms restrict the relation $\succsim$. To see why it is so theoretically informative, suppose the analyst wanted to understand the decision maker's response to an act f which is expressed more finely than $\pi(f)$. Then the description must entail some redundancies, for example, $f^{-1}(p) = E_1 \cup E_2$, but the enumeration separately lists E_1 and E_2 even though they return the same lottery. But, there is a very similar act f' whose minimal expression does require separate expressions for E_1 and E_2: an act which assigns a very close but different lottery p' to E_2. Given the Anscombe–Aumann axioms, the decision maker's utility for the original act f under π is very similar to her utility for the nearby f' under $\pi(f')$.

The defined relation $\succsim$ is generally intransitive, since the frames $\pi(f, g)$, $\pi(g, h)$, and $\pi(f, h)$ required for pairwise comparisons of f, g, and h are generally distinct. One relaxation of transitivity is acyclicity. A preference relation $\succsim$ is acyclic if its strict component $\succ$ does not admit any cycles. Given that $\succsim$ is complete, it is equivalent to the following definition.

Axiom 6* (Acyclicity)**.** For all acts $f_1, \ldots, f_n \in \mathcal{F}$,

$$f_1 \succ f_2, \ldots, f_{n-1} \succ f_n \Longrightarrow f_1 \succsim f_n$$

It is well known that acyclicity of the preference relation is necessary and sufficient for its induced choice rule to be nonempty for any finite choice set. Since the presentation of an entire choice set of many acts will dramatically improve the decision maker's awareness, the interpretation in our setting is less direct. Here, the nonemptiness of the choice rule means that, for any finite set $A \subseteq \mathcal{F}$, we can assign some status quo act $f \in A$ such that if the decision maker is presented the minimal expression of any alternative $g \in A$, she will weakly prefer to keep f, $f \succsim g$. While it might appear innocuous, this assumption is quite strong. The following result shows that assuming Acyclicity precludes any meaningful notion of unawareness, because in the resulting representation, decision making is independent of the minimal level of awareness.

Theorem 1. $\{\succsim_\pi\}_{\pi\in\Pi}$ *admits a partition-independent expected utility representation if and only if it satisfies the Ancsombe–Aumann axioms and Acyclicity.*[5]

Proof. See Appendix A.2.

So, to allow for partition-dependent expected utility, Acyclicity must be further generalized. In particular, some cycles must be admitted. But, we can specify exactly which cycles are still disallowed.

Definition 6. A sequence of events $E_1, E_2, \ldots$ is **sequentially disjoint** if $E_i \cap E_{i+1} = \emptyset$ for all i.

In particular, cycles can only be admitted on simple binary bets across sequentially disjoint events.

Axiom 6 (Binary Bet Acyclicity)**.** For all sequentially disjoint cycle of sets $E_1, \ldots, E_n, E_1$ and lotteries $p_1, \ldots, p_n; q \in \Delta X$,

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_2 & E_2 \\ q & E_2^{\complement} \end{pmatrix}, \ldots, \begin{pmatrix} p_{n-1} & E_{n-1} \\ q & E_n^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_n & E_n \\ q & E_{n-1}^{\complement} \end{pmatrix} \Longrightarrow \begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Binary Bet Acylicity forces the decision maker to consistently evaluate simple likelihoods for disjoint events. Since the compared events are disjoint, there is no issue of relative awareness once these binary bets are presented to the decision maker. It is analogous to Savage's Postulate 4, sometimes called Weak Comparative Probability.[6] Unable to marshal Savage's entire battery of assumptions, we must modify Weak Comparative Probability. First, Axiom 6 admits richer comparisons across consequences. This compensates for dropping Postulate 6 (Small Event Continuity), since S may be finite and a likelihood relation may not identify a quantitative probability. Second, it admits chains of comparisons to compensate for lack of transitivity.

[5]Theorem 1 remains true if Acyclicity is replaced with transitivity of $\succsim$.

[6]Postulate 4, Weak Comparative Probability: for all $A, B \subset S$ and $x, x', y, y' \in X$ such that $x \succ y$ and $x' \succ y'$,

$$\begin{pmatrix} x & A \\ y & A^{\complement} \end{pmatrix} \succsim \begin{pmatrix} x & B \\ y & B^{\complement} \end{pmatrix} \Rightarrow \begin{pmatrix} x' & A \\ y' & A^{\complement} \end{pmatrix} \succsim \begin{pmatrix} x' & B \\ y' & B^{\complement} \end{pmatrix}.$$

Another justification of acyclicity is that it prevents the construction of Dutch book schemes which are strictly profitable at each trade. This justification is more tenuous in in our interpretation of the model as one of awareness, because the preference relations that comprise the sequence in the hypothesis of Binary Bet Acyclicity are all indexed by distinct minimal partitions. If we present the decision maker with the first, the second, and then the third binary bets, she has become aware of more events than had she been presented the second and third bets in isolation. On the other hand, the preference notated in the axiom is really the latter. Perhaps an alternative and similar justification is that if there were n people who exhibit this behavior, we could construct a Dutch book between them by offering them separate choice. This could in principle be tested in a laboratory across subjects.

Once Acyclicity is relaxed to Binary Bet Acyclicity, the classic Sure-Thing Principle of Savage (1954) must be imposed to maintain a form of consistency.

Axiom 7 (Sure-Thing Principle)**.** For all events $E \subset S$ and acts $f, g, h, h; \in \mathcal{F}$,

$$\begin{pmatrix} f & E \\ h & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} g & E \\ h & E^{\complement} \end{pmatrix} \Longrightarrow \begin{pmatrix} f & E \\ h' & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} g & E \\ h' & E^{\complement} \end{pmatrix}$$

The standard justification for the Sure-Thing Principle is in establishing coherent conditional preferences. The evaluation of conditional probabilities for subevents on E should be independent on what happens on $E^{\complement}$. Here, because the expression of acts on $E^{\complement}$ also confers some awareness, this axioms has additional content. As discussed, comparing two acts requires awareness of certain events. When the range of h is disjoint from the ranges of f and g, the awareness needed to make the comparison in the hypothesis can be divided into two parts: conditional awareness of subevents of E generated by f and g, and conditional awareness of subevents of $E^{\complement}$ generated by h. The awareness needed to make the comparison in the conclusion can be similarly divided: conditional awareness of the *same* subevents of E generated by f and g, and awareness of possibly different subevents of $E^{\complement}$ generated by h'. Since the conditional awareness required on E is similar for both comparisons and the acts being compared agree on $E^{\complement}$, the Sure-Thing Principle requires that the preferences are determined by where the acts differ on E.

We can now present the main representation result of the paper:

Theorem 2. $\{\succsim_\pi\}_{\pi \in \Pi}$ *admits a partition-dependent expected utility representation if and only if it satisfies the Anscombe–Aumann axioms, Binary Bet Acyclicity, and the Sure-Thing Principle.*

Proof. See Appendix A.1

4.2 Uniqueness

While the utility function u over lotteries is unique up to positive affine transformations, the uniqueness of ν in the representation is surprisingly delicate. This delicacy also provides some intuition for Theorem 2. Our general strategy for identifying ν is to use an appropriate chain of partitions and betting preferences to calibrate the likelihood ratio $\nu(E)/\nu(F)$. For example, suppose $S = \{a, b, c\}$ and consider the ratio $\nu(\{a, b\})/\nu(\{a\})$. First, examine preferences indexed by the partition $\pi_1 = \{\{a, b\}; \{c\}\}$ to identify the likelihood ratio of $\{a, b\}$ to $\{c\}$. Next, the preferences indexed by $\pi_2 = \{\{a\}; \{b\}; \{c\}\}$ reveal the ratio of $\{c\}$ to $\{a\}$. The Sure-Thing Principle and Binary Bet Acyclicity suggest the following argument: the ratio of $\{a, b\}$ to $\{a\}$ is equal to the ratio of $\{a, b\}$ to $\{c\}$ times the ratio of $\{c\}$ to $\{a\}$, i.e. "the $\{c\}$'s cancel" and the revealed likelihood ratios multiply out. However, if $\{c\}$ is π_1-null, these ratios are undefined. Instead of achieving uniqueness, the state space segregates into equivalence classes of events which reach each other through sequentially disjoint chains of nonnull comparisons. Without further restrictions, ν is unique only up to scale transformations for all such equivalence classes. If all events are nonnull for all partitions, there is one such equivalence class and ν is identified up to constant multiplication. This motivates the following definition.

Axiom 8 (Event Reachability)**.** For any distinct nonnull events E and F different from S, there exists a sequentially disjoint sequence of nonnull events $E_1, \ldots, E_n$ such that $E = E_1$, $F = E_n$.

Event Reachability is immediately satisfied if there are no nonempty null states. The notion of Strict Admissibility is sometimes invoked as a normative condition. It is a strong form of monotonicity or dominance.

Axiom 8* (Strict Admissibility)**.** If $f(s) \succsim g(s)$ for all $s \in S$ and $f(s') \succ g(s')$ for some $s \in S$, then $f \succ g$.

Strict Admissibility readily implies Event Reachability. Also, unlike in the standard Savage model, Strict Admissibility is not a vacuous assumption, even if the state space is very rich. For example, if $\nu(E) > \alpha$ for some $\alpha > 0$, then there will be no null states and Strict Admissibility is satisfied. This bound suggest a decision maker who always put some nontrivial probability on any explicitly mentioned contingency.

On the other hand, as shown in the next example, Event Reachability is strictly weaker and is insufficient to guarantee Strict Admissibility.

Example 3 (Event Reachability $\not\Rightarrow$ Strict Admissibility)**.** Let $S = \{s_1, s_2, s_3\}$ and suppose that $\{\succsim_\pi\}_{\pi \in \Pi}$ has a representation as in Theorem 1, where only the events $\{s_1\}$, $\{s_2\}$, $\{s_3\}$, and $\{s_1, s_2\}$ have strictly positive ν-weight. The specified ν is non-degenerate. Strict Admissibility fails since some non-empty events are null. Event Reachability is satisfied: there is a direct "disjoint path" in between disjoint nonnull events, and $\{s_1, s_2\}$ is linked to the events $\{s_1\}$ and $\{s_2\}$, through $\{s_3\}$.

Of course, the value of ν will be indeterminate on the universal event S, because it will always divide by itself, and on the empty event $\emptyset$, because it never get assigned a consequence. Event Reachability is necessary and sufficient to determine the set function everywhere else up to a scalar multiple. This is the best we can hope for, since this scalar multiple will always divide itself out.

Theorem 3. *Suppose that $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a partition-dependent expected utility representation by (u, ν). The following are equivalent:*

(i) $\{\succsim_\pi\}_{\pi\in\Pi}$ *satisfies Event Reachability.*

(ii) *If* (u', ν') *also represents* $\{\succsim_\pi\}_{\pi\in\Pi}$*, then there exist numbers* $a, c > 0$ *and* $b \in \mathbb{R}$ *such that* $u'(p) = au(p) + b$ *for all* $p \in \Delta X$ *and* $\nu'(E) = c\nu(E)$ *for all* $E \neq \emptyset, S$*.*

Proof. See Appendix A.3

If $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies the uniqueness in Theorem 3, we will say that it admits a unique partition-dependent expected utility representation by (u, ν).

4.3 Monotonicity

We have yet to conclude that ν is monotone with respect to set inclusion, or that $\nu(E) \leq \nu(F)$ whenever $E \subset F$. While it seems very natural that someone would put less weight on a subset of an event, the experimental evidence has repeatedly detected violations of monotonicity. For example, Tversky and Kahneman (1983) document numerous experimental examples of the conjunction fallacy, where subject judge an intersection of different events to be strictly more likely than its components. When estimating the frequency of seven-letter words ending with "ing" versus seven-letter words with "n" as the sixth letter, subjects report a higher frequency for the former set, even though it is a strict subset of the latter. In addition, violations of monotonicity due to the representativeness heuristic, as famously demonstrated by the Linda problem, are remarkably robust despite "a series of increasingly desperate manipulations designed to induce subjects to obey the conjunction rule" (Tversky and Kahneman 1983, p. 299).[7] So, we see no a priori reason to impose monotonicity of the set function.

Nonetheless, we present the characterization of monotonicity for those who are interested. When the set function ν is unique up to a scalar multiple as characterized in Theorem 3, the following condition guarantees that ν is monotone.

Axiom 9 (Monotonicity)**.** For all $E \subset F \subset S$ and $p, q, r, s \in \Delta X$ such that $p \succ q$,

$$s \succsim \begin{pmatrix} p & F \\ q & F^{\mathsf{c}} \end{pmatrix} \Longrightarrow \begin{pmatrix} r & E \\ s & E^{\mathsf{c}} \end{pmatrix} \succsim \begin{pmatrix} r & E \\ p & F \setminus E \\ q & F^{\mathsf{c}} \end{pmatrix}.$$

Theorem 4. *Suppose* $\{\succsim_{\pi\in\Pi}\}_{\pi\in\Pi}$ *admits a unique partition-dependent expected utility representation* (u, ν)*. Then* $\{\succsim_\pi\}_{\pi\in\Pi}$ *satisfies Monotonicity if and only if* ν *is monotone.*

Proof. See Appendix A.4.

[7] In the Linda problem, subjects are told that "Linda is 31 years old, single, outspoken and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations." The subjects believe the event "Linda is a bank teller" is less probable than the event "Linda is a bank teller and is active in the feminist movement" (Tversky and Kahneman 1983, p. 297).

Event Reachability is required because, in general, there could exist one representation where ν is not monotone, but another where ν is monotone. This is demonstrated explicitly in Example 4 of Appendix A.4. What can be guaranteed is that all subevents of null events remain null, i.e. if F is null and $E \subseteq F$, then E is also null.

One interesting and potentially useful property of the Monotonicity axiom is that, once imposed, Event Reachability and Strict Admissibility are equivalent.

Proposition 5. *Suppose $|S| \geq 3$ and $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation and satisfies Monotonicity. Then $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies Event Reachability if and only if it satisfies Strict Admissibility.*

Proof. See Appendix A.5.

5 Measures of unawareness

Here, we introduce behavioral and quantitative characterizations of a decision maker who incompletely corrects for her lack of awareness.

We first review some formalities. Define the binary relation $\geq$ on Π by $\pi' \geq \pi$ if $\sigma(\pi') \supseteq \sigma(\pi)$, i.e. if π' is finer than π. This binary relation defines a lattice on Π, where the meet $\pi \wedge \pi'$ denotes the finest common coarsening of π and π' and the join $\pi \vee \pi'$ denotes the coarsest common refinement of π and π'. Slightly abusing notation, if $E \subset S$ and $\pi'_E \in \Pi_E$, let $\pi \vee \pi'_E$ denote $\pi \vee [\pi'_E \cup \{E^{\complement}\}]$. When it engenders no confusion, given $\pi' \in \Pi$ and $E \in \sigma(\pi)$, let $\pi'_E \in \Pi_E$ denote the restriction of π' to E: $\pi'_E = \{F \in \pi' : F \subset E\}$.

The following definitions of absolute under and overcorrection for unawareness do not depend on the particular utility representation forwarded in the previous section. The decision maker undercorrects for her unawareness of an event if she puts more relative likelihood on the event as she understands its contingencies better. Conversely, she overcorrects if she puts less likelihood on the event as she understands it better. We stress the correction for unawareness because the decision maker can try to adjust her assigned likelihood for the events which are not explicitly mentioned in the framing of the acts. In doing so, she may undershoot or overshoot the desired target. Is she happens to correct for her unawareness precisely, we cannot distinguish her behavior from that of someone who has full awareness.

Definition 7. Suppose $E \in \pi \in \Pi$. $\{\succsim_\pi\}_{\pi\in\Pi}$ **undercorrects for unawareness of π'_E at π** if, for any $p, q, r \in \Delta X$ such that $q \succ r$:

$$\begin{pmatrix} q & E \\ r & E^{\complement} \end{pmatrix} \succsim_\pi p \Longrightarrow \begin{pmatrix} q & E \\ r & E^{\complement} \end{pmatrix} \succsim_{\pi\vee\pi'_E} p$$

Suppose $\pi' \geq \pi$. $\{\succsim_\pi\}_{\pi\in\Pi}$ **undercorrects for unawareness of π' at π** if π undercorrects for unawareness of π'_E for all $E \in \pi$.

Finally, $\{\succsim_\pi\}_{\pi\in\Pi}$ **undercorrects for unawareness** if π undercorrects for unawareness of π' for all $\pi' \geq \pi$.

In words, if the decision maker's certainty equivalent for a bet on the event E increases when she becomes aware of π'_E, then she is undercorrecting at the point when she is unaware of π'_E. One way to consider the definition is that she is more willing to pay more to insure against contingency E as she becomes increasingly aware of its subevents. For example, violations of monotonicity entail severe undercorrections for unawareness.

Definition 8. Suppose $E \in \pi \in \Pi$. $\{\succsim_\pi\}_{\pi\in\Pi}$ **overcorrects for unawareness of** π'_E **at** π if, for any $p, q, r \in \Delta X$ such that $q \succ r$:

$$p \succsim_\pi \begin{pmatrix} q & E \\ r & E^{\complement} \end{pmatrix} \Longrightarrow p \succsim_{\pi\vee\pi'_E} \begin{pmatrix} q & E \\ r & E^{\complement} \end{pmatrix}$$

Suppose $\pi' \geq \pi$. $\{\succsim_\pi\}_{\pi\in\Pi}$ **overcorrects for unawareness of** π' **at** π if π overcorrects for unawareness of π'_E for all $E \in \pi$.

Finally, $\{\succsim_\pi\}_{\pi\in\Pi}$ **overcorrects for unawareness** if π overcorrects for unawareness of π' for all $\pi' \geq \pi$.

In an example given earlier, we considered a car owner who purchases too much warranty protection when she does not understand how her engine works. Such a consumer is overcorrecting to her unawareness.

When preferences admit a unique representation as in Theorems 2 and 3, undercorrection or overcorrection of unawareness is obviously related to the subadditivity or superadditivity of the set function.

Definition 9. A set function ν is **subadditive** if $\nu(A \cup B) \leq \nu(A) + \nu(B)$ whenever $A \cap B = \emptyset$. A set function ν is **superadditive** if $\nu(A \cup B) \geq \nu(A) + \nu(B)$ whenever $A \cap B = \emptyset$.

In the context of pure framing in support theory, Tversky and Koehler (1994) argue for and provide evidence suggesting subadditivity of the support function across disjunctions of hypotheses. Note that subadditivity is strictly weaker than concavity, $\nu(A\cup B)+\nu(A\cap B) \leq \nu(A)+\nu(B)$ for all $A, B \subset S$, and that superadditivity is strictly weaker than convexity, $\nu(A\cup B)+\nu(A\cap B) \geq \nu(A)+\nu(B)$ for all $A, B \subset S$. Concavity and convexity are commonly used in the study of capacities in Choquet integration or in the value functions of cooperative games, but have little behavioral content in terms of unawareness.

Undercorrection and overcorrection are quantitatively characterized by an obvious ratio of the weighing function's value on the event as she is aware and unaware of the subevents of π'_E. If this ratio is larger than unity, than the decision maker puts more likelihood on the event when she is aware of π'_E.

Definition 10. Suppose $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies Axioms 1–8, so is represented by a utility function u and a set function ν. If $E \in \pi$ and $\pi'_E \in \Pi_E$, define the **coefficient of unawareness correction**

of π'_E as

$$\lambda(\pi'_E) = \frac{\sum_{F \in \pi'_E} \nu(F)}{\nu(E)}.$$

The standard notion of risk aversion can be expressed behaviorally in terms of certainty equivalents, as a property of the utility function for wealth, or quantitatively through the Arrow–Pratt coefficient. Our proposed definition of under and overcorrection can be tied to a a structural condition on the set function ν, which can then be tied to a quantitative measure.

Proposition 6. *Suppose $\{\succsim_\pi\}_{\pi \in \Pi}$ satisfy the Anscombe–Aumann axioms, the Sure-Thing Principle, and Binary Bet Acyclicity, hence uniquely represented by (u, ν). The following are equivalent:*

(i) $\{\succsim_\pi\}_{\pi \in \Pi}$ *undercorrects* [overcorrects] *for unawareness;*

(ii) ν *is subadditive* [superadditive]*;*

(iii) $\lambda(\pi'_E) \geq [\leq]\, 1$ *for all $E \subset S$, $\pi'_E \in \Pi_E$.*

We next introduce a relative notion of undercorrection for unawareness.

Definition 11. Suppose (u, ν) and (u', ν') uniquely represent $\{\succsim_\pi\}_{\pi \in \Pi}$ and $\{\succsim'_\pi\}_{\pi \in \Pi}$ in the sense of Theorem 3. Let λ, λ' denote their coefficients of unawareness correction. Then ν is **more underaware** than ν' if $\lambda(\pi_E) \geq \lambda'(\pi_E)$ for all $E \subset S$, $\pi_E \in \Pi_E$.

An obvious deficiency in the comparative definition is its dependence on the particular utility representation of Theorem 2. A more basic definition, which does not refer to a specific functional form, would be superior.[8] However, in the more narrow space where our representation holds, the concept seems like a reasonable one. Referring back to the examples buttresses the intuition.

First, recall Example 1: $\nu(E) = w(\mu(E))$ for some additive probability measure μ and an increasing probability weighting function $w : [0, 1] \to \mathbb{R}_+$. Then $\{\succsim\}_{\pi \in \Pi}$ undercorrects for unawareness if and only if w is a concave transformation, so the absolute definition seems to work here. Also, suppose $(u, w \circ \mu)$ and $(u', w' \circ \mu)$ represent $\{\succsim_\pi\}_{\pi \in \Pi}$ and $\{\succsim'_\pi\}_{\pi \in \Pi}$. Then $\{\succsim_\pi\}_{\pi \in \Pi}$ is more underaware than $\{\succsim'_\pi\}_{\pi \in \Pi}$ if and only if w is a concave transformation of w'.

In Example 2, where $\nu(E) = \alpha\mu(E) + (1 - \alpha)$. When $\alpha = 0$, ν is a constant set function and corresponds with the principle of insufficient reason which puts equal weight on all listed contingencies. Suppose $(u, \alpha\mu + (1 - \alpha))$ and $(u', \beta\mu + (1 - \beta)$ represent $\{\succsim_\pi\}_{\pi \in \Pi}$ and $\{\succsim'_\pi\}_{\pi \in \Pi}$. Then $\{\succsim_\pi\}_{\pi \in \Pi}$ is more underaware than $\{\succsim'_\pi\}_{\pi \in \Pi}$ if and only if $\alpha \leq \beta$. In words, a decision maker who is more biased towards the ignorance prior will exhibit more undercorrection for unawareness.

All the definitions so far are applied to the entire preference or to the set function ν. Moreover, the concept appeal to the decision maker's awareness with respect to partitions. One might be independently interested of the decision maker's awareness of specific events, independent of any partition of the state space. The following provides on extreme notion of unawareness for particular sets.

[8] We have a behavioral characterization of this definition which is independent of any particular utility representation, but at this point it is too complicated to be superior to this one.

Definition 12. $\{\succsim_\pi\}_{\pi\in\Pi}$ is **completely unaware** of $E \subset S$ if E is nonnull and for all partitions $\{E, F, G\}$ of S and $p, q, r \in \Delta X$:

$$\begin{pmatrix} p & E \cup F \\ q & G \end{pmatrix} \sim r \iff \begin{pmatrix} p & F \\ q & E \cup G \end{pmatrix} \sim r.$$

In words, the decision maker never puts any weight on E unless it is explicitly described to her. In the first comparison, she attributes all the likelihood of receiving p to F because she is completely unaware of E; in the second comparison, all the likelihood of q is similarly attributed to G. Due to the framing of both acts, E remains occluded and the certainty equivalents are equal.

Definition 12 begins by distinguishing an event of which the decision maker is completely unaware from an event which the decision maker considers null. The following preference is not precluded by complete unawareness of E:

$$\begin{pmatrix} x_1 & A \cup B \\ x_2 & C \end{pmatrix} \succ \begin{pmatrix} y & A \\ x_1 & B \\ x_2 & C \end{pmatrix}.$$

Here, the presentation of the second act makes the decision maker aware of E, at which point she assigns it some positive likelihood. In contrast, this strict preference is precluded whenever E is a null event, because the decision maker would be indifferent as to whether x_1 or y is assigned to the impossible event. Therefore, the primitives allow the analyst to distinguish unawareness and nullity from preferences over bets.

Proposition 7. *Suppose $|S| \geq 3$ and $\{\succsim_{\pi\in\Pi}\}_{\pi\in\Pi}$ admits a unique partition-dependent expected utility representation (u, ν). Then $\{\succsim_\pi\}_{\pi\in\Pi}$ is completely unaware of all nonempty $E \subset S$ if and only if ν is a constant set function.*

The extreme case of complete unawareness across all events is represented by a constant capacity where $\nu(E) = 1$ for every E. The decision maker places a uniform distribution over the events in her partition; extreme unawareness corresponds to the principle of insufficient reason.

A Appendix

Contrary to the order of presentation, we will prove Theorem 2 before proving Theorem 1.

A.1 Proof of Theorem 2

The necessity of the first five axioms follows immediately from the standard Anscombe–Aumann Expected Utility Theorem. We check the final two axioms.

Claim 1. *If $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-dependent expected utility representation, then $\succsim$ satisfies the Sure-Thing Principle.*

Proof. For any $f, g \in \mathcal{F}$, note that $D(f,g) \equiv \{s \in S : f(s) \neq g(s)\} \in \sigma(\pi(f,g))$, hence:

$$
\begin{aligned}
f \succsim g \qquad & \Longleftrightarrow \qquad f \succsim_{\pi(f,g)} g \\
& \Longleftrightarrow \qquad \int_{D(f,g)} u \circ f \, d\mu_{\pi(f,g)} \geq \int_{D(f,g)} u \circ g \, d\mu_{\pi(f,g)} \\
& \Longleftrightarrow \qquad \sum_{\substack{F \,\in\, \pi(f,\,g)\,: \\ F \,\subset\, D(f,\,g)}} u(f(F))\nu(F) \geq \sum_{\substack{F \,\in\, \pi(f,\,g)\,: \\ F \,\subset\, D(f,\,g)}} \nu(F),
\end{aligned}
$$

where the second equivalence follows from multiplying both sides by $\sum_{F' \in \pi(f,g)} \nu(F')$.

Now, to demonstrate the Sure-Thing Principle, let $E \subset S$ and $f, g, h, h' \in \mathcal{F}$. Let

$$
\hat{f} = \begin{pmatrix} f & E \\ h & E^{\complement} \end{pmatrix}; \qquad \hat{g} = \begin{pmatrix} g & E \\ h & E^{\complement} \end{pmatrix};
$$

$$
\hat{f}' = \begin{pmatrix} f & E \\ h' & E^{\complement} \end{pmatrix}; \qquad \hat{g}' = \begin{pmatrix} g & E \\ h' & E^{\complement} \end{pmatrix}.
$$

Note that $D \equiv D(\hat{f}, \hat{g}) = D(\hat{f}', \hat{g}') \subset E$ and $\pi_D \equiv \{F \in \pi(\hat{f}, \hat{g}) : F \subset D(\hat{f}, \hat{g})\} = \{F \in \pi(\hat{f}', \hat{g}') : F \subset D(\hat{f}', \hat{g}')\}$. Hence by the observation made in the first paragraph:

$$
\begin{aligned}
\hat{f} \succsim \hat{g} \qquad & \Longleftrightarrow \qquad \sum_{F \in \pi_D} u(\hat{f}(F))\nu(F) \geq \sum_{F \in \pi_D} u(\hat{g}(F))\nu(F) \\
& \Longleftrightarrow \qquad \sum_{F \in \pi_D} u(f(F))\nu(F) \geq \sum_{F \in \pi_D} u(g(F))\nu(F) \\
& \Longleftrightarrow \qquad \sum_{F \in \pi_D} u(\hat{f}'(F))\nu(F) \geq \sum_{F \in \pi_D} u(\hat{g}'(F))\nu(F) \\
& \Longleftrightarrow \qquad \hat{f}' \succsim \hat{g}'.
\end{aligned}
$$

□

Claim 2. *If $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a partition-dependent expected utility representation, then $\succsim$ satisfies Binary Bet Acyclicity.*

Proof. Let $E, F \subset S$ be disjoint events and $p, q, r \in \Delta X$ lotteries. Set $\pi = \left\{E, F, (E \cup F)^{\complement}\right\}$, then

$$
\begin{aligned}
\begin{pmatrix} p & E \\ q & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} r & F \\ q & F^{\complement} \end{pmatrix} \qquad & \Longleftrightarrow \qquad [u(p) - u(q)]\mu_\pi(E) \geq [u(r) - u(q)]\mu_\pi(F) \\
& \Longleftrightarrow \qquad [u(p) - u(q)]\nu(E) \geq [u(r) - u(q)]\nu(F)
\end{aligned}
$$

where the second equivalence is obtained by multiplying both sides by $\nu(E) + \nu(F) + \nu\left((E \cup F)^{\complement}\right)$.

To see necessity of Binary Bet Acyclicity, let the events $E_1, \ldots E_n \subset S$ and the lotteries $p_1, p_2, \ldots, p_n; q \in \Delta X$ be such that $E_1 \cap E_2 = E_2 \cap E_3 = \ldots = E_{n-1} \cap E_n = E_n \cap E_1 = \emptyset$ and

$$
\forall i = 1, \ldots n-1 : \qquad \begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix}.
$$

The observation made in the first paragraph implies that $[u(p_1) - u(q)]\nu(E_1) \geq [u(p_2) - u(q)]\nu(E_2) \geq \ldots \geq$

$[u(p_n) - u(q)]\nu(E_n)$. Since $[u(p_1) - u(q)]\nu(E_1) \geq [u(p_n) - u(q)]\nu(E_n)$, we conclude that

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}. \qquad \square$$

We now move to proving the sufficiency of the axioms for the representation. The first five axioms provide a simple generalization of the Anscombe–Aumann Expected Utility Theorem.

Claim 3. *Suppose $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies the Anscombe–Aumann axioms. Then there exists a family of probability measures $\{\mu_\pi\}_{\pi\in\Pi}$ with $\mu_\pi : \sigma(\pi) \to [0,1]$ and an affine utility function $u : \Delta X \to \mathbb{R}$ with $[0,1] \subseteq u(X)$ such that*

$$f \succsim_\pi g \Longleftrightarrow \int_S u \circ f \, d\mu_\pi \geq \int_S u \circ g \, d\mu_\pi$$

Proof. For each $\pi \in \Pi$, Axioms 1–5 guarantee a probability measure μ_π on $(S, \sigma(\pi))$ and a non-constant affine vNM utility function $u_\pi : \Delta X \to \mathbb{R}$ such that $f \succsim_\pi g$ if and only if $\int_S u_\pi \circ f \, d\mu_\pi \geq \int_S u_\pi \circ g \, d\mu_\pi$, for all $f, g \in \mathcal{F}_\pi$. By State Independence, $p \succsim_\pi q$ if and only if $p \succsim_{\pi'} q$, therefore $u_\pi(p) \geq u_\pi(q)$ if and only if $u_{\pi'}(p) \geq u_{\pi'}(q)$. Then the uniqueness component of the standard Anscombe–Aumann Expected Utility Theorem implies that $u_{\pi'}$ is a positive affine transformation of u_π. By appropriately normalizing, we lose no generality by assuming $u_\pi = u_{\pi'} = u$. Nondegeneracy ensures that u is not constant, so we may further assume that its image contains the unit interval, $[0,1] \subset u(X)$, again by appropriately normalizing. $\square$

For any act $f \in \mathcal{F}$, let $\pi(f)$ denote its induced algebra on S, which is the coarsest partition such that $f \in \mathcal{F}_\pi$. We now record two facts which rely only on the Anscombe–Aumann axioms and the Sure-Thing Principle.

Claim 4. *For any events E, F and partitions π, π':*

(i) *If $E \in \pi, \pi'$, then $\mu_\pi(E) = 0 \Leftrightarrow \mu_{\pi'}(E) = 0$.*

(ii) *If $E, F \in \pi, \pi'$ and $E \cap F = \emptyset$, then $\mu_\pi(E)\mu_{\pi'}(F) = \mu_\pi(F)\mu_{\pi'}(E)$*

Proof. To prove part (i), it is enough to show that $E \in \pi, \pi'$, then $\mu_\pi(E) = 0 \Rightarrow \mu_{\pi'}(E) = 0$. Suppose that $\mu_\pi(E) = 0$. Select any two lotteries $p, q \in \Delta X$ satisfying $u(p) > u(q)$ and any two acts h, h' such that $p, q \notin h(S) \cup h'(S)$, $\pi(h) = \pi$, and $\pi(h') = \pi'$. Then

$$\begin{pmatrix} p & E \\ h & E^{\complement} \end{pmatrix} \sim \begin{pmatrix} q & E \\ h & E^{\complement} \end{pmatrix}$$

by Claim 3. Hence

$$\begin{pmatrix} p & E \\ h' & E^{\complement} \end{pmatrix} \sim \begin{pmatrix} q & E \\ h' & E^{\complement} \end{pmatrix}$$

by the Sure-Thing Principle. Since $u(p) > u(q)$, the last indifference can hold only if $\mu_{\pi'}(E) = 0$ by Claim 3.

To prove part (ii), observe that if either side of the desired equality is zero, then part (ii) is immediately implied by part (i). So we may proceed assuming that both sides are strictly positive. Then all of the terms $\mu_\pi(E)$, $\mu_{\pi'}(F)$, $\mu_\pi(F)$, and $\mu_{\pi'}(E) > 0$ are strictly positive. As before, select any two lotteries $p, q \in \Delta X$ such that $u(p) > u(q)$, and let

$$r = \frac{\mu_\pi(E)}{\mu_\pi(E) + \mu_\pi(F)} p + \frac{\mu_\pi(F)}{\mu_\pi(E) + \mu_\pi(F)} q,$$

which is another lottery. Select any two acts h, h' such that $p, q, r \notin h(S) \cup h'(S)$, $\pi(h) = \pi$, and $\pi(h') = \pi'$. By the choice of r and the expected utility representation of $\succsim_\pi$ implied in Claim 3, we have:

$$\begin{pmatrix} p & E \\ q & F \\ h & (E \cup F)^{\complement} \end{pmatrix} \sim \begin{pmatrix} r & E \cup F \\ h & (E \cup F)^{\complement} \end{pmatrix}$$

Hence by the Sure-Thing Principle,

$$\begin{pmatrix} p & E \\ q & F \\ h' & (E \cup F)^{\complement} \end{pmatrix} \sim \begin{pmatrix} r & E \cup F \\ h' & (E \cup F)^{\complement} \end{pmatrix}.$$

This indifference relation, in conjunction with Claim 3, implies that

$$u(r) = \frac{\mu_{\pi'}(E)}{\mu_{\pi'}(E) + \mu_{\pi'}(F)} u(p) + \frac{\mu_{\pi'}(F)}{\mu_{\pi'}(E) + \mu_{\pi'}(F)} u(q)$$

We also have

$$u(r) = \frac{\mu_{\pi}(E)}{\mu_{\pi}(E) + \mu_{\pi}(F)} u(p) + \frac{\mu_{\pi}(F)}{\mu_{\pi}(E) + \mu_{\pi}(F)} u(q)$$

by the definition of r. Subtracting $u(q)$ from each side of the two expressions for $u(r)$ above, we have

$$\frac{\mu_{\pi'}(E)}{\mu_{\pi'}(E) + \mu_{\pi'}(F)} [u(p) - u(q)] = \frac{\mu_{\pi}(E)}{\mu_{\pi}(E) + \mu_{\pi}(F)} [u(p) - u(q)]$$

which further simplifies to $\frac{\mu_{\pi'}(F)}{\mu_{\pi'}(E)} = \frac{\mu_{\pi}(F)}{\mu_{\pi}(E)}$ since both sides of the previous equality are strictly positive. $\square$

By part (i) of Claim 4, any event $E \in \pi, \pi'$ is π-null if and only if it is π'-null. Hence we can change quantifiers in the definitions of null and nonnull events. A nonempty event E is null if and only if E is π-null for *some* partition π with $E \in \pi$. Dually, an event E is nonnull if and only if E is π-nonnull for *every* partition π with $E \in \pi$.[9]

For any two disjoint nonnull events E, F, define the ratio:

$$\frac{E}{F} = \frac{\mu_{\pi}(E)}{\mu_{\pi}(F)}$$

where π is a partition such that $E, F \in \pi$. The value of $\frac{E}{F}$ does not depend on the particular choice of π, by part (ii) of Claim 4. Moreover, $\frac{E}{F}$ is well-defined and strictly positive since E and F are nonnull. Finally, $\frac{F}{E} \times \frac{E}{F} = 1$ by construction. The following appeals to Binary Bet Acyclicity in generalizing this equality. We first show that, given the other axioms, Binary Bet Acyclicity can be strengthened so that the conclusion holds even if the preferences in the hypothesis are weak.

Claim 5. *For all any sequentially disjoint cycle of events $E_1, \ldots, E_n, E_1$ and lotteries $p_1, p_2, \ldots, p_n; q \in \Delta X$*

[9] Obviously, $\emptyset$ is null and S is nonnull by Nondegeneracy. Note that there may exist a nonnull event E, which is π-null for some π for which $E \in \sigma(\pi)$. From the above observation concerning the quantifiers, this can only be possible if E is not a cell in π but a union of its cells. This would correspond to a representation where for example E is a disjoint union of two sub-events $E = E_1 \cup E_2$, and $\nu(E) > 0$ yet $\nu(E_1) = \nu(E_2) = 0$.

such that $q \neq p_i$ for all i,

$$(\forall i = 1, \dots n-1) : \begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix} \Longrightarrow \begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Proof. Suppose $\varepsilon > 0$. For each p_i, pick some $p_i(\varepsilon) \in \Delta X$ such that $u(p_i(\varepsilon)) = u(p_i(\varepsilon)) + \varepsilon^i$. Without loss of generality, we can take ε sufficiently small so $p_i(\varepsilon) \neq q$ for all i, so the minimal awareness is unchanged. The expected utility representation of Claim 3 implies that for sufficiently small ε,

$$\begin{pmatrix} p_i(\varepsilon) & E_i \\ q & E_i^{\complement} \end{pmatrix} \succ \begin{pmatrix} p_{i+1}(\varepsilon) & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix},$$

for $i = 1, \dots, n-1$. By Binary Bet Acyclicity, this implies

$$\begin{pmatrix} p_1(\varepsilon) & E_1 \\ q & E_1^{\complement} \end{pmatrix} \succsim \begin{pmatrix} p_n(\varepsilon) & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Appealing to the continuity of the expected utility representation in the assigned lotteries $f(s)$, and taking $\varepsilon \to 0$ proves the desired conclusion. □

Claim 6. *If $n \geq 2$ and $E_1, \dots, E_n$ are nonnull events such that $E_1 \cap E_2 = E_2 \cap E_3 = \dots = E_{n-1} \cap E_n = E_n \cap E_1 = \emptyset$, then:*

$$\frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \cdots \times \frac{E_{n-1}}{E_n} \times \frac{E_n}{E_1} = 1.$$

Proof. The case where $n = 2$ immediately follows from our definition of event ratios, so assume that $n \geq 3$. Fix $t_1 > 0$, and recursively define

$$t_i = t_1 \times \frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \dots \times \frac{E_{i-1}}{E_i}.$$

for $i = 2, \dots, n$. By selecting a sufficiently small t_i, we may assume that $t_1, \dots t_n \in (0, 1]$. Also note that $\frac{t_{i+1}}{t_i} = \frac{E_i}{E_{i+1}}$ for $i = 1, \dots, n-1$. Recall the range of the utility function u over lotteries contains the unit interval $[0, 1]$, so there exist lotteries $p_1, \dots, p_n, q \in \Delta X$ such that $u(p_i) = t_i$ for $i = 1, \dots, n$ and $u(q) = 0$.

Fix any $i \in \{1, \dots, n-1\}$. Let $\pi = \{E_i, E_{i+1}, (E_i \cup E_{i+1})^{\complement}\}$. Since $\frac{t_{i+1}}{t_i} = \frac{E_i}{E_{i+1}}$, we have $\mu_\pi(E_{i+1})u(p_{i+1}) = \mu_\pi(E_i)u(p_i)$. Hence:

$$\begin{pmatrix} p_i & E_i \\ q & E_i^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_{i+1} & E_{i+1} \\ q & E_{i+1}^{\complement} \end{pmatrix}$$

by the expected utility representation of Claim 3. Since the above indifference holds for any $i \in \{1, \dots, n-1\}$, by two applications of Claim 5, we have

$$\begin{pmatrix} p_1 & E_1 \\ q & E_1^{\complement} \end{pmatrix} \sim \begin{pmatrix} p_n & E_n \\ q & E_n^{\complement} \end{pmatrix}.$$

Hence by the expected utility representation of $\succsim_\pi$ for $\pi = \{E_1, E_n, (E_1 \cup E_n)^{\complement}\}$, $\mu_\pi(E_1)u(p_1) = \mu_\pi(E_n)u(p_n)$, that is $\frac{t_n}{t_1} = \frac{E_1}{E_n}$. By definition of t_n, the latter equality implies the desired conclusion:

$$\frac{E_1}{E_2} \times \frac{E_2}{E_3} \times \dots \times \frac{E_{n-1}}{E_n} = \frac{E_1}{E_n}.$$ □

We can now conclude the proof of sufficiency. Let $\mathcal{E}$ denote the collection of all nonnull events, which is nonempty since Nondegeneracy ensures $S \in \mathcal{E}$. Define the binary relation $\approx$ on $\mathcal{E}$ by $E \approx F$ (we read it as F is reachable from E) if $E = F$ or if there exist nonnull sequence of sequentially disjoint events $E_1, \ldots, E_n$. The relation $\approx$ is obviously reflexive, symmetric, and transitive, defining an equivalence relation on $\mathcal{E}$. For any nonnull $E \in \mathcal{E}$, let $[E] = \{F \in \mathcal{E} : E \approx F\}$ denote the equivalence class of E with respect to $\approx$ (the reach of E). Let $\mathcal{E}/\approx = \{[E] : E \in \mathcal{E}\}$ denote the quotient set of all equivalence classes of $\mathcal{E}$ modulo $\approx$, with a generic class $R \in \mathcal{E}/\approx$.[10] Select a representative event $G_R \in R$ for every equivalence class $R \in \mathcal{E}/\approx$, invoking to the axiom of choice if the quotient is infinite.

We next define ν. For all null E, let $\nu(E) = 0$. For every class $R \in \mathcal{E}/\approx$, *arbitrarily assign* a positive value $\nu(G_R) > 0$ for its representative. We will conclude by defining $\nu(E)$, for any $E \in \mathcal{E} \setminus \{S\}$. If $E = G_{[E]}$, then E represents its equivalence class and $\nu(E)$ has been assigned. Otherwise, whenever $E \neq G_{[E]}$, since $E \approx G_{[E]}$, there exist nonnull sequentially disjoint path of events $E_1, \ldots, E_n$ such that $E = E_1$, $G_{[E]} = E_n$. Then set:

$$\nu(E) = \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

Note that the definition of $\nu(E)$ above is independent of the particular choice of the path $E_1, \ldots, E_n$, because for any other such sequentially disjoint path $E = F_1, \ldots, F_m = G_{[E]}$:

$$\frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \frac{F_m}{F_{m-1}} \times \ldots \times \frac{F_2}{F_1} = 1$$

by Claim 6.

We will next verify that $\nu \colon 2^S \setminus \{S\} \to \mathbb{R}_+$ defined above is a non-degenerate set function satisfying

$$\mu_\pi(E) = \frac{\nu(E)}{\sum_{F \in \pi} \nu(F)} \tag{1}$$

for any event $E \in \pi$ of any partition $\pi \in \Pi \setminus \{\{S\}\}$.

Let $\pi \in \Pi \setminus \{\{S\}\}$. By Nondegeneracy and the expected utility representation of Claim 3 for $\succsim_\pi$, there exists a π-nonnull $F \in \pi$. Then F is nonnull so the denominator on the right hand side of Equation (1) is strictly positive, so the fraction is well-defined. This also implies that ν is a non-degenerate set function. Observe that Equation (1) immediately holds if E is null, since then $\nu(E) = 0$ and $\mu_\pi(E) = 0$ follows from E being π-null. Let $\mathcal{E}_\pi \subset \pi$ denote the nonnull cells of π. To finish the proof of the Theorem, we will show that $\frac{\mu_\pi(E)}{\mu_\pi(F)} = \frac{\nu(E)}{\nu(F)}$ for any distinct $E, F \in \mathcal{E}_\pi$. Along with the fact that $\sum_{E \in \mathcal{E}_\pi} \mu_\pi(E) = 1$, this will prove Equation (1).

Let $E, F \in \mathcal{E}_\pi$ be distinct. Note that $[E] = [F]$ since E and F are disjoint. Suppose first that neither E nor F is $G_{[E]}$. Then there exist nonnull events $E_1, \ldots, E_n$ such that $E = E_1$, $G_{[E]} = E_n$, $E_i \cap E_{i+1} = \emptyset$ for $i = 1, \ldots, n-1$, and:

$$\nu(E) = \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

But then $F, E_1, \ldots, E_n = G_{[E]}$ forms such a path from F to $G_{[E]}$, hence we have:

$$\nu(F) = \frac{F}{E_1} \times \frac{E_1}{E_2} \times \ldots \times \frac{E_{n-1}}{E_n} \times \nu(G_{[E]}).$$

Dividing the term for $\nu(E)$ by the term for $\nu(F)$, we obtain $\frac{E}{F} = \frac{\nu(E)}{\nu(F)}$.

[10] Note that $[S] = \{S\}$ and $E \approx F$ for any disjoint nonnull E, F.

The other possibility is that exactly one of E or F (without loss of generality E) is $G_{[E]}$. Then the nonnull events $F = E_1, E_2 = E$, make up a path from F to $E = G_{[E]}$. Then

$$\nu(F) = \frac{F}{E} \times \nu(E)$$

as desired.

A.2 Proof of Theorem 1

We maintain the notation and the results established in the proof of Theorem 2 in Appendix A.1. Suppose $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a partition-independent expected utility representation. The Anscombe–Aumann axioms follow immediately, so we only check Acyclicity. We have $f \succsim g$ if and only if $\succsim_\pi$ for all $\pi \in \Pi$ such that $f, g \in \pi$. Then if $f \succsim g$ and $g \succsim h$, let π be some partition such that $f, g, h \in \pi$ and $f \succsim h$ because $\succsim_\pi$ is transitive. Thus $\succsim$ is transitive, hence acyclic.

We prove that Acyclicity implies a partition-independent expected utility representation contrapositively. Suppose the Anscombe–Aumann axioms hold, so by Claim 3 there exist a vNM utility function u and a family of measures $\{\mu_\pi\}_{\pi\in\Pi}$ which represent $\{\succsim_\pi\}_{\pi\in\Pi}$. Now suppose that no additive representation can be achieved. Then there exists a partition $\pi \in \Pi \setminus \{S\}$ and $E \in \pi$ such that $\mu_\pi(E) \neq \mu_{\{E,E^\complement\}}(E)$. Without loss of generality, we can assume the range $u(\Delta X)$ contains the interval $[-1, 1]$ by appropriately normalizing. Also, either E or $E^\complement$ must be π-nonnull; we will assume that E is π-nonnull, switching labels if required. Suppose $\mu_\pi(E) > \mu_{\{E,E^\complement\}}(E)$; the other strict inequality is symmetric. Let $p, q \in \Delta X$ be such that $u(p) = 1$, $u(q) = 0$ and define the act h by

$$h = \begin{pmatrix} p & E \\ q & E^\complement \end{pmatrix}.$$

Either $\mu_{\{E,E^\complement\}}(E) \neq 1$ or $\mu_{\{E,E^\complement\}}(E) \neq 0$. We will consider the first case; the second is symmetric. Fix some $\varepsilon \in (0, 1 - \mu_{\{E,E^\complement\}}(E))$. Note that $\alpha p + (1-\alpha)q \succ h$ where $\alpha = \mu_{\{E,E^\complement\}}(E) + \varepsilon$. Let $f \in \mathcal{F}$ be such that $\pi(f) = \pi$ and for all $s \in S$, $u(f(s)) < 0$. Then, for sufficiently small ε, there exists small enough $\delta \in (0, 1)$ such that the act g^δ defined by

$$g^\delta = \begin{pmatrix} p & E \\ (1-\delta)q + \delta f & E^\complement \end{pmatrix}$$

satisfies $\pi(g^\delta) = \pi$ and $g^\delta \succ_\pi \alpha p + (1-\alpha)q$. Then $g^\delta \succ \alpha p + (1-\alpha)q$. By part (i) of Claim 4, this implies $\mu_\pi(E) > 0$. Then, since $u(q) = 0 > u(f(s))$ for all $s \in S$ and E is π-nonnull, we have $h \succ g^\delta$. Collecting relations, we have $\alpha p + (1-\alpha)q \succ h$, $h \succ g^\delta$, and $g^\delta \succ \alpha p + (1-\alpha)q$. Therefore $\succ$ admits a cycle and $\succsim$ violates Acyclicity.

A.3 Proof of Theorem 3

We maintain the notation and the results established in the proof of Theorem 2 in Appendix A.1. Suppose that (u, ν) and (u', ν') are partition-independent expected utility representation of $\{\succsim_\pi\}_{\pi\in\Pi}$ and that Event Reachability is satisfied. For each $\pi \in \Pi$, let μ_π and μ'_π respectively denote the probability distributions derived from ν and ν' by Equation (1):

$$\mu_\pi(E) = \frac{\nu(E)}{\sum_{F\in\pi} \nu(F)}.$$

Applying the uniqueness component of the Anscombe-Aumann Expected Utility Theorem to $\succsim_\pi$, we have $\mu_\pi = \mu'_\pi$ and $u' = au + b$ for some $a > 0$ and $b \in \mathbb{R}$.

Note that if E, F are two disjoint nonnull events, then

$$\frac{\nu(E)}{\nu(F)} = \frac{\mu_\pi(E)}{\mu_\pi(F)} = \frac{E}{F} = \frac{\mu'_\pi(E)}{\mu'_\pi(F)} = \frac{\nu'(E)}{\nu'(F)}.$$

We will next extend the equality $\frac{\nu(E)}{\nu(F)} = \frac{\nu'(E)}{\nu'(F)}$ to any pair of distinct (but not necessarily disjoint) nonnull events E and F different from S, in order to conclude that $\nu' = c\nu$ for some $c > 0$. Let E and F be two distinct nonnull events different from S. By Event Reachability, there exist nonnull events $E_1, \ldots, E_n$ such that $E = E_1$, $F = E_n$, $E_i \cap E_{i+1} = \emptyset$ for $i = 1, \ldots, n-1$. Then:

$$\frac{\nu(E)}{\nu(F)} = \frac{\nu(E_1)}{\nu(E_2)} \times \ldots \times \frac{\nu(E_{n-1})}{\nu(E_n)} = \frac{\nu'(E_1)}{\nu'(E_2)} \times \ldots \times \frac{\nu'(E_{n-1})}{\nu'(E_n)} = \frac{\nu'(E)}{\nu'(F)}$$

where the second equality follows from E_i and E_{i+1} being disjoint for $i = 1, \ldots, n-1$. Thus ν' is a scalar multiple of ν, determined by the constant $c = \nu(E)/\nu'(E)$ for any nonnull set E.

A.4 Proof of Theorem 4

Note that if $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies Axioms 1–7, so that $\{\succsim_\pi\}_{\pi\in\Pi}$ is represented by a pair (u, ν) as in Theorem 1. Then for any events $E \subset F \subset S$ and $p, q, r, s \in \Delta X$:[11]

$$s \succsim \begin{pmatrix} p & F \\ q & F^{\complement} \end{pmatrix} \quad \Leftrightarrow \quad u(s)[\nu(F) + \nu(F^{\complement})] \geq u(p)\nu(F) + u(q)\nu(F^{\complement})$$

$$\begin{pmatrix} r & E \\ s & E^{\complement} \end{pmatrix} \succsim \begin{pmatrix} r & E \\ p & F \setminus E \\ q & F^{\complement} \end{pmatrix} \quad \Leftrightarrow \quad u(s)[\nu(F \setminus E) + \nu(F^{\complement})] \geq u(p)\nu(F \setminus E) + u(q)\nu(F^{\complement}).$$

The next Claim shows that the existence of a partition-dependent expected utility representation with a monotone set function ν implies Monotonicity. This is true even without Event Reachability, or without the uniqueness of ν, hence is a stronger version of the necessity of the axiom required in Theorem 4.

Claim 7. *If $\{\succsim_\pi\}_{\pi\in\Pi}$ admits a (not necessarily unique) partition-dependent expected utility representation by (u, ν) and ν is monotone, then $\{\succsim_\pi\}_{\pi\in\Pi}$ satisfies Monotonicity.*

Proof. Let $E \subset F \subset S$ and $p, q, r, s \in \Delta X$ such that $p \succ q$ and $u(s)[\nu(F)+\nu(F^{\complement})] \geq u(p)\nu(F)+u(q)\nu(F^{\complement})$. If $\nu(F\setminus E)+\nu(F^{\complement}) = 0$, then the desired conclusion holds. Otherwise $\nu(F\setminus E)+\nu(F^{\complement}) > 0$ and $\nu(F)+\nu(F^{\complement}) > 0$ by nondegeneracy of ν. Since $\nu(F) \geq \nu(F \setminus E)$ by monotonicity, we also have:

$$\frac{\nu(F)}{\nu(F) + \nu(F^{\complement})} \geq \frac{\nu(F \setminus E)}{\nu(F \setminus E) + \nu(F^{\complement})}.$$

But then since $u(p) > u(q)$, the inequality:

$$u(s) \geq \frac{\nu(F)}{\nu(F) + \nu(F^{\complement})} u(p) + \frac{\nu(F^{\complement})}{\nu(F) + \nu(F^{\complement})} u(q)$$

[11] For notational convenience arbitrarily fix $\nu(S) > 0$.

implies

$$u(s) \geq \frac{\nu(F \setminus E)}{\nu(F \setminus E) + \nu(F^{\complement})} u(p) + \frac{\nu(F^{\complement})}{\nu(F \setminus E) + \nu(F^{\complement})} u(q).$$

Therefore $\{\succsim_\pi\}_{\pi \in \Pi}$ satisfies Axiom 9. $\square$

The next Example shows that in the absence of Event Reachability, we cannot guarantee monotonicity of ν for *every* partition-dependent expected utility representation (u, ν) of $\{\succsim_\pi\}_{\pi \in \Pi}$. It requires a state space with at least three elements because otherwise, any set function ν with $\nu(\emptyset) = 0$ is trivially monotone.

Example 4. Consider an arbitrary state space S with $|S| \geq 3$, and fix an nonempty event $A \subsetneq S$. Define ν by

$$\nu(E) = \begin{cases} 1 & \text{if } E \cap A \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

for any event $E \neq S$. Note that ν is nondegenerate. Let $\{\succsim_\pi\}_{\pi \in \Pi}$ be represented by (u, ν) for some non-constant u. Any event B such that $A \subset B \subsetneq S$ has nonempty intersection with all nonnull events, hence it can not be linked to any other nonnull set through sequentially disjoint nonnull sets: In the notation of the proof of Theorem 1, such an event B's reachability class $[B]$ consists of only B. Hence although ν itself is monotone, it is straightforward to verify that ν' obtained from ν by changing $\nu(B)$ to $\frac{1}{2}$ continues to represent the same preference. Moreover if we choose B such that $|B| \geq 2$, then there exists a C such that $C \subsetneq B$ and $\nu'(C) = 1 > \frac{1}{2}\nu'(B)$, so ν' is not monotone.

We show in the next claim that it is possible to guarantee a weaker version of monotonicity of the set function from the Monotonicity of $\{\succsim_\pi\}_{\pi \in \Pi}$: subsets of null events should also be null.

Claim 8. *Suppose $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a (not necessarily unique) partition-dependent expected utility representation by (u, ν). If $\{\succsim_\pi\}_{\pi \in \Pi}$ satisfies Monotonicity, then:*

$$E \subset F \subsetneq S \;\&\; \nu(F) = 0 \;\Rightarrow\; \nu(F \setminus E) = 0.$$

Proof. Suppose that there exist events E, F such that $E \subset F \subsetneq S$ and $\nu(F \setminus E) > \nu(F) = 0$. Since $\nu(F) = 0$, by non-degeneracy of ν, we have $\nu(F^{\complement}) > 0$. Let $p, q, s \in \Delta X$ be such that $u(p) > u(q) = u(s)$. Then $u(s)[\nu(F) + \nu(F^{\complement})] = u(p)\nu(F) + u(q)\nu(F^{\complement})$, so by Monotonicity, we should have $u(s)[\nu(F \setminus E) + \nu(F^{\complement})] \geq u(p)\nu(F \setminus E) + u(q)\nu(F^{\complement})$. However the latter inequality is not possible, since $u(p) > u(q) = u(s)$ and $\frac{\nu(F \setminus E)}{\nu(F \setminus E) + \nu(F^{\complement})} > 0$, a contradiction. $\square$

In the next Claim, we prove the sufficiency of the Monotonicity axiom for the existence of a monotone representation in Theorem 4, given Event Reachability and the uniqueness of the set function up to scalar multiples.

Claim 9. *Suppose $\{\succsim_\pi\}_{\pi \in \Pi}$ admits a unique partition-dependent expected utility representation by (u, ν). If $\{\succsim_\pi\}_{\pi \in \Pi}$ satisfies Monotonicity, then ν is monotone.*

Proof. We prove the contrapositive. Suppose that ν is not monotone. Then there exist events E, F such that $E \subset F \subsetneq S$ and $\nu(F \setminus E) > \nu(F)$. By Claim 8, we can assume that $\nu(F) > 0$. We also have that $\nu(F^{\complement}) > 0$, because otherwise by Claim 8, any subevent of $F^{\complement}$ is null, hence F and $F \setminus E$ are nonnull events that can not be linked by sequentially disjoint non-events, contradicting Monotonicity.

Let $p, q, s \in \Delta X$ be such that $u(p) > u(q)$ and

$$s = \frac{\nu(F)}{\nu(F) + \nu(F^{\complement})} p + \frac{\nu(F^{\complement})}{\nu(F) + \nu(F^{\complement})} q.$$

Then $u(s)[\nu(F) + \nu(F^{\complement})] = u(p)\nu(F) + u(q)\nu(F^{\complement})$, so by Monotonicity, we have $u(s)[\nu(F \setminus E) + \nu(F^{\complement})] \geq u(p)\nu(F \setminus E) + u(q)\nu(F^{\complement})$. Together with $u(p) > u(q)$, these imply:

$$\frac{\nu(F)}{\nu(F) + \nu(F^{\complement})} \geq \frac{\nu(F \setminus E)}{\nu(F \setminus E) + \nu(F^{\complement})},$$

a contradiction to $\nu(F \setminus E) > \nu(F)$ and $\nu(F^{\complement}) > 0$. □

A.5 Proof of Proposition 5

We maintain the notation from the proof of Theorem 4. Specifically, recall Claim 8, which guarantees that subsets of null events are null.

Given the existence of a partition-dependent expected utility representation, Strict Admissibility is equivalent to to all nonempty events being nonnull. The "if" part is immediate. We proceed contrapositively to prove the "only if" part. Let $\{\succsim_\pi\}_{\pi \in \Pi}$ be represented by (u, ν). Now suppose that there is a nonempty null event E. By nondegeneracy of ν, $E \neq S$ and $E^{\complement}$ is nonnull. By Claim 8, all subevents of E are null. If there is an event B such that $E^{\complement} \subset B \subsetneq S$, then B is nonnull by Claim 8. Hence $E^{\complement}$ and B are two nonnull events that are not linked by sequentially disjoint nonnull sets, so Event Reachability fails. If there is no such event B, then since $|S| \geq 3$, $E^{\complement}$ must consist of at least two elements. In this case, let $E^{\complement} = E_1 \cup E_2$, where E_1 and E_2 are nonempty and disjoint. Then $\{E_1, E_2, E\}$ is a partition of S where E is null, so one of the other two events, say E_i, is nonnull by nondegeneracy of ν. But then $E^{\complement}$ and E_i are two nonnull events that are not linked by sequentially disjoint nonnull sets, so again Event Reachability fails.

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