

# Economics 202A

## Lecture Outline #3 (version 1.0)

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### Steady State of the Ramsey-Cass-Koopmans Model

In the last few lectures we have seen how to set up the Ramsey-Cass-Koopmans Model in discrete time, and with an infinite horizon. We have reviewed the solution method for (bivariate) difference equation systems and derived an exact solution to the model in a special case. I have not yet developed the main qualitative implications of the model however. These can perhaps be drawn out most conveniently by defining the model's steady state (or balanced growth allocation) and linearizing the model around that long-run destination. As we saw, when  $f''(k) = 0$  as in the last lecture, the model has no steady state in  $c$  and  $k$ , but in the customary case with  $f''(k) < 0$  a well-defined steady state exists. It is instructive to examine its properties.

Let  $\bar{c}$  and  $\bar{k}$  denote the steady-state values. Then they must satisfy the intertemporal Euler equation

$$u'(\bar{c}) = \beta [1 + f'(\bar{k}) - \delta] u'(\bar{c}),$$

which is equivalent to

$$f'(\bar{k}) = \frac{1 - \beta}{\beta} + \delta. \quad (1)$$

Intuitively, this states that the *net* marginal product of steady-state capital,  $f'(\bar{k}) - \delta$ , equals the rate of pure time preference.<sup>1</sup>

Steady state values must also ensure that

$$\bar{k} = \frac{f(\bar{k}) + (1 - \delta)\bar{k} - \bar{c}}{1 + n},$$

or, solving for  $\bar{c}$ , that

$$\bar{c} = f(\bar{k}) - (n + \delta)\bar{k}. \quad (2)$$

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<sup>1</sup>If  $\beta = 1/(1 + \theta)$ , then we call  $\theta$  the rate of pure time preference. With this notation eq. (1) becomes

$$f'(\bar{k}) = \theta + \delta.$$

Differentiation of eq. (2) shows that per capita consumption is maximized when the constant capital stock equals  $k^*$ , where

$$f'(k^*) = n + \delta. \quad (3)$$

This is the “golden rule” point where the marginal product of capital just equals replacement needs. Because the model assumed  $(1+n)\beta < 1$ , however, it follows that

$$n < \frac{1 - \beta}{\beta}.$$

Because  $f''(k) < 0$ , a comparison of (3) with (1) shows that

$$\bar{k} < k^*.$$

Given the consumer optimization assumed in this model, there is no possibility of a dynamically inefficient steady state with capital over-accumulation. This is one difference compared to the Solow model.

An excellent exercise is to linearize this model in the neighborhood of  $(\bar{c}, \bar{k})$  and investigate its dynamic properties, in particular showing that the two characteristics roots are, respectively, greater than and less than 1. The first question on Problem Set 2 involves an example like that one, so I will not pursue the linearization here. Instead, I will look in greater detail at the continuous-time version of this model, using that as a springboard to a discussion of optimal control theory.

### The Ramsey-Cass-Koopmans Model in Continuous Time

The first tool we need is compound interest. Please bear with me if this is familiar ground.

Suppose you invest \$1 for a year at 3% interest, compounded annually. After a year you will have

$$\left(1 + \frac{.03}{1}\right)^1 = \$1.03.$$

What if, instead, interest is compounded every six months? In other words, the interest plus principal accrued after six months is reinvested for another six months at a 3% annualized rate. In that case, after a year you will have (approximately)

$$\left(1 + \frac{.03}{2}\right)^2 \approx \$1.030225.$$

Clearly you get more money if interest is compounded periodically. and the more frequently it is compounded, the more you earn. For example, with quarterly compounding you end the year holding

$$\left(1 + \frac{.03}{4}\right)^4 \approx \$1.03033.$$

Weekly compounding gets you yet more, about \$1.03044.

The numbers are approaching a limit as the time intervals become progressively finer. That limit is the (hypothetical) case of continuous compounding, which yields

$$\lim_{h \rightarrow 0} (1 + .03h)^{1/h} \equiv e^{.03} \approx \$1.03045.$$

With this background, let's imagine time is measured in intervals of length  $h$ , so that  $t = 0, h, 2h, 3h, 4h$ , etc. The intertemporal objective to be maximized is now

$$\sum_{t=0}^{\infty} (1 + nh)^{t/h} (1 + \theta h)^{-t/h} h u(c_t),$$

where we multiply  $u(c_t)$  by  $h$  on the theory that the flow of utility from consumption at rate  $c_t$  over a period of length  $h$  is proportional to the length of the period. In the limit as  $h \rightarrow 0$ , this objective takes the form of an integral,

$$\int_0^{\infty} e^{-(\theta-n)t} u[c(t)] dt$$

Maximization is carried out subject to the constraint

$$k_{t+h} = \frac{hf(k_t) + k_t - \delta h k_t - hc_t}{1 + nh}.$$

Above,  $f(k_t)$  is interpreted as the rate of output flow and  $\delta$  as the rate of depreciation (per unit time).

The Lagrangian for the optimization problem is

$$\mathcal{L} = \sum_{t=0}^{\infty} \left(\frac{1 + nh}{1 + \theta h}\right)^{t/h} \{hu(c_t) + \lambda_t [hf(k_t) + k_t - \delta h k_t - hc_t - (1 + nh)k_{t+h}]\}.$$

First-order conditions are

$$\frac{\partial \mathcal{L}}{\partial c_t} = \left( \frac{1+nh}{1+\theta h} \right)^{t/h} h [u'(c_t) - \lambda_t] = 0,$$

$$\frac{\partial \mathcal{L}}{\partial k_{t+h}} = - \left( \frac{1+nh}{1+\theta h} \right)^{t/h} (1+nh)\lambda_t + \left( \frac{1+nh}{1+\theta h} \right)^{(t+h)/h} \lambda_{t+h} [hf'(k_{t+h}) + 1 - \delta h] = 0.$$

These can be simplified to read

$$u'(c_t) = \lambda_t \tag{4}$$

(a familiar condition) and

$$\frac{\lambda_{t+h} - \lambda_t}{h} = \frac{\lambda_t [\theta + \delta - f'(k_{t+h})]}{1 + hf'(k_{t+h}) - \delta h}.$$

Going to the limit of continuous time (and assuming that  $\lambda(t)$  has a right-hand derivative) gives us

$$\lim_{h \rightarrow 0} \frac{\lambda_{t+h} - \lambda_t}{h} = \dot{\lambda}(t) = \lambda(t) \{ \theta + \delta - f'[k(t)] \}. \tag{5}$$

Because the accumulation equation can be rewritten as

$$\frac{k_{t+h} - k_t}{h} = \frac{f(k_t) - (n + \delta)k_t - c_t}{1 + nh},$$

its continuous-time limit is

$$\dot{k}(t) = f[k(t)] - c(t) - (n + \delta)k(t). \tag{6}$$

Also necessary is the transversality condition

$$\lim_{t \rightarrow \infty} e^{-(\theta-n)t} u'[c(t)] k(t) = 0.$$

To reduce this all to a differential equation system in  $c$  and  $k$ , note that because  $u'(c_t) = \lambda_t$ ,

$$\dot{\lambda}(t) = u''[c(t)] \dot{c}(t).$$

By condition (5),  $u''[c(t)] \dot{c}(t) = u'[c(t)] \{ \theta + \delta - f'[k(t)] \}$ , or

$$\frac{\dot{c}(t)}{c(t)} = -\frac{u'[c(t)]}{c(t)u''[c(t)]} \{f'[k(t)] - (\theta + \delta)\}.$$

This equation and eq. (6) constitute the desired system in  $c$  and  $k$ . To simplify the notation, I assume that  $u(c)$  has the isoelastic form  $u(c) = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$ , in which case the last equation becomes

$$\frac{\dot{c}(t)}{c(t)} = \sigma \{f'[k(t)] - (\theta + \delta)\}. \quad (7)$$

In this equation (which is the continuous-time analog of the dynamic Euler equation),  $\sigma$ , as noted earlier in this course, is the *intertemporal elasticity of substitution*. Together with eq. (6), eq. (7) defines the system's potential dynamic paths.

The parameter  $\sigma$  helps determine the response of consumption to changes in the interest rate (or marginal product of capital). When  $\sigma$  is high, marginal utility declines more gently when consumption rises. Thus, for example, if  $f'[k(t)] - \delta$  rises above  $\theta$ ,  $c(t)$  will fall sharply, making  $\frac{\dot{c}(t)}{c(t)}$  strongly positive. We will look more closely at the relationship among  $\sigma$ , saving, and interest rates later in the course.

The system consisting of (7) and (6) has a useful phase diagram representation; I borrow the diagrams from David Romer's book. The steady state point in this diagram is a *saddle point*; given  $k(0)$ , there is a unique value of  $c(0)$  that places the economy on the unique convergent *saddle path*. In the infinite-horizon case, the saddle path defines the unique equilibrium consumption level.

Why? Given a  $k(0)$ . imagine that we start off at  $c(0)$  *above* the saddle. Then the dynamics of the system will cause the economy to crash into the  $y$ -axis, as  $k$  goes to zero. At that point  $c$  must drop abruptly to zero, which cannot be optimal. What if, given  $k(0)$ ,  $c(0)$  starts off *below* the saddle. Notice that the solution for eq. (7) is:

$$c(t) = c(0)e^{\int_0^t \sigma \{f'[k(s)] - (\theta + \delta)\} ds}.$$

(Why? Be sure you know how to differentiate this expression.) Then

$$\begin{aligned} \lim_{t \rightarrow \infty} e^{-(\theta-n)t} u'[c(t)] k(t) &= \lim_{t \rightarrow \infty} e^{-(\theta-n)t} c(0)^{-\frac{1}{\sigma}} e^{-\int_0^t \{f'[k(s)] - (\theta + \delta)\} ds} k(t) \\ &= c(0)^{-\frac{1}{\sigma}} \lim_{t \rightarrow \infty} k(t) e^{-\int_0^t \{f'[k(s)] - (n + \delta)\} ds}. \end{aligned}$$

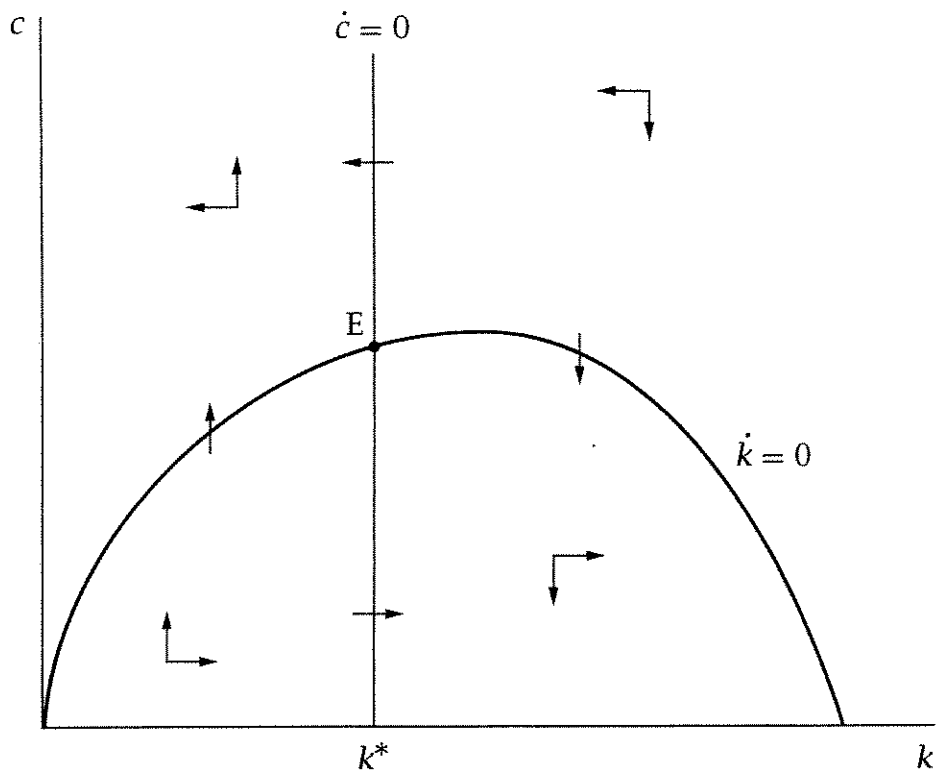


FIGURE 2.3 The dynamics of  $c$  and  $k$

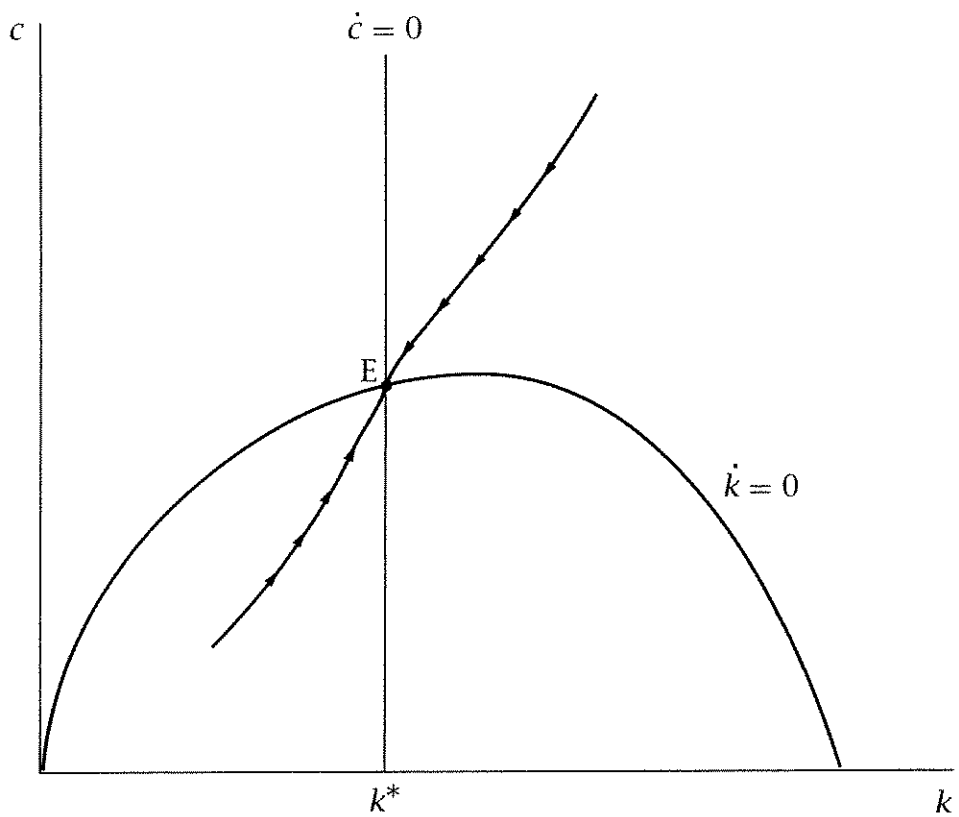


FIGURE 2.5 The saddle path

This limit is zero along the saddle path (because  $f'(k) \rightarrow \theta + \delta > n + \delta$ ), but along the rising capital, declining consumption paths,  $f'(k)$  eventually falls below  $n + \delta$  and the previous limit therefore is strictly positive. Thus, the transversality condition is violated.

Perhaps surprisingly, the diagram can also be used to study the finite-horizon ( $T < \infty$ ) case, but in that situation the equilibrium path looks different. Equations (7) and (6) still apply, but the terminal condition is  $k(T) = 0$ . Thus, the economy appears to be on an unstable path of the diagram, with  $c(0)$  determined by the requirement that  $k$  hit 0 exactly at time  $T$ . Obviously,  $c(0)$  will always be above the saddle path in a finite-horizon problem. But the bigger is  $T$ , the closer is  $c(0)$  to its saddle value and the closer does the economy draw to  $(\bar{c}, \bar{k})$  before veering off toward  $k(T) = 0$  relatively shortly before the terminal time  $T$ .

### Exogenous Growth

We can easily add to the model an exogenous rate of labor-augmenting technological advance,  $g = \dot{A}/A$ . For an individual the intertemporal Euler equation is the same. However, if we define consumption per efficiency labor unit as  $c^E \equiv c/A$ , then we have

$$\begin{aligned} \frac{\dot{c}^E}{c^E} &= \sigma \{ f'(k^E) - (\theta + \delta) \} - g \\ &= \sigma \left\{ f'(k^E) - \left( \theta + \delta + \frac{g}{\sigma} \right) \right\}. \end{aligned}$$

In addition, from  $\dot{k}(t) = f[k(t)] - c(t) - (n + \delta)k(t)$  and  $k^E \equiv k/A$  we derive

$$\begin{aligned} \frac{\dot{k}^E}{k^E} &= \frac{\dot{k}}{k} - g = \frac{F(K/N, A)}{K/N} - \frac{C/N}{K/N} - (n + \delta) - g \\ &= \frac{f(k^E)}{k^E} - \frac{c^E}{k^E} - (n + \delta + g), \end{aligned}$$

so that

$$\dot{k}^E = f(k^E) - c^E - (n + \delta + g)k^E.$$

Comparing the last two dynamic equations with (7) and (6), we see that there is a balanced growth path (steady state)  $(\bar{c}^E, \bar{k}^E)$  where  $c$  and  $k$  both grow at rate  $g$ . As  $g$  is determined outside the model, we have exogenous growth, just as in the Solow model.

## Optimal Control and the Maximum Principle

The previous model illustrates a more general approach to solving continuous-time dynamic optimization problems. That approach is called the *maximum principle*. You will see that it gives precisely the same answers we derived above by passing to the continuous-time limit.

A generic problem would be to maximize with respect to the path of the control variable  $c(t)$  the objective function

$$\int_0^{\infty} e^{-\rho t} v [c(t), k(t)] dt$$

subject to the transition equation for the state variable,  $k, \dot{k}(t) = G [c(t), k(t)]$ , with  $k(0)$  given. (In general  $c$  and  $k$  can be vectors of controls and states.)

To implement it, set up the (present-value) *Hamiltonian*

$$H [c(t), k(t), \lambda(t)] = e^{-\rho t} (v [c(t), k(t)] + \lambda(t) \{G [c(t), k(t)]\}).$$

The multiplier  $\lambda$  is called the costate variable (it is a vector if  $k$  is a vector). It has an interpretation as the “shadow price” of the stock  $k$ , making  $H$  the present shadow value of the flow of consumption plus stock accumulation. Necessary conditions for an optimum are as follows.

- Optimality of the control:

$$\frac{\partial H}{\partial c} = e^{-\rho t} (v_c + \lambda G_c) = 0.$$

- Equation of motion for the costate variable:

$$\frac{d}{dt} e^{-\rho t} \lambda = -\frac{\partial H}{\partial k}.$$

- Equation of motion for the state variable:

$$\dot{k} = G [c(t), k(t)].$$

- Initial condition  $k(0)$ .

- Transversality condition:

$$\lim_{t \rightarrow \infty} e^{-\rho t} \lambda(t) k(t) = 0.$$



The first of these conditions implies, of course, that

$$v_c + \lambda G_c = 0,$$

while the second can be expressed as

$$\dot{\lambda} = \rho\lambda - (v_k + \lambda G_k),$$

or, more intuitively, as the arbitrage formula

$$\frac{\dot{\lambda} + v_k + \lambda G_k}{\lambda} = \rho.$$

An interpretation of the latter differential equation comes from its integral solution,

$$\lambda(t) = \int_t^\infty e^{-\rho(s-t)} \{v_k [c(s), k(s)] + \lambda(s)G_k [c(s), k(s)]\} ds,$$

which gives  $\lambda$  as a present discounted value of future marginal returns to  $k$ .

Let us ask how to fit the (discounted) Ramsey problem into this framework. The Hamiltonian is

$$H [c(t), k(t), \lambda(t)] = e^{-(\theta-n)t} (u [c(t)] + \lambda(t) \{f [k(t)] - c(t) - (n + \delta)k(t)\}).$$

The first-order condition with respect to the control  $c$  is

$$\frac{\partial H}{\partial c} = 0 \iff u'(c) - \lambda = 0.$$

The dynamic equation for the costate variable  $\lambda$  is

$$\begin{aligned} \dot{\lambda} &= (\theta - n)\lambda - \lambda G_k \\ &= (\theta - n)\lambda - \lambda [f'(k) - (n + \delta)] \\ &= \lambda [\theta + \delta - f'(k)]. \end{aligned}$$

These correspond exactly to eqs. (4) and (5) above.