Economics 202A

Problem Set #4

1. *An endogenous growth model based on human capital.* Consider an economy with a fixed labor force. Output per worker is given by

$$y = Ak^{\alpha}(uh)^{1-\alpha}$$

where k is physical capital (per worker), h is human capital (per worker), and $u \in [0,1]$ is the fraction of the human capital stock allocated to production of output. The rest of the human capital is used to produce new human capital, which depreciates at rate δ:

$$\dot{h} = B(1-u)h - \delta h.$$

Here, A and B are constant. The stocks k and h are predetermined state variables as, therefore, is their ratio,

$$\omega \equiv k/h.$$

The representative household maximizes

$$\int_0^{\infty} v[c(t)]e^{-\theta t}dt$$

subject to the preceding two equations and

$$\dot{k} = y - c - \delta k,$$

where $v(c) = (1-\sigma^{-1})c^{1-\sigma^{-1}}$ and σ is the intertemporal substitution elasticity.

(a) Show via the Maximum Principle that the intertemporal Euler equation for the household's consumption is

$$\frac{\dot{c}}{c} = \sigma\left[\alpha A u^{1-\alpha}\omega^{-(1-\alpha)} - \delta - \theta\right].$$

(b) A second control variable in this optimization problem is u. Define $\chi \equiv c/k$. Show that the Euler equation for u has the form

$$\frac{\dot{u}}{u} = -\chi + Bu + B\left(\frac{1-\alpha}{\alpha}\right).$$

(c) Define $z \equiv Au^{1-\alpha}\omega^{-(1-\alpha)}$. Use the $\dot{c}/c$ and $\dot{k}$ equations above to conclude:

$$\frac{\dot{\chi}}{\chi} = (\alpha\sigma - 1)z + \chi - [\sigma\theta + (\sigma-1)\delta].$$

(d) Recalling that $\omega = k/h$, show that

$$\frac{\dot{\omega}}{\omega} = z - \chi - B(1-u).$$

(e) Use this last equation and the equation for $\dot{u}/u$, together with the definition of z, to derive:

$$\frac{\dot{z}}{z} = (1-\alpha)\left(\frac{B}{\alpha} - z\right).$$

(f) Suppose we considered the differential equation system consisting of the preceding equations of motion for the three variables z, χ, and u. This (self-contained) system is enough to describe the economy. To see why, note that, in effect, the system is allowing us to track χ, u, and $\omega = u(A/z)^{1/(1-\alpha)}$. But at any time, h and k are given by past investment and education decisions, and so $\omega = k/h$ is also a predetermined state variable. Thus, from the model-implied initial value of $\chi(0) = c(0)/k(0)$ we can infer $c(0)$, along with $u(0)$, and thereby track c, u, h, and k.

In a steady state, there is a constant fraction of labor in manufacturing (u), a constant ratio of consumption to capital (c/k), and a constant ratio of physical to human capital (ω). Find the steady state values $\bar{z}$, $\bar{\chi}$, and $\bar{u}$ from the preceding differential equations [and notice that $\bar{\omega} = \bar{u}(A/\bar{z})^{1/(1-\alpha)}$].

(g) Because y, c, k, and h all rise together over time, we have endogenous growth. Using the consumption Euler equation calculate the steady state growth rate of these variables. What is the intuition behind the solution? [Hint: Think back to the Solow model.]

(h) Linearize the system in z, χ, and u around the steady state of part (f), and calculate its characteristic roots, showing that one is negative and two are positive. Is this what you expected? Why or why not?