Econ 202A, 10/30/2012 Guest Lecture

Professor Romer covered already consumption theory under these assumptions:

- certainty, constant r
- uncertainty about labor income, constant r

Today we do a third case: certainty and variable r.

Adding uncertainty takes us into asset pricing, which Professor Romer will cover later.

Unlike the book, I will look at the case of continuous time.

I will use two solution methods, one analogous to section 8.1 and one based on optimal control theory, and show they give the same answer.

Notation

T: time horizon
$Y(t)$: labor income
$C(t)$: consumption
$A(t)$: financial (non-human capital) wealth
$r(t)$: instantaneous rate of interest

The individual problem is to maximize

$$\int_0^T \mathrm{e}^{-\delta t} u[C(t)] \mathrm{d}t,$$

where $u'(C) > 0$, $u''(C) < 0$, $\lim_{t \to 0} u'(C) = \infty$, subject to

$$\dot{A}(t) = r(t)A(t) + Y(t) - C(t),$$

where the individual takes $A(0)$ and the paths $\{r(t)\}$, $\{Y(t)\}$ as given.

Parallel to what was done in the (discrete-time) setting of the government budget constraint earlier in the course, we work with the constraint that the PDV of consumption $\leq$ PDV of lifetime resources. Define the market discount factor as

$$R(t) \equiv \int_0^t r(\tau)\mathrm{d}t$$

(so that $\mathrm{e}^{-R(t)}$ is the present value at time 0 of a unit of output available at time t). Then the lifetime budget constraint is

$$\int_0^T \mathrm{e}^{-R(t)} C(t)\mathrm{d}t \leq A(0) + \int_0^T \mathrm{e}^{-R(t)} Y(t)\mathrm{d}t. \qquad (*)$$

Approach I: Calculus

We know it is optimal for the budget constraint to hold with equality. Set up the Lagrangean

$$\mathcal{L} = \int_0^T \mathrm{e}^{-\delta t} u[c(t)]\mathrm{d}t + \lambda \left[A(0) + \int_0^T \mathrm{e}^{-R(t)} Y(t)\mathrm{d}t - \int_0^T \mathrm{e}^{-R(t)} C(t)\mathrm{d}t \right].$$

The first-order condition for $C(t)$ is

$$\mathrm{e}^{-\delta t} u'[C(t)] = \lambda \mathrm{e}^{-R(t)}$$

where I have canceled the $\mathrm{d}t$ that multiplies both sides.

Note that λ is *not* a function of time here because there is a single constraint, which depends on the PDV of lifetime resources and not its particular time-path. And notice that only if $R(t) = \delta t$ for every t is the path of consumption going to be flat.

Since

$$u'[C(t)] = \lambda \mathrm{e}^{\delta t - R(t)},$$

we can write

$$\ln u'[C(t)] = \ln \lambda + \delta t - R(t) = \ln \lambda + \delta t - \int_0^t r(\tau) \mathrm{d}t$$

and so we may differentiate using the chain rule and the fundamental theorem of calculus to get

$$\frac{u"[C(t)]}{u'[C(t)]}\dot{C}(t) = \delta - r(t) \Rightarrow$$

$$\frac{\dot{C}(t)}{C(t)} = \left\{ -\frac{C(t)u"[C(t)]}{u'[C(t)]} \right\}^{-1} [r(t) - \delta] . \qquad (**)$$

We saw this same relationship in the Cass-Koopmans-Ramsey model. Recall that

$$\begin{aligned} -\frac{C(t)u''[C(t)]}{u'[C(t)]} &= \text{coefficient of relative risk aversion} \\ &= \text{inverse of intertemporal substitution elasticity.} \end{aligned}$$

When this coefficient is big, the utility function is more sharply curved, meaning that the marginal utility of consumption drops off quickly as consumption rises.

So consumption is:

- rising whenever $r > \delta$
- falling whenever $r < \delta$
- stationary whenever $r = \delta$ (this last being our old result).

The response of consumption growth to a deviation between r and δ depends on the curvature of the utility function. It is smaller when the utility function is more highly curved (low intertemporal substitution elasticity).

Of course, at an optimum, lifetime budget constraint $(*)$ has to hold as an *equality*. Using that PDV budget constraint and the equation for $\dot{C}$ just derived, one could solve explicitly for $C(0)$. In general the solution is messy, but I will derive it later in a simplified special (infinite-horizon) case.

It is worth making explicit something that may or may not be obvious to you already, just in case it is not. The way we derive $(*)$ from the flow constraint $\dot{A}(t) = r(t)A(t) + Y(t) - C(t)$ is by integrating it forward in time to solve for $A(t)$ (in analogy to the iterative forward substitution procedure we followed in discrete time). The solution is

$$A(t) = \int_0^t [Y(s) - C(s)]\, \mathrm{e}^{R(t-s)}\mathrm{d}s + \mathrm{e}^{R(t)}A(0),$$

where $R(t-s) \equiv \int_s^t r(\tau)\mathrm{d}\tau$, as you can check by differentiating.Multiply this through by $\mathrm{e}^{-R(t)}$, rearrange, and conclude that

$$\int_0^t C(s)\mathrm{e}^{-R(s)}\mathrm{d}s + \mathrm{e}^{-R(t)}A(t) = A(0) + \int_0^t Y(s)\mathrm{e}^{-R(s)}\mathrm{d}s.$$

Setting $t = T$, we see that the constraint that $A(T) \geq 0$ is the same as constraint $(*)$. Furthermore, requiring that constraint $(*)$ holds as an equality is exactly the same as requiring that $A(T) = 0$.

Aside on the infinite-horizon case: As you saw earlier, applying the no-Ponzi game condition that

$$\lim_{t\to\infty} \mathrm{e}^{-R(t)}A(t) \geq 0$$

leads to equation $(*)$ above with $T = \infty$, which states that the PDV of C cannot exceed the PDV of Y plus initial financial assets $A(0)$. (Recall Chapter 2.2 of Romer, *Advanced Macroeconomics*.)

Approach II: Maximum Principle

You have seen this before. The optimal control problem is to find

$$\max_{\{C(t)\}} \int_0^T \mathrm{e}^{-\delta t}u[C(t)]\mathrm{d}t,$$

subject to the same "givens" as before and

$$\begin{aligned} \dot{A}(t) &= r(t)A(t) + Y(t) - C(t), \\ A(T) &\geq 0, \end{aligned}$$

where the second inequality constraint states that you cannot plan to die owing money.

We write the Hamiltonian as:

$$\mathcal{H}[C(t), A(t), \mu(t)] = \mathrm{e}^{-\delta t} \left\{ u[C(t)] + \mu(t) \left[r(t)A(t) + Y(t) - C(t) \right] \right\}.$$

As we saw earlier, the costate variable $\mu(t)$ can be interpreted as the marginal contribution of the state variable (financial wealth) to lifetime utility, discounted to date t, whereas $\mathrm{e}^{-\delta t}\mu(t)$ is the same value discounted to the initial time 0.

Necessary conditions for an optimum are:

- optimality of the control C: $\dfrac{\partial \mathcal{H}}{\partial C} = 0, \, \forall t \Leftrightarrow u'[C(t)] = \mu(t), \, \forall t$
- equation of motion for the costate: $\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{-\delta t}\mu(t)\right] = -\dfrac{\partial \mathcal{H}}{\partial A}, \, \forall t \Leftrightarrow \dfrac{\dot{\mu}(t)}{\mu(t)} = \delta - r(t), \, \forall t$
- terminal necessary condition: $\mu(T)A(T) = 0$.

The first of these three equations says that if I raise consumption by a little bit at time t and reap $u\prime[C(t)]$, the opportunity cost is the marginal value of wealth at that same time t, $\mu(t)$. The second equation is actually the intertemporal Euler equation (in continuous time). If I reduce consumption and raise saving on date 0 by $\mathrm{d}C$, thereby forgoing $u'[C(0)]\mathrm{d}C$, then if I am optimizing, it does not matter at what time t I choose to consume the proceeds of my saving (including interest), thereby reaping $\mathrm{e}^{-\delta t}u'[C(t)]\mathrm{e}^{R(t)}\mathrm{d}C$. In particular, then, $\mathrm{e}^{-\delta t}u'[C(t)]\mathrm{e}^{R(t)}$ is constant for all t and equal to $u'[C(0)]$. As a result,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathrm{e}^{R(t)-\delta t}u'[C(t)]\right\} = 0,$$

and if we denote $u'[C(t)] = \mu(t)$, the result is

$$\frac{\dot{\mu}(t)}{\mu(t)} = \delta - r(t),$$

as derived above. As for the third necessary condition, we saw how to derive it in discrete time earlier using the Kuhn-Tucker theorem, and unsurprisingly, it holds in continuous time as well. Unless the terminal marginal value of wealth, $\mu(T)$, is zero, it can never be optimal to die holding positive wealth – one should plan to drive terminal wealth $A(T)$ down precisely to zero. As I pointed out above, this means that constraint $(*)$ must hold as an equality.

In the infinite-horizon setting, the analogous terminal condition to $\mu(T)A(T) = 0$ (recall the Cass-Koopmans-Ramsey model) is the transversality condition that $\lim_{t\to\infty} \mathrm{e}^{-\delta t}\mu(t)A(t) = 0$. The intertemporal Euler equation, however, tells us that $u'[C(0)] = \mathrm{e}^{R(r)-\delta t}u'[C(t)] = \mathrm{e}^{R(t)-\delta t}\mu(t)$, $\forall t$, so we may express the transversality condition equivalently as

$$u'[C(0)] \lim_{t\to\infty} \mathrm{e}^{-R(t)}A(t) = 0 \Leftrightarrow \lim_{t\to\infty} \mathrm{e}^{-R(t)}A(t) = 0.$$

We can conclude that whereas the no-Ponzi game condition imposes the constraint that $\lim_{t\to\infty} \mathrm{e}^{-R(t)}A(t) \geq 0$, the transversality condition tells us that this constraint will have to bind at an optimum. Thus, also for the case $T \to \infty$, constraint $(*)$ must hold as an equality at an optimum.

Observe finally that $\dfrac{\dot{\mu}(t)}{\mu(t)} = \dfrac{u''[C(t)]}{u'[C(t)]}\dot{C}(t)$. Thus, our two approaches yield precisely the same result.

An Infinite-Horizon Example

Let us take

$$u(C) = \frac{C^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$$

and also assume that r is fixed, but not necessarily equal to δ. We also let $T \to \infty$. Then the lifetime budget constraint (which we can assume will hold as an equality at an optimum) is:

$$\int_0^\infty \mathrm{e}^{-rt}C(t)\mathrm{d}t = A(0) + \int_0^\infty \mathrm{e}^{-rt}Y(t)\mathrm{d}t.$$

Optimal consumption, we know follows

$$\frac{\dot{C}(t)}{C(t)} = \sigma\left(r - \delta\right).$$

How can we calculate $C(0)$ in this case? (C won't be constant unless $r = \delta$.) Notice that the preceding differential equation has the solution

$$C(t) = C(0)\mathrm{e}^{\sigma(r-\delta)t}.$$

Substitution into the preceding intertemporal budget constraint gives

$$\begin{aligned}
\int_0^\infty \mathrm{e}^{-rt}C(0)\mathrm{e}^{\sigma(r-\delta)t}\mathrm{d}t &= C(0)\int_0^\infty \mathrm{e}^{[(\sigma-1)r-\sigma\delta]t}\mathrm{d}t \\
&= A(0) + \int_0^\infty \mathrm{e}^{-rt}Y(t)\mathrm{d}t,
\end{aligned}$$

or, integrating the expression $\int_0^\infty \mathrm{e}^{[(\sigma-1)r-\delta]t}\mathrm{d}t$, assuming that $(\sigma - 1)\, r - \sigma\delta < 0$,

$$C(0) = [\sigma\delta - (\sigma - 1)r]\left[A(0) + \int_0^\infty \mathrm{e}^{-rt}Y(t)\mathrm{d}t\right].$$

Here we see very clearly the substitution, income, and wealth effects on consumption (and saving) of a change in the interest rate r.

But you are probably wondering: why did we have to assume that $(\sigma - 1)\, r - \sigma\delta < 0$? This inequality will always hold if $\sigma < 1$, but if $\sigma > 1$ and r is sufficiently higher than δ, it may not, in which case we cannot compute a finite value for the integral $\int_0^\infty \mathrm{e}^{[(\sigma-1)r-\sigma\delta]t}\mathrm{d}t$. The problem is that consumption optimally is growing at the proportional rate $\sigma(r - \delta)$ and if this exceeds r, so that $(\sigma - 1)\, r - \sigma\delta \geq 0$, then the present value of consumption is *not finite*: the budget constraint is therefore violated.

By investigating a little further this case in which $(\sigma - 1)\, r - \sigma\delta \geq 0$, we can gain an independent verification of our optimal consumption function, as well as some other insights. To simplify notation, let $Y(t) = Y$, a constant, for all t, so that the PDV of income is simply Y/r.

Let's assume that consumption grows at the arbitrary proportional rate $g < r$, so that the PDV of consumption is well defined.[1] I would like to figure out what level of lifetime utility this policy yields. In this case, date t consumption will be $C(t) = C(0)\mathrm{e}^{gt}$ and therefore from the budget constraint,

$$C(0) = (r - g)\left[A(0) + \frac{Y}{r}\right].$$

On the other hand, lifetime utility is

$$U(0) = \frac{C(0)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}\int_0^{\infty} \mathrm{e}^{-\delta t}\left(\mathrm{e}^{gt}\right)^{1-\frac{1}{\sigma}}\,\mathrm{d}t = \int_0^{\infty} \mathrm{e}^{\left[\left(1-\frac{1}{\sigma}\right)g-\delta\right]}\mathrm{d}t.$$

If also $\left(1-\frac{1}{\sigma}\right)g - \delta < 0$, this implies that lifetime utility is

$$U(0) = \frac{(r-g)^{1-\frac{1}{\sigma}}}{\left(1-\frac{1}{\sigma}\right)\left[\delta - \left(1-\frac{1}{\sigma}\right)g\right]}\left[A(0) + \frac{Y}{r}\right]^{1-\frac{1}{\sigma}}.$$

Take natural logs and differentiate with respect to g to get

$$\frac{\mathrm{d}\ln U(0)}{\mathrm{d}g} = -\left(1-\frac{1}{\sigma}\right)\frac{1}{r-g} + \left(1-\frac{1}{\sigma}\right)\frac{1}{\delta - \left(1-\frac{1}{\sigma}\right)g},$$

so that in the case $\sigma > 1$ that is of interest here,

$$\frac{\mathrm{d}\ln U(0)}{\mathrm{d}g} \propto \frac{(r-g) - \left[\delta - \left(1-\frac{1}{\sigma}\right)g\right]}{(r-g)\left[\delta - \left(1-\frac{1}{\sigma}\right)g\right]} = \frac{\sigma(r-\delta) - g}{\sigma\,(r-g)\left[\delta - \left(1-\frac{1}{\sigma}\right)g\right]},$$

where the symbol $\propto$ signifies "is proportional to." Notice first that setting $\frac{\mathrm{d}\ln U(0)}{\mathrm{d}g} = 0$ yields the implication that if it so happens that $\sigma(r-\delta) < r$, then the optimum growth rate of consumption – the one that maximizes $U(0)$ – is $g = \sigma(r-\delta)$, which we derived above. This is yet another way of deriving our earlier results.

[1] You should recall from equation (∗∗) that this will be true only if the utility function is in the CRRA class (equivalently, constant intertemporal substitution elasticity).

But if $\sigma(r-\delta) \geq r$, we have a problem. In that case, setting $g = \sigma(r-\delta)$ would violate the intertemporal budget constraint, so we cannot plan to have that high a growth rate of consumption. So what should the consumer do?

To see the answer, notice that when $\delta - \left(1 - \frac{1}{\sigma}\right) g > 0$ as we have assumed, the preceding proportionality shows that lifetime utility is *increasing* in g for any $g < r \leq \sigma(r-\delta)$. So in this case, *no optimum value of g exists.* In other words, for any value of g that is consistent with the intertemporal budget constraint, we can always increase utility by choosing g to be slightly higher.[2]

The lesson of this example is that when addressing a maximization problem, one cannot always take it for granted that a maximum actually exists! In the case $\sigma > 1$, the utility function $u(C)$ is not bounded above, so it is plausible there might be conditions in which the infinite-horizon maximization problem cannot be solved. In contrast, $u(C)$ is bounded above (by 0) when $\sigma < 1$, and in that case there is a well-defined maximum.

[2]Notice that if $g = \sigma(r-\delta) < r$, then $\left(1 - \frac{1}{\sigma}\right) g - \delta = (\sigma-1)(r-\delta) - \delta = (\sigma-1)r - \sigma\delta < 0$, so the utility integral necessarily converges.