OPTIMAL PORTFOLIOS AND THE MUTUAL FUND SEPARATION THEOREM

Set-up

An agent has wealth W. There's a riskless asset, which pays a return of zero for sure, and N risky assets. Asset N pays return r_i. The r's have means $\mu_1, \mu_2, \ldots, \mu_N$ and variance-covariance matrix Σ. Σ is N x N, and we denote its ij element by σ_{ij}.

Thus, the agent's consumption is $C = B + x_1(1 + r_1) + x_2(1 + r_2) + \ldots + x_N(1 + r_N)$, where B is the agent's holdings of the riskless asset and x_i is holdings of risky asset i. Using the budget constraint, $B + x_1 + x_2 + \ldots + x_N = W$, we can rewrite this as

$$C = W + x_1 r_1 + x_2 r_2 + \ldots + x_N r_N.$$

The agent's objective function is assumed to depend only on the mean and variance of consumption (and to be increasing in the mean and decreasing in the variance). One case where this arises is quadratic utility, as in class. Another case is constant absolute risk aversion utility and normally distributed returns, along the lines of Problem 8.14 in the book.

Since the individual cares only about the mean and variance of consumption, then the optimal allocation of the individual's wealth will have the lowest possible variance of consumption given its mean. (If not, it would be possible to lower the variance without changing the mean, which would make the agent better off.) Thus, rather the solving the full optimization problem, we will focus on minimizing variance for a given mean.

Case 1: Two risky assets, solved with algebra

With two risky assets, $C = W + x_1 r_1 + x_2 r_2$. Thus the mean of C is $C = W + x_1 \mu_1 + x_2 \mu_2$. To find the variance of C, note that $C - E[C] = (r_1 - \mu_1)x_1 + (r_2 - \mu_2)x_2$. Thus,

$$E[(C - E[C])^2] = E[((r_1 - \mu_1)x_1 + (r_2 - \mu_2)x_2)^2]$$

$$= x_1^2 E[(r_1 - \mu_1)^2] + x_2^2 E[(r_2 - \mu_1)^2] + 2x_1 x_2 E[(r_1 - \mu_1)(r_2 - \mu_2)]$$

$$= x_1^2 \sigma_{11} + x_2^2 \sigma_{22} + 2x_1 x_2 \sigma_{12}.$$

So the Lagrangian for the problem of minimizing variance subject to achieving some target level of expected consumption, Z, is

$$L = x_1^2\sigma_{11} + x_2^2\sigma_{22} + 2x_1x_2\sigma_{12} + \lambda[Z - (W + x_1\mu_1 + x_2\mu_2)].$$

The first-order conditions for x_1 and x_2 are

$$2x_1^*\sigma_{11} + 2x_2^*\sigma_{12} = \lambda\mu_1,$$

$$2x_2^*\sigma_{22} + 2x_1^*\sigma_{12} = \lambda\mu_2.$$

Solving these two linear equations for x_1^* and x_2^* gives us

$$(1) \quad x_1^* = \frac{\sigma_{22}\mu_1 - \sigma_{12}\mu_2}{\sigma_{11}\sigma_{22} - \sigma_{12}^2}\frac{\lambda}{2},$$

$$(2) \quad x_2^* = \frac{\sigma_{11}\mu_2 - \sigma_{12}\mu_1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2}\frac{\lambda}{2}.$$

Discussion

The Mutual Fund Separation Theorem

Notice what happens as λ changes – that is, as the agent puts more or less weight on the mean relative to the variance. x_1 and x_2 change in the same proportion. Thus, agents who differ in their attitudes toward risk will hold different amounts of the riskless asset, but their mix of the risky assets (that is, their ratio of x_1 to x_2) will be the same.

This is the Mutual Fund Separation Theorem. We can construct an optimal mix (that is, an optimal mutual fund) of risky assets. Depending on their risk preferences, agents will choose different combinations of the safe asset and the mutual fund; but they will not choose different mixes of risky assets.

An Attempt at Intuition for the Mutual Fund Separation Theorem

Consider a portfolio of risky assets. If the agent holds none of the portfolio and puts all his or her wealth into the safe asset, his or her mean consumption is W, and its standard deviation is zero. As the agent moves out of the riskless asset into the portfolio, both the mean and standard deviation of consumption change linearly with the amount invested in the portfolio. Thus, as the agent shifts out of the riskless asset into the portfolio, his or her mean consumption and its standard deviation move along a ray (in standard deviation of consumption–mean consumption space) from the point (0,W). Thus, each portfolio gives the agent access to a ray of points out of (0,W) in standard deviation–mean space. Every agent prefers to be on a higher ray than to be on a lower one. So every agent chooses the portfolio with the highest slope in this space – that is, the portfolio with the highest ratio of expected excess return (return minus that on the safe asset) to standard deviation. Agents' risk attitudes then determine where on the line they choose to be – that is, how much of the portfolio they hold.

<u>The Determinants of the Mix of the Two Assets</u>

Equations (1) and (2) also show what determines how much of the two assets the agent holds.

For example, suppose $\sigma_{12} = 0$. Then $x_1 = \lambda\mu_1/2\sigma_{11}$, $x_2 = \lambda\mu_2/2\sigma_{22}$. Thus whether the agent holds a positive, negative, or zero amount of the asset is determined by whether the asset's expected excess return is positive, negative, or zero. Holdings of an asset are proportional to its expected excess return, inversely proportional to its variance, and increasing in the importance the agent attaches to the mean of his or her consumption relative to its variance.

Consider also a small increase in σ_{12} starting from $\sigma_{12} = 0$. The marginal effect is to reduce the agent's holdings of asset 1 if his or her holdings of asset 2 are positive, and to increase his or her holdings of asset 1 if his or her holdings of asset 2 are negative.

<u>Case 2: N risky assets, solved with linear algebra</u>

Here $C = W + X'r$, where $X = [x_1\ x_2\ \ldots\ x_N]'$ and $r = [r_1\ r_2\ \ldots\ r_N]'$. Thus, $E[C] = W + X'\mu$ (where $\mu = [\mu_1\ \mu_2\ \ldots\ \mu_N]')$, and $Var(C) = X'\Sigma X$.

The Lagrangian is $L = X'\Sigma X + \lambda[Z - (W + X'\mu)]$. The derivative of $X'\Sigma X$ with respect to X is $(X'\Sigma)' + \Sigma X$, which equals $\Sigma'X + \Sigma X$, or $2\Sigma X$. (To see this, consider the derivative with respect to x_1. x1 appears twice in $X'\Sigma X$, and so there are two terms. The first term is $X'\Sigma$ times the vector $[1\ 0\ 0\ \ldots\ 0]$, which is X' times $[\sigma_{11}\ \sigma_{21}\ \ldots\ \sigma_{N1}]$, which is the first element of $\Sigma'X$ (and hence the first element of ΣX, since Σ is symmetric). The second term is the vector $[1\ 0\ 0\ \ldots\ 0]'$ times ΣX, which is $[\sigma_{11}\ \sigma_{12}\ \ldots\ \sigma_{1N}]'$ times X, which is also the first element of ΣX. Thus the derivative of $X'\Sigma X$ with respect to x_1 is 2 times the first element of ΣX.)

Thus, the Nx1 vector of first-order conditions is $2\Sigma X^* = \lambda\mu$, which implies

$$X^* = \frac{\lambda}{2}\Sigma^{-1}\mu.$$

The Mutual Fund Separation Theorem holds here: when λ changes, all the elements of X^* change by the same proportion. The intuition (such as it is) is the same as for the case of two assets.