

FORCES LIMITING THE EXTENT TO WHICH SOPHISTICATED INVESTORS ARE WILLING TO MAKE TRADES THAT MOVE ASSET PRICES BACK TOWARD FUNDAMENTALS

As described in lecture, researchers have identified three factors that limit the extent to which sophisticated investors are willing to buy assets that are undervalued relative to fundamentals, and sell (or sell short) assets that are overvalued relative to fundamentals: fundamental risk, noise-trader risk, and performance-based risk.¹ What follows is a simple model, based loosely on DeLong, Shleifer, Summers and Waldmann (1990) and Shleifer and Vishny (1997), that captures these ideas.

Note: Relative to the version in lecture, there are some changes that I hope are improvements. First, I've tried to make the notation a little more intuitive. Second, I've changed signs so that N is noise traders' demand rather than their supply. This avoids the need to carry around a lot of negative signs. Third, and most important, I've found a way of modeling performance-based risk that I like a little better than the version with momentum traders I covered in class. These notes also cover the case of momentum traders, however.

Assumptions

There are three periods, denoted 0, 1, and 2.

There are two assets. The first is a safe asset in perfectly elastic supply. For simplicity, its rate of return is normalized to zero. Thus, one unit of the economy's single good in period 0 can be invested in a way that yields one unit of the good for sure in period 1; likewise, one unit of the good invested in this asset in period 1 yields one unit for sure in period 2.

The second is a risky asset. Its payoff, which is realized in period 2, is $1 + F_1 + F_2$, where F_t is distributed normally with mean 0 and variance V_t^F . F_1 is observed in period 1, and F_2 is observed in period 2. For simplicity, this asset is in zero net supply. Thus equilibrium requires that the sum across agents of the quantity of the asset demanded is zero.

There are potentially three types of traders. The first type is intended to capture individual investors subject to fads. They are the source of the shocks that potentially move asset prices away from their fundamental values. The second type is intended to capture sophisticated private investors. They have rational expectations, but short horizons. The third type is intended to capture sophisticated institutional investors, such as hedge fund managers. They have longer horizons, but they face a cost associated with short-term losses. We can think of this as arising from the fact that they obtain funds from investors with less information about the determinants

¹ A potential fourth factor is "model-based risk": sophisticated investors cannot be certain that their estimates of fundamental values are in fact the best estimates given the available information. Modeling model-based risk raises deep and hard issues, so we will not pursue it.

of asset prices, and who therefore withdraw or add funds based on the short-term performance of the hedge fund's portfolio. We will not model this explicitly, however.

Specifically, the first group of traders buy or sell the risky asset for random reasons unrelated to anything else happening in the economy. Their actions convey no information about fundamental values, so if the forces pushing asset prices toward fundamentals were strong enough, the actions would not affect prices. These traders are referred to as "noise traders."

The noise traders demand quantity N_0 of the asset in period 0, and $N_0 + N_1$ in period 1, where N_t is distributed normally with mean 0 and variance V_t^N . (N_0 is simply an exogenous parameter, though we could easily model it as random.) F_1 , F_2 , and N_1 are independent.

The sophisticated, short-term traders maximize expected utility and have rational expectations. A_0 sophisticated traders are born in period 0, and A_1 are born in period 1, where A_0 and A_1 are exogenous and certain. We will refer to these traders as "sophisticated investors."

Sophisticated investors live for two periods (0 and 1 for the ones born in period 0; 1 and 2 for the ones born in period 1). They care only about consumption in the second period of their life, and have constant absolute risk aversion (CARA) utility, $U(C) = -e^{-2\gamma C}$, $\gamma > 0$. Each sophisticated investor has wealth W at birth. Sophisticated investors act as price-takers.

The sophisticated long-term traders are born in period 0 and care only about consumption in period 2. Like the sophisticated short-term traders, they have CARA utility, $U(C) = -e^{-2\gamma C}$, $\gamma > 0$; each one has wealth W at birth; and they are price-takers. There are A_H of them. They are referred to as "hedge fund managers."

The hedge fund managers participate in the market for the risky asset in period 0, and do not make any additional trades in period 1. However, they face a cost if they incur short-term losses, and gain a reward if they obtain short-term gains. Specifically, if a hedge fund manager purchases amount H of the risky asset, he or she receives $aH(P_1 - E_0[P_1])$ in period 1, where $a > 0$ and where P_t is the price of the risky asset in period t . The manager then holds this payment in the safe asset from period 1 to period 2, and so it adds to (or subtracts from) his or her period-2 consumption.

Preliminaries

1. Let S_t denote each sophisticated short-term investor's holdings of the risky asset in period t , and let H denote each hedge fund manager's holdings of the risky asset. (Because the hedge fund managers cannot trade in period 1, their asset holdings are the same in period 0 and period 1.) Then equilibrium in the market for the risky asset requires $N_0 + S_0A_0 + HA_H = 0$ in period 0, and $N_0 + N_1 + S_1A_1 + HA_H = 0$ in period 1. In period 2, the payoff to the asset is realized and the holders consume the proceeds; there are no trades.

2. What are fundamental values in this economy? If there were any risk-neutral agents who could buy and sell in unlimited quantities, the price of the risky asset would have to be 1 in period 0, and $1 + F_1$ in period 1. Call these prices P_0^* and P_1^* . If the price in period t were less than P_t^* ,

selling one unit of the safe asset and investing the proceeds in the risky asset would raise expected consumption; if the price were more than this, selling one unit of the risky asset and investing the proceeds in the safe asset would raise expected consumption.

In the absence of noise traders, the price of the risky asset would again be P_0^* and P_1^* in the two periods. At those prices, the expected rate of return on the risky asset would be the same as that on the safe asset. Thus, sophisticated investors and hedge fund managers would not want to hold either positive or negative quantities of the asset. Since the supply of the asset is zero, the market would clear.

Thus, fundamental values are $P_0^* = 1$ and $P_1^* = F_1$. Our question concerns departures of actual prices from these values.

3. Before proceeding, it is useful to say more about utility maximization. We will see that because the underlying shocks are normally distributed, each individual's consumption will be normally distributed. And recall the rule for the mean of a variable that is distributed lognormally: if x is distributed normally with mean μ and variance V , $E[e^x] = e^{\mu + V/2}$. Thus if C is distributed normally with mean $E[C]$ and variance $\text{Var}(C)$, the expectation of $e^{-2\gamma C}$ is $e^{-2\gamma E[C] + 2\gamma^2 \text{Var}(C)}$. To maximize expected utility, the individual will therefore want to make $-2\gamma E[C] + 2\gamma^2 \text{Var}(C)$ as small as possible. Equivalently, he or she will maximize $E[C] - \gamma \text{Var}(C)$.

Equilibrium in period 1 and "fundamental risk"

Many of the messages of the model are clearest if we start with the case where the hedge fund managers are absent ($H = 0$).

In this case, the condition for equilibrium in period 1 is $N_0 + N_1 + S_1 A_1 = 0$. The sophisticated investors born in period 1 care about consumption in period 2. The representative investor's period-2 consumption is his or her holdings of the safe asset, which are $W - P_1 S_1$, plus the product of his or her holdings of the risky asset, S_1 , and the payoff of each unit of the asset, $1 + F_1 + F_2$: $C = W - P_1 S_1 + S_1(1 + F_1 + F_2)$. In period 1, F_1 has already been realized (and investors can observe the price of the asset, P_1). Thus, the expectation of their consumption given period-1 information is $W - P_1 S_1 + S_1(1 + F_1)$, and its variance is $V_2^F S_1^2$. The problem of the representative sophisticated investor in period 1 is therefore

$$\max_{S_1} W - P_1 S_1 + S_1(1 + F_1) - \gamma V_2^F S_1^2.$$

The first-order condition for the investor's choice of S_1 is

$$-P_1 + (1 + F_1) - 2\gamma V_2^F S_1 = 0.$$

Market-clearing requires $S_1 = -(N_0 + N_1)/A_1$. Substituting this into the first-order condition and rearranging gives us:

$$(*) \quad P_1 - (1 + F_1) = \left(\frac{2\gamma}{A_1}\right) V_2^F (N_0 + N_1).$$

The left-hand side of this expression is the departure of the price of the asset from its fundamental value. Consider the three terms on the right-hand side of this expression:

- $N_0 + N_1$ is noise traders' demand for the asset. In the model, if agents enter the market and demand some of the asset for reasons unrelated to economic fundamentals, the price of the asset rises: without risk-neutral investors, prices can deviate from fundamentals.
- V_2^F is the variance of fundamentals in period 2. What deters the sophisticated investors from fully eliminating the mispricing is that the realized value of the asset may differ from its expected value – that is, there is “fundamental risk.” That is, *fundamental risk prevents sophisticated investors from taking infinite positions (and thereby eliminating departures of prices from fundamentals)*.
- $A_1/(2\gamma)$ is the “depth” of the market: when there are more sophisticated investors or they are less risk averse, prices depart less from fundamentals.

Equilibrium in period 0 and “noise-trader risk”

Continue to assume that the hedge fund managers are absent from the model. To find the price of the asset in period 0, we therefore need to analyze the behavior of the sophisticated investors born in period 0.

The period-1 consumption of a representative sophisticated investor born in period 0 is $W - P_0S_0 + P_1S_0$. Note that it depends not on the ultimate realization of the value of the risky asset, but on its price in period 1. The investor's expected consumption is therefore $W - P_0S_0 + E_0[P_1]S_0$, and the variance of his or her consumption is $S_0^2 Var(P_1)$. Proceeding along similar lines as before, one can show that the resulting first-order condition for the investor's choice of S_0 is:

$$(**) \quad -P_0 + E_0[P_1] - 2\gamma S_0 Var(P_1) = 0.$$

We can then use expression (*) above to find the mean and variance of P_1 given the information available at time 0 (that is, to find $E_0[P_1]$ and $Var(P_1)$). Substituting those expressions into (**) and then into the market-clearing condition, $S_0A_0 + N_0 = 0$, and then performing algebra gives us an expression for the departure of the period-0 price of the asset from fundamentals:

$$(***) \quad P_0 - 1 = \left\{ \frac{V_2^F}{A_1} + \frac{1}{A_0} \left[\left(\frac{2\gamma V_2^F}{A_1} \right)^2 V_1^N + V_1^F \right] \right\} 2\gamma N_0.$$

As before, fundamental risk causes the sophisticated investors to not fully undo the impact of the noise traders' actions on price, and so allows the price to depart from its fundamental value.

The key new result, however, involves the V_1^N term. This term shows that the response of P_0 to the period-0 noise traders is larger when V_1^N is larger. Intuitively, the sophisticated investors in period 0 risk losses not only from the fact that the fundamental value of the risky asset is likely to change by the time they need to sell, but also from the fact that the difference between the actual value and the fundamental value is also likely to change. This makes them more reluctant

to made trades to correct departures of the price from its fundamental value. That is, *the risk created by the possibility of future departures of prices from fundamentals magnifies those departures today – “noise traders create their own space.”*

Reintroducing the hedge fund managers and “performance-based risk”

We now bring the hedge fund managers back into the model. It turns out that the algebra is much simpler if we drop the period-0 sophisticated investors from the model. That is, we set $A_0 = 0$.

Consider first period 1. In period 1, the demand of the hedge fund managers is fixed and does not respond to the price of the asset. As a result, their demand enters in the same way as that of the noise traders. Thus, analysis paralleling that used to derive equation (*) shows that P_1 is now given by

$$P_1 - (1 + F_1) = \left(\frac{2\gamma}{A_1}\right) V_2^F (N_0 + N_1 + HA_H).$$

Recall that the hedge fund managers purchase the asset in period 0 and hold it until period 2. In addition, they face a penalty or earn a reward depending on the performance of the investment from period 0 to period 1. (Recall this is intended to be a short-cut way of modeling the idea that even if their investment strategy is sound for the long run, they may lose access to funds if the short-run performance of their investments is poor.²) As described above, the amount a hedge fund manager receives (which may be negative) is $aH(P_1 - E_0[P_1])$ ($a > 0$). Thus the manager’s consumption is $W - HP_0 + H(1 + F_1 + F_2) + aH(P_1 - E_0[P_1])$. It follows that the mean of the manager’s consumption, for a given P_0 and H , is $W + H(1 - P_0)$. And using the expression above for P_1 , the departure of consumption from its mean is $H(F_1 + F_2) + aH(F_1 + \emptyset N_1)$, where $\emptyset \equiv \left(\frac{2\gamma}{A_1}\right) V_2^F$. Thus, the representative manager’s problem is

$$\max_H W + H(1 - P_0) - \gamma[(1 + a)^2 V_1^F + V_2^F + a^2 \emptyset^2 V_2^N] H^2.$$

The first-order condition for the manager’s choice of H is therefore

$$(1 - P_0) - 2\gamma[(1 + a)^2 V_1^F + V_2^F + a^2 \emptyset^2 V_2^N] H = 0.$$

Equilibrium in period 0 requires $A_H H + N_0 = 0$. Straightforward algebra then gives us

$$(***) \quad P_0 - 1 = \frac{2\gamma[(1+a)^2 V_1^F + V_2^F + a^2 \emptyset^2 V_2^N]}{A_H} N_0.$$

The key message of this extension of the model is that the coefficient multiplying N_0 is increasing in a . That is, performance-based risk – the fact that even long-horizon investors are affected by the short-term performance of their investments – increases the impact of the noise

² Explicitly making the amount of funds they are able to invest change in response their short-run performance would cause their consumption to not be a linear function of P_1 , and thus to not be normally distributed. That is the reason for taking the short-cut of assuming a financial penalty rather than a loss of funds for poor short-run performance. See Problem 3 below.

traders in period 0 on the price of the asset. Thus, *forces that punish long-run investors based on their short-run performance magnify departures of prices from fundamentals today*. Or as Keynes famously put it (or is famously reputed to have put it), “The market can remain irrational longer than you can remain solvent”

Adding a variation on performance-based risk – momentum traders

Another extension of the model that is somewhat in the spirit of performance-based risk is to add some “return-chasers” or “momentum investors.” These are investors who buy in period 1 if the price of the risky asset has risen, and sell if it falls. They therefore exacerbate the effects of further departures from fundamentals, and so, like performance-based risk, make sophisticated investors more wary of trading against departures from fundamentals.

Concretely, return to the version of the model without the hedge fund managers, and assume an additional component of the demand for the risky asset in period 1 that takes the form $M_1 = m[P_1 - E_0[P_1]]$.

With this change, expression (*) for P_1 becomes

$$P_1 - (1 + F_1) = \left(\frac{2\gamma}{A_1}\right) V_2^F (N_0 + N_1 + M_1).$$

Since $E_0[M_1] = 0$, $E_0[P_1] = 1 + (2\gamma/A_1)V_2^F N_0$. Thus,

$$\begin{aligned} P_1 - E_0[P_1] &= F_1 + \frac{2\gamma}{A_1} V_2^F (N_1 + M_1) \\ &= F_1 + \frac{2\gamma}{A_1} V_2^F N_1 + \frac{2\gamma}{A_1} V_2^F m [P_1 - E_0[P_1]]. \end{aligned}$$

Solving this expression for $P_1 - E_0[P_1]$ yields:

$$P_1 - E_0[P_1] = \frac{1}{1 - \frac{2\gamma V_2^F m}{A_1}} \left[F_1 + \frac{2\gamma}{A_1} V_2^F N_1 \right].$$

Note that the presence of the momentum traders makes the price of the asset more responsive to both F_1 and N_1 . Thus, it increases the variance of P_1 . (We assume that $\frac{2\gamma V_2^F m}{A_1} < 1$; otherwise the momentum traders are so extreme that the model is unstable.)

Using this expression to find the variance of P_1 , solving the maximization problem of the period-0 sophisticated investors, and then substituting into the expression for market-clearing in period 0 leads (after lots of algebra!) to:

$$P_0 - 1 = \left\{ \frac{V_2^F}{A_1} + \frac{1}{A_0} \frac{1}{\left(1 - \frac{2\gamma V_2^F}{A_1} m\right)^2} \left[\left(\frac{2\gamma V_2^F}{A_1}\right)^2 V_1^N + V_1^F \right] \right\} 2\gamma N_0.$$

The point of all this algebra is that the expression in curly brackets is larger when m is larger – that is, the presence of the momentum traders in period 1 increases the impact of the noise traders in period 0 on the price of the asset. Thus, *the presence of forces (performance-based evaluation or momentum traders) that magnify future departures of prices from fundamentals magnifies price departures today.*

Problems (for edification only)

1. Suppose that the demand of the period 0 noise traders is not fully persistent, so that the market-clearing condition in period 1 is $\rho N_0 + N_1 + S_1 A_1 = 0$, $\rho < 1$. The market-clearing condition in period 0 remains $N_0 + S_0 A_0 = 0$. How, if at all, does this affect expression (***) for how the noise traders affect the price in period 0? What happens if $\rho = 0$?

2. In our short-cut approach to performance-based risk, we assume that the amount the hedge fund managers receive in period 0 is a function of $P_1 - E_0[P_1]$. But one can make a reasonable argument that it is better to model it as a function of $P_1 - P_0$. Specifically, suppose the payout is $aH(P_1 - P_0)$ (where, as before, $a > 0$).

a. With this change in the model, prove (by providing an example) or disprove the following claim: for some parameter values, performance-based risk makes P_0 *less* responsive to noise traders – that is, that the coefficient multiplying N_0 in the expression analogous to (***) is smaller when $a > 0$ than when $a = 0$. (Hints: (1) I have not solved this problem; I have only gotten far enough to know that the algebra is messy. (2) Assume that each hedge fund manager takes the total demand of hedge fund managers, A_{HH} , as given in choosing his or her own demand.)

b. Explain in words: (1) Why it is reasonable to assume that the payment is a function of $P_1 - P_0$; (2) Why this introduces a force tending to cause performance-based risk to dampen the response of prices to forces pushing them away from their fundamental values.

3. Two features of the various versions of the model that are critical to their tractability are that the shocks are normally distributed and consumption is linear in the shocks. These features cause consumption to be normally distributed, and thus allow us to find agents' expected utilities. This problem asks you to show that with some natural approaches to modeling performance-based risk, this no longer occurs.

a. Suppose that the representative hedge fund manager, rather than receiving a payment or incurring a cost in period 1, is forced to sell quantity $b(P_1 - E_0[P_1])H$, $b > 0$, of the risky asset in period 1. Show that in this case, the manager's consumption is not linear in F_1 .

b. Consider the version of the model without the hedge fund managers, and suppose that A_1 is not exogenous but depends on the success of the period 0 sophisticated investors: $A_1 = \bar{A} + b(P_1 - E_0[P_1])S_0$, $b > 0$. Show that in this case, the consumption of the sophisticated investors born in period 0 is not linear in F_1 .