CONTINUOUS-TIME STOCHASTIC PROCESSES AND SOME APPLICATIONS

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Let $\{X(t)\}$ be a stochastic process such that

$$\Delta X(t) = X(t) - X(t-1) \sim N(\mu,\sigma^2)$$

Case of interest: $\Delta X(t)$ can be written as a sum (integral) of very small (infinitesimal) independent normal increments.

Carve up interval $[t-1, t]$ into n disjoint subintervals of length $h = 1/n$.

For every $i = 1, 2,\ldots, n$, let $v(i)$ be $N(0,1)$ with $Ev(i)v(j) = 0$ for $i \neq j$. If defined by

$$\Delta X(t) = \Sigma_i [\mu h + \sigma h^{1/2} v(i)],$$

$\Delta X(t)$ is of the desired form: $E\Delta X(t) = n\mu h = \mu$ and

$$\mathrm{Var}\Delta X(t) = \sigma^2 \Sigma_i \Sigma_j Ev(i)v(j)h = \sigma^2 \Sigma_i Ev(i)^2/n = \sigma^2.$$

Take the limit of this process as $h \to 0$; we denote it as

$$dX(t) = \mu dt + \sigma dz(t)$$

where $dz(t) = \lim_{h\to 0} h^{1/2} v(t)$. The result is called a *Gaussian diffusion process*.

Interpretation:

● If $\sigma = 0$, we are back in case where X(t) follows a differentiable path with constant slope μ. Under uncertainty, however, X(t) follows a continuous-time random walk with predictable drift (per unit time) of μ and variance (per unit time) of σ^2.

● *This stochastic process is nowhere differentiable*. An expression like dz(t)/dt, which one might be tempted to define as the limit as $h \to 0$ of $[z(t+h) - z(t)]/h$, has no meaning because the latter limit does not exist. This is why we write the diffusion in differential form as dX(t).

● To see this point about differentiability more technically, notice that $X(t+h) - X(t)$ is normal with variance $h\sigma^2$, by construction. Therefore, $[X(t+h) - X(t)]/h$ has variance σ^2/h, which $\to \infty$ as $h \to 0$.

● More general forms of Gaussian diffusion process are easy to write down and analyze, for example,

$$dX = \mu(X,t)dt + \sigma(X,t)dz,$$

in which the conditional mean and variance can evolve through time. (Observe the simplified notation.)

Multiplication rules for stochastic differentials:

Just as in ordinary differential calculus, terms of order h^k disappear for $k > 1$.

For example, let's compute dy/dx for $y = x^2$: it is the limit as $h \to 0$ of 1/h times $(x + h)^2 - x^2 = 2xh + h^2$. We can ignore the squared h in computing the derivative, 2x.

Thus, in stochastic calculus,

$$(dt)^2 = \lim_{h \to 0} h^2 = 0.$$

That is, the squared time interval goes to zero faster than the time interval, and therefore can be ignored.

Terms like the product (dz)(dt) are limits of $h^{3/2}v$ and also go to zero faster than h. Thus, we write

$$(dz)(dt) = 0.$$

Terms like $(dz)^2$ are of order h, because they are limits of hv^2. They *cannot* be ignored. In fact, because the *variance* of hv^2 is of order h^2, it goes to zero in the continuous-time limit and so hv^2 converges in probability to its mean, dt, as $h \to 0$. Thus, we have the rule

$$(dz)^2 = dt.$$

Itô's Lemma

Let X(t) follow a diffusion and let f(•) be a twice continuously differentiable function. We'd like to know the process followed by the stochastic process f[X(t)].

If X(t) were a purely *deterministic* function of time, the chain rule of calculus would give us the answer that

$$df(X) = f'(X)dX = f'(X)\mu dt.$$

It may be tempting simply to plug in for dX as above when X(t) follows a diffusion, too. This would give

$$f'(X)\mu dt + f'(X)\sigma dz$$

But the conditional mean of df(X) is *not* $f'(X)\mu dt$, i.e., it is *not* true in general that $E[df(X)] = f'(X)E(dX)$. This follows from *Jensen's inequality*. If f(X) is strictly convex, E[df(X)] will be somewhat higher than this formula indicates, and if f(X) is strictly concave that mean will be somewhat lower. Only if f(X) is linear will the "naive" formula above work.

In general, Itô's Lemma (in its univariate form) states that

$$df(X) = f'(X)dX + \tfrac{1}{2}f''(X)(dX)^2.$$

The extra term captures the convexity or concavity of f(X).

Heuristic argument for Itô's Lemma

By Taylor's Theorem, $\exists$ a number $\xi(h) \in [0, 1]$ such that

$$f[X(t+h)] - f[X(t)] = f'[X(t)][X(t+h) - X(t)]$$

$$+ \tfrac{1}{2}f''\{X(t) + \xi(h)[X(t+h) - X(t)]\}[X(t+h) - X(t)]^2.$$

The first right-hand side terms goes to $f'(X)dX$ as $h \to 0$. The second right-hand side term goes to $\tfrac{1}{2}f''(X)(dX)^2$.

In the case we've been looking at,

$$(dX)^2 = \mu^2(dt)^2 + 2\mu\sigma(dz)(dt) + \sigma^2(dz)^2 = \sigma^2 dt.$$

Thus, Itô's Lemma takes the form

$$df[X(t)] = f'[X(t)]\mu dt + f'[X(t)]\sigma dz + \tfrac{1}{2}f''[X(t)]\sigma^2 dt.$$

- <u>Exercise</u> What if the function is $f(X,t)$?

TARGET-ZONE MODELS OF EXCHANGE RATES

Consider the exchange-rate model described by

$$x(t) = k(t) + \alpha E_t[dx(t)]/dt$$

where x(t) is the spot exchange rate, k(t) the "fundamentals."

As a general fact, the equilibrium exchange rate (absent rational bubbles) is

$$x(t) = (1/\alpha) \int_t^\infty e^{-(s-t)/\alpha} E_t[k(s)]ds.$$

But if k(t) follows a complex process, this integral can be hard to compute directly.

Easy example: Free float with random-walk fundamentals

Let k(t) follow $dk = \sigma dz$ *always*. Then $E_t k(s) = k(t)$ for all $s \geq t$. Thus,

$$x(t) = k(t)$$

Target-zone case:

The fundamentals k(t) follow $dk = \sigma dz$ *only* so long as k(t) stays in the range $[k_0, k^0]$. At lower (upper) boundary, the authorities intervene to raise (lower) k by the amount $\delta > 0$. Furthermore, $[k_0, k^0]$ and δ are chosen to restrict the exchange rate to the *target zone* $[x_0, x^0]$. We need a short cut to calculate the integral above!

Let's assume the solution is of the form

$$x = S(k).$$

By Itô's Lemma, as long as we are not at an intervention point (that is, strictly *within* $[k_0, k^0]$), $dk = \sigma dz$, so

$$dx = S'(k)dk + \tfrac{1}{2}S''(k)(dk)^2 = S'(k)\sigma dz + \tfrac{1}{2}S''(k)\sigma^2 dt$$

Combining this with the first equation of the model above,

$$S(k) = k + \tfrac{1}{2}S''(k)\alpha\sigma^2$$

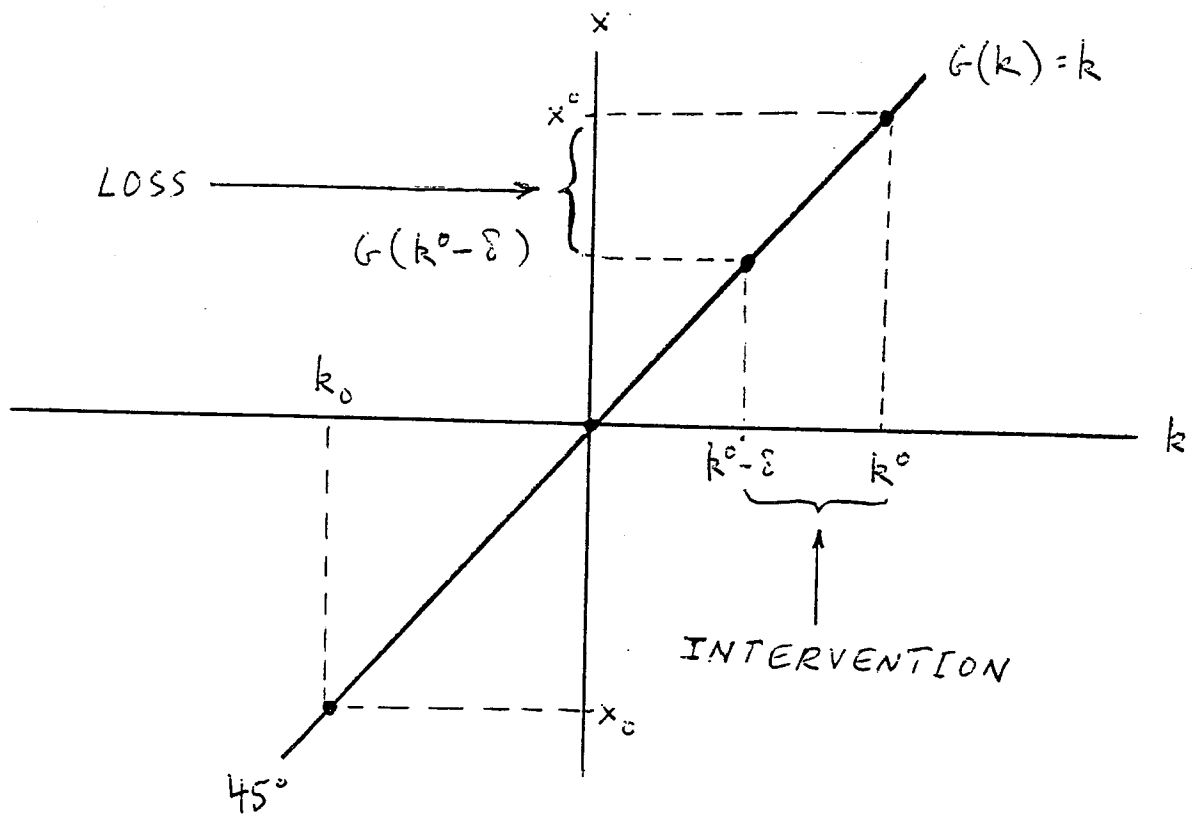
This is just a differential equation in k. Its general solution takes the form (A_1 and A_2 are arbitrary constants):

$$G(k) = k + A_1 e^{\lambda k} + A_2 e^{-\lambda k}$$

Plugging this into the preceding differential equation shows that λ must be given by

$$\lambda = (2/\alpha\sigma^2)^{1/2}.$$

For example, if $A_1 = A_2 = 0$, we have $G(k) = k$. But that cannot be the solution $S(k)$, because it would imply discrete anticipated capital losses (gains) at k^0 (at k_0).



The value-matching boundary conditions

To avoid such arbitrage opportunities we need *boundary conditions* on S(k) that rule them out. These boundary conditions are what pin down the constants A_1 and A_2.

The appropriate *value-matching conditions* are:

$$S(k^0) = S(k^0 - \delta)$$

$$S(k_0) = S(k_0 + \delta)$$

To solve, let's make life easier by assuming the symmetric case in which $k_0 = -k^0$. Then $A_1 = -A_2 = A$, which we can solve for using

$$k^0 + A[\exp(\lambda k^0) - \exp(-\lambda k^0)]$$

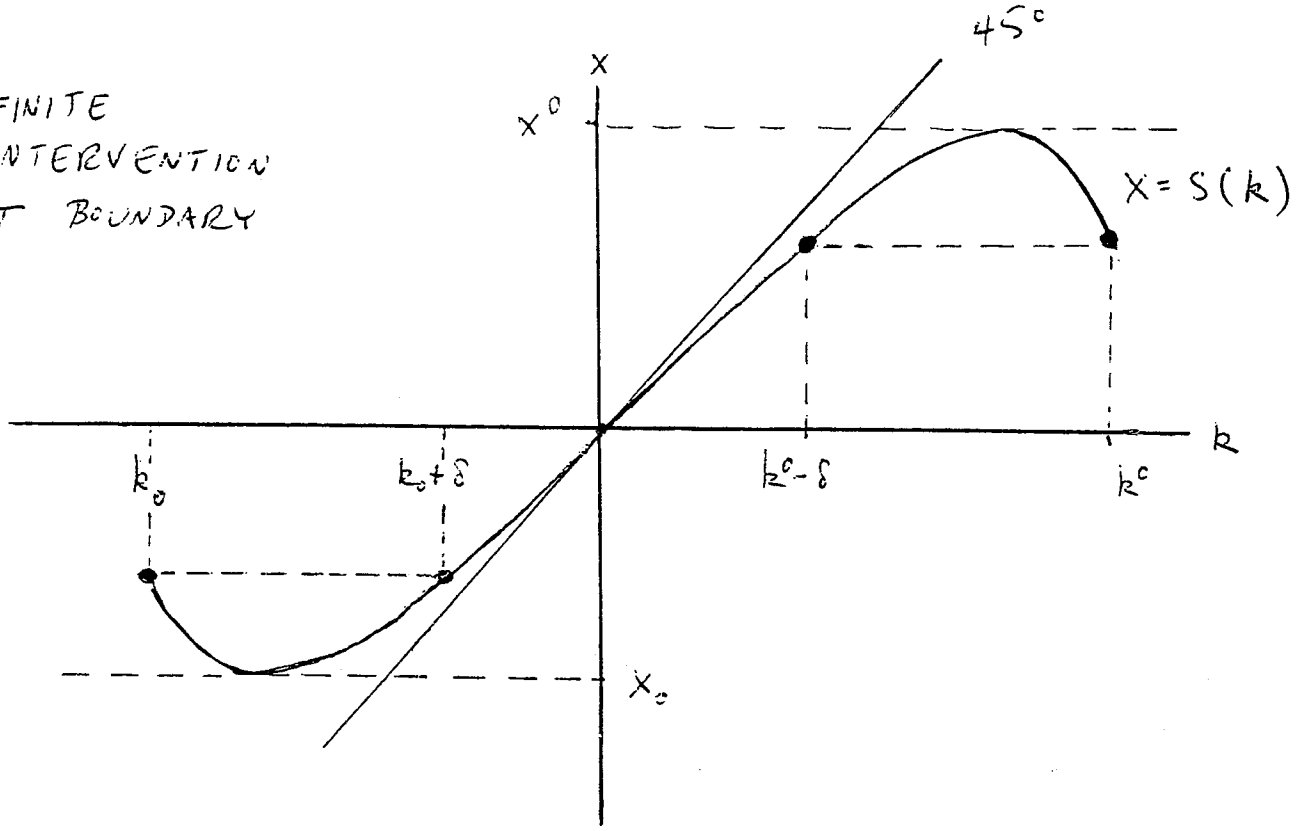
$$= k^0 - \delta + A[\exp(\lambda(k^0 - \delta)) - \exp(-\lambda(k^0 - \delta))]$$

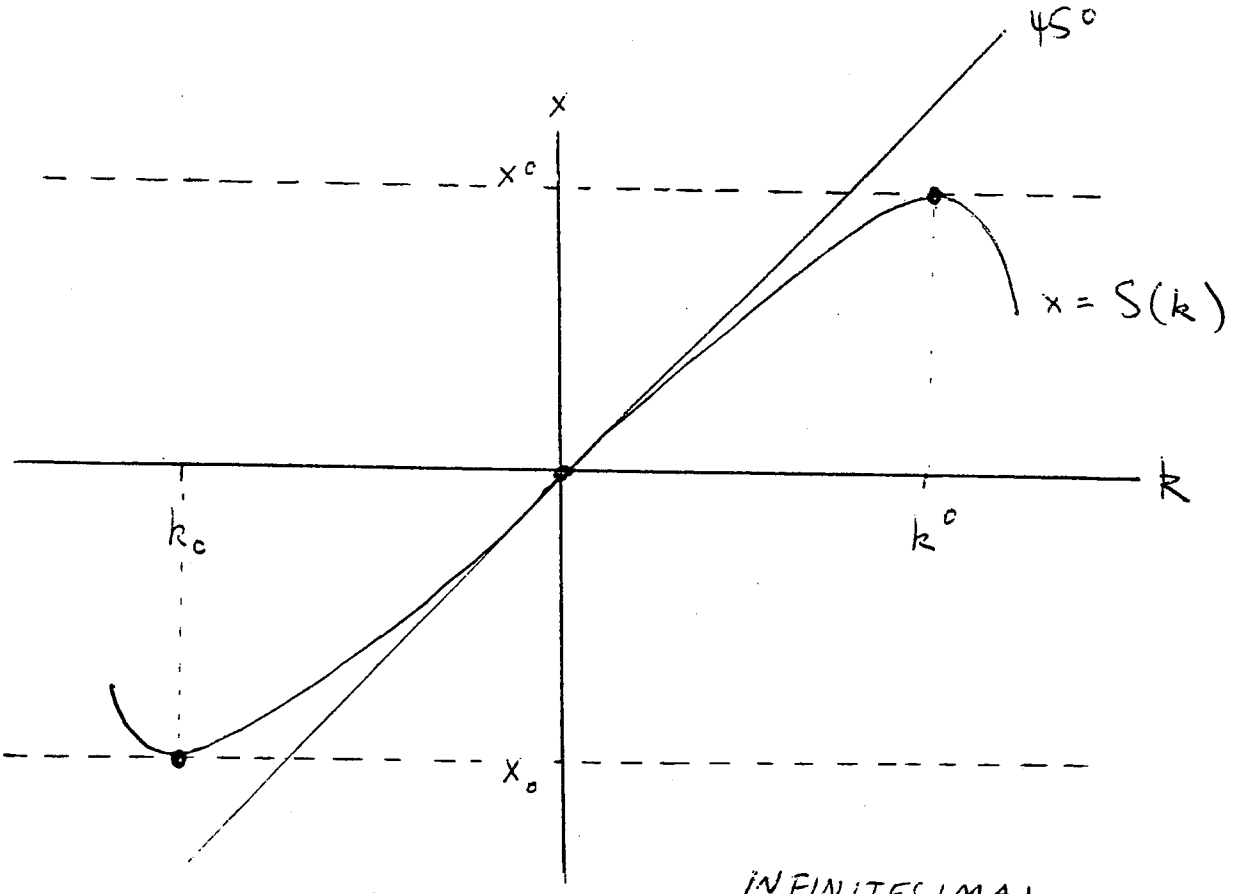
The implied exchange-rate bands are the maximum and minimum of the resulting S(k) function.

In the limit of *infinitesimal* intervention, the boundary conditions are

$$S'(k^0) = S'(k_0) = 0.$$

FINITE INTERVENTION AT BOUNDARY





INFINITESIMAL INTERVENTION AT BOUNDARY

In the symmetric case, these derivative conditions imply

$$1 + A\lambda\exp(\lambda k^0) - A\lambda\exp(-\lambda k^0) = 0$$

or

$$A = -1/[\lambda\exp(\lambda k^0) - \lambda\exp(-\lambda k^0)]$$

This negative coefficient indicates the *stabilizing* effect of the target zone. We also can solve for x^0.

We have looked at the case of a *completely credible* target zone: policy interventions keep the exchange rate within the zone with probability 1. If there is some chance that the zone will be realigned or widened, different boundary conditions determining A_1 and A_2 apply. For discussion, see the items by Bertola, Krugman and Miller, and Svensson in the "Further reading" section at the end of these notes. Svensson contains a particularly nice review of empirical work on target-zone models.

Stochastic dynamic programming in continuous time: application to portfolio selection

There are two investments, safe and risky. Output invested in the safe activity grows in value according to:

$$dV^B = rV^B dt$$

Output invested in the risky activity grows according to:

$$dV^K = \alpha V^K dt + \sigma V^K dz \qquad (\alpha > r)$$

All output is generated by these two types of capital.

Let ζ be the share of wealth invested in the safe asset. Then wealth follows the process:

$$dW = \zeta rWdt + (1-\zeta)\alpha Wdt + (1-\zeta)\sigma Wdz - Cdt$$

Let J(W) be maximized lifetime expected utility, given W:

$$J[W(t)] = \max E_t \int_t^\infty e^{-\delta(s-t)} u[C(s)]ds.$$

The stochastic Bellman equation

The *Bellman equation* for the time interval [t, t+h] is:

$$J[W(t)] = \max_{C(t)} \{u[C(t)]h + e^{-\delta h} E_t J[W(t+h)]\}$$

subject to the wealth-accumulation identity.

Subtract J[W(t)] from both sides and approximate the discount factor by 1 - δh. The result is

$$0 = \max_{C(t)} \{u[C(t)]h + (1-\delta h)E_t J[W(t+h)] - J[W(t)]\},$$

subject to the wealth constraint.

In the limit of continuous time we have

$$0 = \max_{C} \{u(C)dt + E_t dJ(W) - \delta J(W)dt\}$$

By Itô's Lemma, however:

$$dJ(W) = J'(W)dW + \tfrac{1}{2}J''(W)(dW)^2$$

$$dW^2 = \sigma^2(1 - \zeta)^2 W^2 dt$$

Substitution and division by dt yields the continuous-tme Bellman equation:

$$0 = \max_C \{u(C) + J'(W)[\zeta r W + (1-\zeta)\alpha W - C] + \tfrac{1}{2}J''(W)\sigma^2(1 - \zeta)^2 W^2 - \delta J(W)\} \qquad (*)$$

First-order conditions with respect to C and ζ are:

$$u'(C) - J'(W) = 0$$

$$(\alpha - r)J'(W) + WJ''(W)\sigma^2(1 - \zeta) = 0$$

Consumption and portfolio rules

Let $u(C) = c^{1-R}/(1 - R)$. Let's make the (informed) guess

$$J(W) = [m/(1-R)]W^{1-R}$$

for some constant m. Then our first-order conditions imply:

$$c = m^{-1/R}W$$

$$1 - \zeta = (\alpha - r)/R\sigma^2$$

To find m, notice that, upon substitution of the conjectured value function, the consumption rule $c = m^{-1/R}W$, and the portfolio share above, eq. (*) $\Rightarrow$

$$m = (1/R)\{\delta - (1 - R)[r + (\alpha - r)^2/2R\sigma^2]\}$$

As usual, $R = 1$ (log preferences) gives $m = \delta$.

Further reading

Bertola, G. "Continuous-Time Models of Exchange Rates and Intervention." In F. van der Ploeg, ed., *Handbook of International Macroeconomics*. Blackwell, 1994.

A. Dixit and R. Pindyck. *Investment under Uncertainty*. Princeton University Press, 1994.

P. Krugman and M. Miller, eds. *Exchange Rate Targets and Currency Bands*. Cambridge University Press, 1992.

A. Malliaris and W. Brock. *Stochastic Methods in Economics and Finance*. North-Holland, 1982.

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L. Svensson. "An Interpretation of Recent Research on Exchange Rate Target Zones." *Journal of Economic Perspectives* 6 (Fall 1992).