Notes on PTM Model with Interest Rule

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We start with the interest-rate rules

$$\begin{aligned}\log(1+i_t) &= \bar{\imath} + \beta p_t - \alpha_{\mathrm{W}} u_t^{\mathrm{W}} - \alpha_{\mathrm{D}} u_t^{\mathrm{D}}, \\ \log(1+i_t^*) &= \bar{\imath} + \beta p_t^* - \alpha_{\mathrm{W}}^* u_t^{\mathrm{W}} + \alpha_{\mathrm{D}}^* u_t^{\mathrm{D}}\end{aligned}$$

where the technology processes follow

$$\begin{aligned}a_{t+1} &= \lambda a_t + u_{t+1}, \\ a_{t+1}^* &= \lambda a_t^* + u_{t+1}^*,\end{aligned}$$

and the innovations share the same variance σ_u^2.

Pricing and ex ante consumption (using complete markets, no nontradables case)

$$\frac{P(h)}{P(f)} = \frac{\mathrm{E}\left(\frac{C}{A}\right)}{\mathrm{E}\left(\frac{C}{A^*}\right)}, \quad \frac{P(f)^*}{P(h)^*} = \frac{\mathrm{E}\left(\frac{C^*}{A^*}\right)}{\mathrm{E}\left(\frac{C^*}{A}\right)}$$

$$1 = \frac{\theta\kappa}{\theta-1}\frac{\sqrt{\mathrm{E}\left(\frac{C}{A}\right)\mathrm{E}\left(\frac{C}{A^*}\right)}}{\mathrm{E}\left(C^{1-\rho}\right)} = \frac{\theta\kappa}{\theta-1}\frac{\sqrt{\mathrm{E}\left(\frac{C^*}{A}\right)\mathrm{E}\left(\frac{C^*}{A^*}\right)}}{\mathrm{E}\left(C^{1-\rho}\right)}$$

$C^{*\,1-\rho}$

From the last two equations we can derive

$$\mathrm{E}_t c_{t+1} = \frac{1}{\rho}\left[\log\left(\frac{\theta-1}{\theta\kappa}\right) + \mathrm{E}_t a_{t+1}^{\mathrm{W}} - \rho\left(1-\frac{\rho}{2}\right)\sigma_c^2 - \frac{1}{2}\sigma_u^2 + \sigma_{cu^{\mathrm{W}}}\right],$$

in which the variances will be constant, with endogenous values to be derived below.

Solutions for price level and ex post consumption

Start from the Euler equation

$$\frac{C_t^{-\rho}}{P_t} = \beta(1+i_t)\mathrm{E}_t\left\{\frac{C_{t+1}^{-\rho}}{P_{t+1}}\right\}$$

As P_{t+1} is predetermined on the basis of date t information, we may write this (after substitution of the interest-rate policy rule) as

$$\rho\left(\mathrm{E}_t c_{t+1} - c_t\right) = \log\beta + \bar{\imath} - \left[p_{t+1} - (1+\gamma)p_t\right] + \frac{\rho^2}{2}\sigma_c^2 - \alpha_{\mathrm{W}} u_t^{\mathrm{W}} - \alpha_{\mathrm{D}} u_t^{\mathrm{D}}$$

Taking expectations based on date $t-1$ information yields

$$\rho\left(\mathrm{E}_{t-1} c_{t+1} - \mathrm{E}_{t-1} c_t\right) = \log\beta + \bar{\imath} - \left[\mathrm{E}_{t-1} p_{t+1} - (1+\gamma)p_t\right] + \frac{\rho^2}{2}\sigma_c^2$$

which can be solved for p_t to yield the first-order difference equation

$$p_t = \frac{1}{1+\gamma}\mathrm{E}_{t-1}p_{t+1} + \frac{1}{1+\gamma}\left\{\rho\left(\mathrm{E}_{t-1}c_{t+1} - \mathrm{E}_{t-1}c_t\right) - \left(\log\beta + \bar{\imath} + \frac{\rho^2}{2}\sigma_c^2\right)\right\}$$

Notice that

$$\mathrm{E}_{t-1}c_{t+1} - \mathrm{E}_{t-1}c_t = \frac{1}{\rho}\left(\lambda^2 - \lambda\right)a_{t-1}^{\mathrm{W}}$$

so

$$p_t = \frac{1}{1+\gamma}\mathrm{E}_{t-1}p_{t+1} + \frac{1}{1+\gamma}\left\{\lambda(\lambda-1)a_{t-1}^{\mathrm{W}} - \left(\log\beta + \bar{\imath} + \frac{\rho^2}{2}\sigma_c^2\right)\right\}$$

Solving this equation forward (and ruling out p bubbles) gives

$$p_t = \frac{\lambda(\lambda-1)}{1+\gamma-\lambda}a_{t-1}^{\mathrm{W}} - \frac{1}{\gamma}\left(\log\beta + \bar{\imath} + \frac{\rho^2}{2}\sigma_c^2\right)$$

Consider next ex post consumption. We again use the Euler equation to write

$$c_t = \mathrm{E}_t c_{t+1} - \frac{1}{\rho}\left[\log\beta + \log(1+i_t) - (p_{t+1} - p_t) + \frac{\rho^2}{2}\sigma_c^2\right]$$

Here i_t is predetermined unless it reacts to current shocks; assume temporarily it does not. Then we can assess the response of c_t to an innovation in a_t^{W} from

$$\begin{aligned} c_t &= \frac{\lambda}{\rho}a_t^{\mathrm{W}} + (\text{constants}) \\ &\quad -\frac{1}{\rho}\left[\log\beta + \log(1+i_t) - \left(\frac{\lambda(\lambda-1)}{1+\gamma-\lambda}a_t^{\mathrm{W}} - p_t\right) + \frac{\rho^2}{2}\sigma_c^2\right] \end{aligned}$$

so that

$$\frac{\mathrm{d}c_t}{\mathrm{d}u_t^{\mathrm{W}}} = \frac{1}{\rho}\left[\lambda + \frac{\lambda(\lambda-1)}{1+\gamma-\lambda}\right] = \frac{\gamma\lambda}{\rho(1+\gamma-\lambda)} > 0,$$

which is zero if $\lambda = 0$.

Now let i_t respond to current shocks and write consumption in tems of innovations:

$$c_t - \mathrm{E}_{t-1}c_t = \frac{\gamma\lambda}{\rho(1+\gamma-\lambda)}u_t^{\mathrm{W}} + \frac{1}{\rho}\left(\alpha_{\mathrm{W}}u_t^{\mathrm{W}} + \alpha_{\mathrm{D}}u_t^{\mathrm{D}}\right) = \frac{\gamma\lambda + \alpha_{\mathrm{W}}(1+\gamma-\lambda)}{\rho(1+\gamma-\lambda)}u_t^{\mathrm{W}} + \frac{\alpha_{\mathrm{D}}u_t^{\mathrm{D}}}{\rho}$$

The corresponding innovation in Foreign consumption equals:

$$c_t^* - \mathrm{E}_{t-1}c_t^* = \frac{\gamma\lambda + \alpha_{\mathrm{W}}^*(1+\gamma-\lambda)}{\rho(1+\gamma-\lambda)}u_t^{\mathrm{W}} - \frac{\alpha_{\mathrm{D}}^*u_t^{\mathrm{D}}}{\rho}$$

We may now compute the conditional variances and covariances

$$\begin{aligned} \sigma_c^2 &= \left[\frac{\gamma\lambda + \alpha_{\mathrm{W}}(1+\gamma-\lambda)}{\rho(1+\gamma-\lambda)}\right]^2\sigma_{u^{\mathrm{W}}}^2 + \left(\frac{\alpha_{\mathrm{D}}}{\rho}\right)^2\sigma_{u^{\mathrm{D}}}^2 \\ \sigma_{c^*}^2 &= \left[\frac{\gamma\lambda + \alpha_{\mathrm{W}}^*(1+\gamma-\lambda)}{\rho(1+\gamma-\lambda)}\right]^2\sigma_{u^{\mathrm{W}}}^2 + \left(\frac{\alpha_{\mathrm{D}}^*}{\rho}\right)^2\sigma_{u^{\mathrm{D}}}^2 \\ \sigma_{cu^{\mathrm{W}}} &= \frac{\gamma\lambda + \alpha_{\mathrm{W}}(1+\gamma-\lambda)}{\rho(1+\gamma-\lambda)}\sigma_{u^{\mathrm{W}}}^2 \\ \sigma_{c^*u^{\mathrm{W}}} &= \frac{\gamma\lambda + \alpha_{\mathrm{W}}^*(1+\gamma-\lambda)}{\rho(1+\gamma-\lambda)}\sigma_{u^{\mathrm{W}}}^2 \end{aligned}$$

Expected utility

We can write one-period Home expected utility as

$$\begin{aligned}
& \mathrm{E}_{t-1}\left\{\frac{C_t^{1-\rho}}{1-\rho}-\frac{1}{2}\left(\frac{\theta-1}{\theta}\right)\left(C_t^{1-\rho}+C_t^{*1-\rho}\right)\right\} \\
= \quad & \mathrm{E}_{t-1}\left\{\frac{\theta-\frac{1}{2}(\theta-1)(1-\rho)}{(1-\rho)\theta}C_t^{1-\rho}-\frac{1}{2}\left(\frac{\theta-1}{\theta}\right)C_t^{*1-\rho}\right\} \\
= \quad & \frac{\theta-\frac{1}{2}(\theta-1)(1-\rho)}{(1-\rho)\theta}\exp\left\{(1-\rho)\mathrm{E}_{t-1}c_t+\frac{(1-\rho)^2}{2}\sigma_c^2\right\} \\
& -\frac{1}{2}\left(\frac{\theta-1}{\theta}\right)\exp\left\{(1-\rho)\mathrm{E}_{t-1}c_t^*+\frac{(1-\rho)^2}{2}\sigma_{c^*}^2\right\}
\end{aligned}$$

The first part of this (Home welfare) boils down to

$$-\frac{\sigma_c^2}{2}+\frac{\sigma_{cu^w}}{\rho}$$