# Equilibrium Portfolios and External Adjustment under Incomplete Markets*

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**Abstract**

Recent evidence on the importance of cross-border equity flows calls for a rethinking of the standard theory of external adjustment. We introduce equity holdings and portfolio choice into an otherwise conventional open-economy dynamic equilibrium model. Our model is simple and admits an exact closed-form solution regardless of whether financial markets are complete or incomplete. We derive a necessary and sufficient condition under which the current account is different from zero and shed light on the relationship between market incompleteness and the current account dynamics. Furthermore, we revisit the current debate on the relative importance of the standard vs. the capital-gains-based (or "valuation") channels of the external adjustment and establish that in our framework they are congruent. Our model's implications are consistent with a number of intriguing stylized facts documented in the recent empirical literature.

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# 1. Introduction

An unprecedented rise in cross-border equity holdings over the past two decades has generated a source of income previously disregarded in the national accounts: capital gains on equity holdings. The current practice incorporates capital gains only after they are redeemed, and this lack of marking to market may result in a significant misrepresentation of the extent of external imbalances worldwide—especially in the US, most of Europe, and Japan. Burgeoning empirical literature has emphasized that capital gains on the industrialized countries' net foreign asset (NFA) positions—or "valuation changes"—have indeed become sizable.[1] These exciting new empirical developments call for a modification of the standard external adjustment theory that includes valuation changes.[2]

The goal of this paper is to incorporate portfolio choice and asset pricing into an otherwise standard open economy macro model and to investigate how fluctuations in asset prices influence external accounts—in particular, how the standard current account dynamics are affected by the inclusion of capital gains on financial assets. We do this in the context of a multiple-asset two-good two-country stochastic general equilibrium model under complete and incomplete markets. We make enough simplifying assumptions so that our model admits an exact, closed-form solution. From the methodological viewpoint, the paper brings in some powerful solution techniques developed in the asset pricing literature in Finance that could prove to be very useful in open economy macroeconomics.

Recent literature offers several ambitious attempts to study similar questions. The complexity of the proposed models, however, rules out analytical solutions and hence the literature has concentrated on developing sophisticated approximate or numerical methods to be able to analyze the dynamic properties of such economies.[3] Moreover, it has been suggested that to get nontrivial current account dynamics (otherwise, there is no international adjustment to speak of!), one needs to move away from the complete markets paradigm, which complicated the matters further. By

[1] Prominent examples of papers belonging to this strand of literature include Gourinchas and Rey (2007a), Gourinchas and Rey (2007b), Lane and Milesi-Ferretti (2001), Lane and Milesi-Ferretti (2007), Tille (2003), and Tille (2005).

[2] For example, in his Harms Lecture at the Kiel Institute, Obstfeld (2004) remarks that the standard international adjustment models "now look manifestly inadequate to describe the dynamics of net foreign assets in "the brave new world of huge two-way diversification flows" and stresses the need for a new view of external adjustment.

[3] See, e.g., Ghironi, Lee, and Rebucci (2006) and Kollmann (2006) for the first-generation analyses that employ standard first-order approximation techniques. The second-generation methodologies were developed by Devereux and Sutherland (2006), Evans and Hnatkovska (2007), and Tille and van Wincoop (2007) who solve stochastic portfolio models with incomplete markets using more complex higher-order approximations. Cavallo and Tille (2006) propose a shortcut solution method in which optimal portfolios are specified to be an (exogenous) fraction of trade flows.

contrast, our model admits an exact analytical solution, even under incomplete markets.

An advantage of our approach is that it provides a theoretical framework in which we can examine (and clarify) some of the conjectures made in the literature. First, we reassess the role that incomplete markets play in generating a nontrivial current account. We find that the current account is different from zero if and only if the optimal portfolios are such that countries' bondholdings are different from zero. We provide examples in which this property occurs under complete and under incomplete markets. In other words, whether or not the current account is nontrivial depends on the hedging demands of agents and not on market incompleteness per se.

Second, we revisit the traditional intertemporal approach to the current account that says that, for the budget constraint to be satisfied, a country's current negative NFA position must be compensated by future trade surpluses (Obstfeld and Rogoff (1996)). A new view that has recently emerged in the literature criticizes the traditional approach for neglecting the possibility that changes in asset returns may lead to changes in the discount factor that could raise the *present value* of future trade surpluses without the need to actually adjust the trade balance.[4] Surprisingly, we find that in our model, once the endogenous responses of asset prices to underlying shocks are taken into account, any shock to the NFA position ends up being financed by adjusting the current trade balance—thus making the traditional and the new views of the external adjustment congruent.

Finally, because we are able to fully characterize the equilibrium in our economy, we describe the adjustment process, and what role portfolio reallocations, changes in expected returns, unexpected capital gains, and the trade balance play. We study the behavior of these and other variables in response to supply and demand shocks and explain the economic mechanisms behind the patterns that we find.

We have already cited the papers that are conceptually related to this work. Methodologically, the most closely related works are He and Pearson (1991), Cuoco and He (1994), and Basak and Cuoco (1998). At a partial equilibrium level, He and Pearson derive a solution to a consumption-portfolio problem under incomplete markets. Cuoco and He develop a method for solving for equilibrium under incomplete markets via a "planner" with stochastic weights. Basak and Cuoco

[4] As pointed out by Gourinchas and Rey (2007b). See also Hausmann and Sturzenegger (2006) and Tille (2003) for related arguments. This new view sheds a fresh light on the question of the sustainability of the US external imbalances, suggesting that the widening current account deficit in the US could be part of the normal adjustment process and does not necessarily spell any economic disaster. This conclusion is contested by the proponents of the traditional view who believe that a significant adjustment of the trade balance and in particular a large US dollar depreciation needs to take place (see e.g., Edwards (2005), Frankel (2006), Obstfeld (2004), Obstfeld and Rogoff (2007), Roubini and Setser (2004)).

were the first to apply this method to study financial markets with frictions (restricted participation, in their case). None of these papers, however, offers a model with multiple risky assets and incomplete markets that can be analyzed analytically. The model that we develop builds on Cole and Obstfeld (1991), Helpman and Razin (1978), Pavlova and Rigobon (2007), and Zapatero (1995). All these are tractable multi-asset multi-good models like ours, but in contrast to our work, in each of these papers markets are complete or effectively complete.

The rest of the paper is organized as follows. Section 2 describes the model and characterize its equilibrium. Section 3 derives a number of implications of our model for the current account and its dynamics. Section 4 presents several special cases of our economy in which the dynamics of portfolios and the current account simplify significantly. Section 5 discusses several caveats and desirable extensions, and Section 6 concludes.

# 2. The Model

## 2.1. The Economic Setting

We consider a continuous-time pure-exchange world economy with a finite horizon, $[0, T]$ along the lines of Pavlova and Rigobon (2007). Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, on which is defined a standard four-dimensional Brownian motion $\vec{w}(t) = (w(t), w^*(t), w^\alpha, w^\beta)^\top$, $t \in [0, T]$. All stochastic processes are assumed adapted to $\{\mathcal{F}_t; t \in [0, T]\}$, the augmented filtration generated by $\vec{w}$. All stated (in)equalities involving random variables hold $P$-almost surely. In what follows, given our focus, we assume all processes introduced to be well-defined, without explicitly stating regularity conditions ensuring this.

There are two countries in the world economy: Home and Foreign. The Home country represents a large industrialized country, while Foreign stands for the rest of the world. Each country is endowed with a Lucas tree producing a strictly positive amount of a country-specific perishable good:

$$dY(t) = \mu_Y(t)\, Y(t)\, dt + \sigma_Y(t)\, Y(t)\, dw(t) \qquad \text{(Home)}, \tag{1}$$

$$dY^*(t) = \mu_{Y^*}(t)\, Y^*(t)\, dt + \sigma_{Y^*}(t)\, Y^*(t)\, dw^*(t) \qquad \text{(Foreign)}, \tag{2}$$

where $\mu_Y$, $\mu_Y^*$, $\sigma_Y > 0$, and $\sigma_Y^* > 0$ are arbitrary adapted processes. The claims to the trees, Home and Foreign stocks $S$ and $S^*$, respectively, are available for trade by all investors and are in fixed supply of one share each. The prices of the Home and Foreign goods are denoted by $p$ and $p^*$, respectively. We fix the world numeraire basket to contain $a \in (0,1)$ units of the Home good and (1-$a$) units of the Foreign good, and normalize the price of this basket to be equal to unity. The terms of trade, $q$, are defined as the price of the Home good relative to that of the Foreign good: $q \equiv p/p^*$.

In addition to the stocks $S$ and $S^*$, there is also the "world" bond $B$ available for investment, which is a money market account locally riskless in units of the numeraire. The bond is in zero net supply. Since there are four independent Brownian motions driving the economy and only three investment opportunities in place, financial markets are incomplete. To fix notation, the posited dynamics of the investment opportunity set of the agents is given by

$$dB(t) = B(t)r(t)dt, \tag{3}$$

$$dS(t) + p(t)Y(t)dt = S(t)[\mu_S(t)dt + \sigma_S(t)d\vec{w}(t)], \tag{4}$$

$$dS^*(t) + p^*(t)Y^*(t)dt = S^*(t)[\mu_{S^*}(t)dt + \sigma_{S^*}(t)d\vec{w}(t)], \tag{5}$$

where the interest rate $r$, the stocks expected returns $\mu_S$ and $\mu_{S^*}$ and their volatilities $\sigma_S$ and $\sigma_{S^*}$ are to be determined in equilibrium. The *volatility matrix* of the stock returns is then defined as $\sigma = \begin{bmatrix} \sigma_S \\ \sigma_{S^*} \end{bmatrix}$.

The initial shareholdings of a representative consumer-investor of each country consist of no shares of the bond and a total supply of the stock market of his country. Thus, the initial wealth of the Home resident is $W_H(0) = S(0)$ and that of the Foreign resident is $W_F(0) = S^*(0)$. Each consumer $i$, $i \in \{H, F\}$, chooses nonnegative consumption of each good $(C_i(t), C_i^*(t))$ and a portfolio of the available securities $x_i(t) \equiv (x_i^S(t), x_i^{S^*}(t))$, where $x_i^j$ denotes the fraction of wealth of consumer $i$ invested in asset $j$. The dynamic budget constraint of each consumer has the standard form

$$\begin{aligned} dW_i(t) &= \Big[W_i(t)\,r(t) + x_i^S(t)W_i(t)(\mu_S(t) - r(t)) + x_i^{S^*}(t)W_i(t)(\mu_{S^*}(t) - r(t))\Big]dt \\ &\quad + \Big[x_i^S(t)W_i(t)\sigma_S(t) + x_i^{S^*}(t)W_i(t)\sigma_{S^*}(t)\Big]d\vec{w}(t) - \Big[p(t)C_i(t) + p^*(t)C_i^*(t)\Big]dt\,, \end{aligned} \tag{6}$$

where $W_i(T) \geq 0$, $i \in \{H, F\}$. Preferences of consumer $i$, are represented by a time-additive utility function defined over consumption of both goods:

$$E\left[\int_0^T e^{-\rho t} u_i(C_i(t), C_i^*(t))\, dt\right], \qquad \rho > 0, \quad i \in \{H, F\}, \tag{7}$$

where

$$u_H(C_H(t),\, C_H^*(t)) \;=\; \alpha_H(t)\log C_H(t) + \beta_H(t)\log C_H^*(t),$$

$$u_F(C_F(t),\, C_F^*(t)) \;=\; \beta_F \log C_F(t) + \alpha_F \log C_F^*(t).$$

In our specification of the countries' utilities, we allow for the possibility of preference shifts towards the home or the foreign good (or "demand shocks"), modeled along the lines of Dornbusch, Fischer, and Samuelson (1977). The role of this assumption is twofold. First, in the absence of the demand shocks, free trade in goods makes stock prices perfectly correlated and financial markets irrelevant (Helpman and Razin (1978), Cole and Obstfeld (1991), Zapatero (1995)). Second, empirical evidence indicates that demand shocks are important drivers of the real-world dynamics. For example, Stockman and Tesar (1995) calibrate preference shocks to be roughly 85% of the size of supply shocks, while of Pavlova and Rigobon (2007) estimate a similar model and conclude that they have about the same volatility as supply shocks. Formally, we assume that $\alpha_H$ and $\beta_H$ are positive adapted stochastic processes, martingales, and have dynamics

$$d\alpha_H(t) = \sigma_{\alpha_H}(t)d\vec{w}(t), \quad d\beta_H(t) = \sigma_{\beta_H}(t)d\vec{w}(t).$$

In the analysis that follows, we consider primarily two types of demand shocks: (i) demand shocks that are completely independent of the supply (output) shocks $w$ and $w^*$ and (ii) demand shocks that are allowed to be correlated with the supply shocks. For simplicity, we assume that there are no demand shocks at Foreign, but our model can be easily extended to accommodate these.[5]

## 2.2. Countries' Portfolio Choice

The first step in our solution procedure is to derive the countries' optimal portfolios at a partial equilibrium level. To do so, we are going to employ techniques developed in the portfolio choice literature. However, relative to that literature, there are two non-standard ingredients in the optimization problem that the countries are facing: multiple consumption goods and incomplete markets. We address them in turn.

For concreteness, we focus our exposition on the Home consumer. The portfolio of the Foreign consumer is derived analogously. Following the early literature in finance (Breeden (1979), Adler

[5]At this point, for generality, we are not requiring that each country has a stronger preference for the home good ($\alpha_i > \beta_i$, $i \in \{H, F\}$). However, a realistic calibration of a model like ours would typically incorporate such a preference bias.

and Dumas (1983)), we decompose the problem of maximizing his utility (7) subject to the budget constraint (6) into two parts. First, at each $t$, we derive the consumer's demands for the Home and the Foreign goods, keeping the overall consumption expenditure fixed. Second, we derive his optimal consumption expenditure process and the optimal portfolio.

The first step is the standard static consumer problem under certainty:

$$\max_{\{C_H(t),\, C_H^*(t)\}} \alpha_H(t) \log C_H(t) + \beta_H(t) \log C_H^*(t)$$
$$\text{s.t.} \qquad p(t)C_H(t) + p^*(t)C_H^*(t) \le \mathbb{C}_H(t),$$

where $\mathbb{C}_H(t)$ denotes overall consumption expenditure at time $t$. Solving this problem, we obtain the following demands for the individual goods as fractions of the overall expenditure:

$$\overline{C}_H(t) = \frac{\alpha_H(t)}{\alpha_H(t) + \beta_H(t)} \frac{\mathbb{C}_H(t)}{p(t)}, \quad \overline{C}_H^*(t) = \frac{\beta_H(t)}{\alpha_H(t) + \beta_H(t)} \frac{\mathbb{C}_H(t)}{p^*(t)}. \tag{8}$$

The indirect utility function defined as $U_H\left(\mathbb{C}_H(t);\, p(t), p^*(t)\right) \equiv u_H\left(\overline{C}_H(t),\, \overline{C}_H^*(t)\right)$ is then given by

$$U_H\left(\mathbb{C}_H(t);\, p(t), p^*(t)\right) = \left(\alpha_H(t) + \beta_H(t)\right) \log \mathbb{C}_H(t) + F(\alpha_H(t),\, \beta_H(t),\, p(t),\, p^*(t)),$$

where $F(\cdot)$ is a function the form of which does not affect our analysis. This function $F$ depends only on the variables that are exogenous from the viewpoint of the consumer and therefore, because of the separability of the indirect utility, it drops out of his portfolio choice.

The second step is to reformulate the portfolio choice problem of the consumer in terms of his indirect utility:

$$\max_{x_H^S,\, x_H^{S^*},\, \mathbb{C}_H} E\left[\int_0^T e^{-\rho t} (\alpha_H(t) + \beta_H(t)) \log \mathbb{C}_H(t)\, dt\right] \tag{9}$$

$$\begin{aligned} \text{s.t.} \qquad dW_H(t) = & \left[W_H(t)\, r(t) + x_H^S(t) W_H(t)(\mu_S(t) - r(t)) + x_H^{S^*}(t) W_H(t)(\mu_{S^*}(t) - r(t))\right] dt \\ & + \left[x_H^S(t) W_H(t) \sigma_S(t) + x_H^{S^*}(t) W_H(t) \sigma_{S^*}(t)\right] d\vec{w}(t) - \mathbb{C}_H(t) dt\,. \end{aligned} \tag{10}$$

The optimization problem is thus formally equivalent to a familiar single-good consumption-investment problem, with consumption expenditure $\mathbb{C}_H$ replacing consumption. Consumption of individual goods can then be recovered from (8). It is important to note that the prices of the individual goods, $p$ and $p^*$, and hence the terms of trade have dropped out of the optimization problem. This implies that fluctuations in the terms of trade do not pose a risk that the consumer desires to hedge. In contrast, one would generally expect him to hedge against the preference shifts $\alpha_H$ and $\beta_H$, which enter as state variables in his optimization problem.

The next issue we need to address is market incompleteness. A technique for solving such problems in a single-good framework via martingale methods has been developed in a seminal contribution of He and Pearson (1991). These authors show that, just like for the case of complete markets, one can replace the dynamic optimization problem (9)–(10) by the following static variational problem:

$$\max_{\mathbb{C}_H} E\left[\int_0^T e^{-\rho t}(\alpha_H(t)+\beta_H(t))\log \mathbb{C}_H(t)\,dt\right] \tag{11}$$

$$\text{s.t.} \qquad E\left[\int_0^T \xi_\nu(t)\mathbb{C}_H(t)dt\right] \le W_H(0), \tag{12}$$

where $\xi_\nu$ denotes an appropriate state price density—i.e., an Arrow-Debreu state price per unit of probability $P$. The difficulty arises from the fact that in incomplete markets, there is an infinite number of such state price densities consistent with no arbitrage and hence potentially an infinite number of static budget constraints (12). However, this set of budget constraints is known to possess some special structure. Let $m$ denote the market price of risk process

$$m(t) \equiv \sigma(t)^\top(\sigma(t)\sigma(t)^\top)^{-1}(\mu(t)-r(t)\mathbf{1}), \tag{13}$$

where $\mu \equiv (\mu_S, \mu_{S^*})^\top$ and $\mathbf{1}$ is a two-dimensional vector of ones. Then the set of state price densities can be represented as (He and Pearson, Proposition 1):

$$d\xi_\nu(t) = -r(t)\xi_\nu(t)dt - (m(t)+\nu(t))^\top \xi_\nu(t)d\vec{w}(t), \tag{14}$$

with $\nu(t) \in \mathbb{R}^4$ satisfying $\sigma(t)\nu(t) = 0$, $\forall t \in [0,T]$ and $\int_0^T ||\nu(t)||^2 dt < \infty$.[6] It is easy to see that if the volatility matrix sigma is a nondegenerate square matrix, the condition $\sigma(t)\nu(t) = 0$ can be satisfied only for $\nu(t) = \mathbf{0}$, where $\mathbf{0}$ is a four-dimensional vector of zeros. This is precisely the case when markets are complete: the state price density is unique and $\nu(t) = \mathbf{0}$ at all $t$. If, however, the volatility matrix has has fewer rows than there are Brownian motions (and hence columns), many possible $\nu(t)$'s can satisfy the restriction $\sigma(t)\nu(t) = 0$. This is the case when markets are intrinsically incomplete.

He and Pearson go on to prove that there exists a unique individual-specific $\nu$, which we denote by $\nu_H$, that minimizes the maximum expected utility in (11). We derive the expression for it in the

[6] The notation $||z||^2$ stands for the dot product $z \cdot z$.

proposition below. The only relevant budget constraint in (12) is then the one corresponding to $\nu_H$. Establishing the portfolio that solves the optimization problem (11)–(12) is then straightforward. We report this portfolio, as well as the portfolio of Foreign, in the following proposition.

**Proposition 1.** *(i) The fractions of wealth $x_H$ and $x_F$ invested in the risky stocks by the Home and the Foreign country, respectively, are given by*

$$x_H(t) = \underbrace{(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\, m(t)}_{\text{mean-variance portfolio}} + \underbrace{(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\, \frac{\sigma_{\alpha_H}(t)^\top + \sigma_{\beta_H}(t)^\top}{\alpha_H(t) + \beta_H(t)}}_{\text{hedging portfolio}} \tag{15}$$

$$x_F(t) = (\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\, m(t). \tag{16}$$

*The fractions of wealth invested in the bond by Home and Foreign are given by $1 - \mathbf{1}^\top x_H(t)$ and $1 - \mathbf{1}^\top x_F(t)$, respectively.*

*(ii) The processes $\nu_H$ and $\nu_F$, entering the specification of the personalized state price densities of Home and Foreign, respectively, are given by*

$$\nu_H(t)^\top = -\frac{\sigma_{\alpha_H}(t) + \sigma_{\beta_H}(t)}{\alpha_H(t) + \beta_H(t)}(I_{4\times 4} - \sigma(t)^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)) \quad \text{and} \quad \nu_F(t) = \mathbf{0}, \tag{17}$$

*where $I_{4\times 4}$ is a $4 \times 4$ identity matrix.*

Consider first the portfolio of the Home consumer. It consists of two parts: the mean-variance efficient portfolio and the hedging portfolio. This decomposition is standard in the portfolio choice literature. The optimal mean-variance portfolio was first derived by Markowitz (1952) in a one-period setting and later generalized by Merton (1971) to a continuous-time stochastic environment. Furthermore, Merton shows that in addition to the mean-variance portfolio an investor optimally selects a hedging portfolio whose role is to offset fluctuations in the state variables in his optimization problem. As is well-known, investors with logarithmic preferences do not wish to hedge against changes in their investment opportunity set (stock and bond price dynamics)—in that sense they behave myopically. However, they do wish to hedge against fluctuations in the state variables entering their preferences, namely the preference shifts. When markets are complete (or effectively complete), the gains made by the hedging portfolio are perfectly positively correlated with the fluctuations in state variable Home desires to hedge: $\alpha_H + \beta_H$. (This is the state variable entering Home's objective function (9)). When markets are incomplete, not every payoff can be replicated and so it is typically not possible to construct a portfolio whose gains are perfectly correlated with a state variable. In that case, the Home investor chooses the portfolio most highly correlated with $\alpha_H + \beta_H$.

In contrast, the Foreign investor demands no hedging portfolio. This is because the term $\alpha_F + \beta_F$ entering his objective function is non-stochastic. Consequently, the inability to hedge perfectly under incomplete markets does not hurt the Foreign investor: in contrast to that of the Home investor, his personalized $\nu_F$ remains the same as it would be under complete markets.

As we elaborate later, (heterogeneous) hedging demands are key vehicles for generating trade between consumers in equilibrium. For example, in the absence of preference shifts, agents have no hedging demands and hence they have no reason to trade assets.

## 2.3. Characterization of Equilibrium

An equilibrium in our economy is defined in a standard way: it is a collection of goods and asset prices ($p$, $p^*$, $S$, $S^*$, $B$) and consumption-investment policies ($C_i(t)$, $C_i^*(t)$, $x_i^S(t)$, $x_i^{S^*}(t)$), $i \in \{H, F\}$ such that (i) each consumer-investor maximizes his utility (7) subject to the budget constraint (6) and (ii) goods, stock, and bond markets clear.

In the economy with incomplete markets the equilibrium allocation would not be Pareto optimal. Hence, the usual construction of a representative agent's (planner's) utility as a weighted sum, with constant weights, of individual utility functions is not possible. Instead, we are going to employ a fictitious representative agent with stochastic weights (introduced in an important contribution by Cuoco and He (1994)), with these stochastic weights reflecting the effects of market incompleteness.[7] This fictitious representative agent maximizes his utility subject to the resource constraints:

$$\max_{\{C_H, C_H^*, C_F, C_F^*\}} \quad E\left[\int_0^T e^{-\rho t}\left(u_H(C_H(t),\, C_H^*(t)) + \lambda(t)\, u_F(C_F(t),\, C_F^*(t))\right)dt\right]$$

$$\text{s. t.} \quad C_H(t) + C_F(t) = Y(t)\,,$$

$$C_H^*(t) + C_F^*(t) = Y^*(t)\,,$$

where we have normalized the weight on the Home consumer to be equal to one and assigned the weight $\lambda$ to the Foreign consumer. The possibly stochastic weighting process $\lambda$ will be linked to the wealth distribution in the economy and will be determined as part of the equilibrium. In the

[7] Alternatively, we could have solved for equilibrium directly from the system of equilibrium equations. We prefer the method we are presenting because of the clarity of the ensuing intuitions. The construction of a representative agent with stochastic weights has been employed extensively in dynamic asset pricing models with financial market frictions. See, for example, Basak and Croitoru (2000), Basak and Cuoco (1998), and Detemple and Serrat (2003). A related approach is the extra-state-variable methodology of Kehoe and Perri (2002). For the original solution method utilizing weights in the representative agent, see Negishi (1960).

event that in an equilibrium $\lambda$ ends up being a constant (we encounter this situation in some of the special cases we consider later), the allocation is Pareto optimal. This situation corresponds to the case of complete or so-called effectively complete financial markets.

Solving the representative agent's optimization problem, we obtain the sharing rules

$$C_H(t) = \frac{\alpha_H(t)}{\alpha_H(t) + \lambda(t)\beta_F} Y(t), \quad C_H^*(t) = \frac{\beta_H(t)}{\beta_H(t) + \lambda(t)\alpha_F} Y^*(t), \tag{18}$$

$$C_F(t) = \frac{\lambda(t)\beta_F}{\alpha_H(t) + \lambda(t)\beta_F} Y(t)\,, \quad C_F^*(t) = \frac{\lambda(t)\alpha_F}{\beta_H(t) + \lambda(t)\alpha_F} Y^*(t)\,. \tag{19}$$

We can now derive the terms of trade that prevail in a competitive equilibrium. They are identified with the ratio of either country's marginal utilities of the Home and Foreign goods:

$$q(t) = \frac{\alpha_H(t) + \lambda(t)\beta_F}{\beta_H(t) + \lambda(t)\alpha_F} \frac{Y^*(t)}{Y(t)}\,. \tag{20}$$

We next use the no-arbitrage valuation principle to obtain stock prices and equilibrium wealth of the countries.

**Lemma 1.** *Equilibrium stock prices in our economy are given by*

$$S(t) = \frac{1 - e^{-\rho(T-t)}}{\rho} \frac{q(t)}{aq(t) + 1 - a} Y(t)\,, \tag{21}$$

$$S^*(t) = \frac{1 - e^{-\rho(T-t)}}{\rho} \frac{1}{aq(t) + 1 - a} Y^*(t) \tag{22}$$

*and the wealth of the countries by*

$$W_H(t) = \frac{\alpha_H(t) + \beta_H(t)}{\alpha_H(t) + \lambda(t)\beta_F} S(t)\,, \quad W_F(t) = \frac{\lambda(t)(\alpha_F + \beta_F)}{\beta_H(t) + \lambda(t)\alpha_F} S^*(t)\,. \tag{23}$$

Lemma 1 yields a simple interpretation of the weight $\lambda$. One can see that

$$\lambda(t) = \frac{W_F(t)(\alpha_H(t) + \beta_H(t))}{W_H(t)(\alpha_F + \beta_F)}. \tag{24}$$

That is, incomplete markets enrich the dynamics of the economy with an additional state variable $\lambda$, which is related to the wealth distribution, but not given exactly by the wealth distribution unless $\alpha_H(t) + \beta_H(t)$ is constant. We have already encountered the expression $\alpha_H(t) + \beta_H(t)$ earlier in our analysis: it was the state variable giving rise to the hedging portfolio held by Home.

Lemma 1 allows us to characterize the dynamics of stock returns and the market price of risk in equilibrium, which are tedious but straightforward to compute. Equation (24) lets us pin down the weight $\lambda$. We relegate the details of the necessary calculations to the Appendix, and report the resulting dynamics of $\lambda$ below.

**Proposition 2.** *(i) In an equilibrium, the weight of the Foreign country in the fictitious representative agent follows*

$$d\lambda(t) = -\lambda(t)\, \nu_H(t)\, d\vec{w}(t), \quad \textit{with} \;\; \lambda(0) = \beta_H(0)/\beta_F.$$

*(ii) When such equilibrium exists, the volatility matrix $\sigma$ and the market price of risk $m$ can be computed as functions of exogenous state variables. They are reported in the Appendix.*

Note that our characterizations the terms of trade, consumption, and stock prices presented in this section all involve the exogenous state variables of the model and one endogenous quantity: the weight $\lambda$. With this weight $\lambda$ now characterized in Proposition 2, we can then pin down these equilibrium quantities and their dynamics. Moreover, the countries' portfolios held in equilibrium are also fully determined now, with the volatility matrix of stock returns and the market price of risk characterized fully in terms of exogenous state variables (see the Appendix). Admittedly, the equilibrium characterizations of the portfolios are not particularly transparent. To develop intuitions, in Section 4 we are going to consider several special cases in which the expressions for the portfolios are simple. The analysis of these special cases relies in part on the result of the following lemma.

**Lemma 2.** *The countries hold no bond in their portfolios if and only if the value of the hedging portfolio demanded by Home is equal to zero.*

**Proof.** Suppose that bondholdings of the countries are zero. This is equivalent to saying that the fraction of wealth each country invests in the stocks is equal to one:

$$\begin{aligned}
\mathbf{1}^\top x_H(t) &= \mathbf{1}^\top \left[(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\, m(t) + (\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\, \frac{\sigma_{\alpha_H}(t) + \sigma_{\beta_H}(t)}{\alpha_H(t) + \beta_H(t)}\right] = 1 \\
\mathbf{1}^\top x_F(t) &= \mathbf{1}^\top (\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\, m(t) = 1,
\end{aligned}$$

where we have substituted the formulas for the portfolios derived in Proposition 1. This can happen only if $\mathbf{1}^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\, \frac{\sigma_{\alpha_H}(t)+\sigma_{\beta_H}(t)}{\alpha_H(t)+\beta_H(t)} = 0$—i.e., the fraction of wealth invested in the hedging portfolio is zero.

Conversely, if the value of the hedging portfolio is zero, then $\mathbf{1}^\top x_H(t) = \mathbf{1}^\top x_F(t) = \mathbf{1}^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)\, m(t)$. Bond market clearing then implies that $\mathbf{1}^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)m(t) = 1$. ■

## 3. External Accounts

From the theoretical and practical point of view, the measure of external sustainability of countries has evolved through time. The original measure reflected simply the trade balance of goods, then it got revised to reflect the trade balance of goods and services, then switched to the (conventionally-defined) current account, and now it is changing again so as to better capture changes in net foreign asset positions. In policy circles, it is not uncommon to disregard all previous measures of external imbalances once a new measure comes to fore.[8]

In this section, we define the trade balance, the conventional current account, and the capital-gains adjusted current account in the context of our model. The first one refers to the balance of imported and exported goods, the second one adds dividend and interest payments to the trade balance, and the last one adds the balance of capital gains to the conventional current account. Our goal is to study the relationships among all these definitions of sustainability. We are going to show that the trade balance and the capital-gains adjusted current account are closely intertwined, and link the conventional current account to the value of the countries' hedging portfolios.

### 3.1. The Trade Balance and the Conventional Current Account

In our model, the *trade balance*—defined as exports minus imports—is simply

$$TB_H(t) = p(t)(Y(t) - C_H(t)) - p^*(t)C_H^*(t).$$

The conventional measure of the *current account* differs from the trade balance in that it also includes net dividend and interest payments (but not capital gains). For expositional simplicity, let us concentrate on the Home country. The conventionally-defined current account in our model is given by

$$CA_H(t) = \Big[TB_H(t) + \underbrace{s_H^{S^*}(t)p^*(t)Y^*(t) - s_F^S(t)p(t)Y(t)}_{\text{Net Dividend Payments}} + \underbrace{s_H^B(t)B(t)r(t)}_{\text{Net Interest Payments}}\Big]dt, \tag{25}$$

where $s_i^j$ denotes the number of shares of asset $j$ held by country $i$. The second and the third terms in (25) are dividend receipts from foreign assets minus dividend payments to Foreign, and the last term is the interest paid on current bondholdings. Recall that each of the above quantities in our

[8] As the former US Treasury Secretary Paul O'Neill once memorably remarked, the (conventional) current account had become a "meaningless concept."

model is defined as a rate (e.g., the export rate, the dividend rate, etc.) and hence need to be scaled by a time increment. This is the reason behind the term "dt" appearing in (25).

An often cited shortcoming of pure-exchange models with log-linear preferences is that they are unable to generate a nontrivial current account. Having a current account equal to zero at all times would clearly hinder any quantitative analysis of current account deficits that we intend to undertake in this paper. It is therefore worth highlighting the situations under which the current account is zero in our model.

**Lemma 3.** *The current account of the Home country can be represented as follows:*

$$CA_H(t) = s_H^B(t)B(t)\left(r(t) - \frac{\rho}{1-e^{-\rho(T-t)}}\right)dt. \tag{26}$$

**Proof.** Note that

$$\begin{aligned}
s_H^B(t)B(t) &= W_H(t) - s_H^S(t)S(t) - s_H^{S^*}(t)S^*(t) \\
&= \frac{\alpha_H(t)+\beta_H(t)}{\alpha_H(t)+\lambda(t)\,\beta_F}S(t) - (1-s_F^S(t))S(t) - s_H^{S^*}(t)S^*(t) \\
&= \frac{1-e^{-\rho(T-t)}}{\rho}\left[p(t)Y(t)\frac{\beta_H(t)-\lambda(t)\,\beta_F}{\alpha_H(t)+\lambda(t)\,\beta_F} - s_H^{S^*}(t)p^*(t)Y^*(t) + s_F^S(t)p(t)Y(t)\right],
\end{aligned}$$

where the second equality follows from Lemma 1 and stock market clearing ($s_H^S(t) = 1 - s_F^S(t)$), and the last one, again, from Lemma 1. On the other hand, by substituting (18) into (25) and simplifying, one can show that

$$\begin{aligned}
CA_H(t) &= \left[-p(t)Y(t)\frac{\beta_H(t)-\lambda(t)\,\beta_F}{\alpha_H(t)+\lambda(t)\,\beta_F} + s_H^{S^*}(t)p^*(t)Y^*(t) - s_F^S(t)p(t)Y(t) + s_H^B(t)B(t)r(t)\right]dt \\
&= s_H^B(t)B(t)\left(-\frac{\rho}{1-e^{-\rho(T-t)}} + r(t)\right)dt.
\end{aligned}$$

This proves the statement in (26). ■

This lemma reveals that the first sufficient condition for the current account to be equal to zero is that the Home country (and hence the Foreign) holds no bonds. While it is indeed a common implication of models with log-linear preferences to have zero net bond holdings in equilibrium (e.g., Pavlova and Rigobon (2007), Pavlova and Rigobon (2008)), nothing in our model prevents the bond holdings from being different from zero. That is, preference shifts may potentially introduce enough heterogeneity among the countries so that they are willing to trade in all available financial assets for risk-sharing purposes. The second condition under which the current account is zero is when the interest rate $r(t)$ is equal to $\rho/(1-e^{-\rho(T-t)})$. The latter quantity is deterministic, while the interest rate is a stochastic process. Hence, it is true only on the measure zero set of parameter values.

### 3.2. The Capital-Gains Adjusted Current Account

Defining the current account as the change in the net foreign asset position of a country, we have

$$CGCA_H(t) = d\Big[s_H^{S^*}(t)S^*(t) - s_F^S(t)S(t) + s_H^B(t)B(t)\Big], \tag{27}$$

where the first two terms in the square brackets are Home's investment in the Foreign stock minus Foreign' investment in the Home stock, and the last term is Home's balance on the bond account. The label "$CGCA$" stands for "*capital-gains adjusted current account*," reflecting the fact that the above measure properly accounts for the capital gains on the NFA positions.

Note that, by market clearing, $s_F^S(t) = 1 - s_H^S(t)$ and that, by definition, Home's financial wealth equals its portfolio value, $W_H(t) = s_H^S(t)S(t) + s_H^{S^*}(t)S^*(t) + s_H^B(t)B(t)$. Hence, we can rewrite (27) as

$$CGCA_H(t) = dW_H(t) - dS(t). \tag{28}$$

### 3.3. Congruence between NFA and Trade Balance

To conclude this section, we derive the expression for NFA in our model and draw a connection between the NFA position and the trade balance. Note that

$$\begin{aligned} NFA_H(t) &= W_H(t) - S(t) \\ &= \frac{1}{\xi_{\nu_H}(t)} E_t\left[\int_t^T \xi_{\nu_H}(s)\,(p(s)\,C(s) + p^*(s)\,C^*(s))ds\right] - \frac{1}{\xi_{\nu_H}(t)} E_t\left[\int_t^T \xi_{\nu_H}(s)\,p(s)\,Y(s)\,ds\right]. \end{aligned}$$

Hence, by definition of a trade balance,

$$NFA_H(t) = -\frac{1}{\xi_{\nu_H}(t)} E_t\left[\int_t^T \xi_{\nu_H}(s)\,TB_H(t)\,ds\right]. \tag{29}$$

Equation (29) is nothing else but the familiar statement that the NFA position is given by the present value of the future trade deficits. The traditional intertemporal approach to external adjustment—that ignores changes in the state price density (or the stochastic discount factor) $\xi_{\nu_H}$—says that, for example, for a country with a negative NFA position, adjustment must come through future trade surpluses. Recent literature challenges this conclusion and draws attention to the "valuation channel" of the external adjustment that operates precisely through changes in the stochastic discount factor. It is argued that such changes are large and volatile, and hence the valuation channel should have a substantial contribution to the NFA dynamics. Surprisingly, it turns out that in our model, after the endogenous responses of asset prices and hence the stochastic discount factor to underlying shocks are taken into account, the NFA adjustment takes place instantaneously and entirely through the trade balance. In that sense, the traditional and the new views are not at all inconsistent.[9]

**Lemma 4 (Congruence between $NFA$ and trade balance).** *The relationship between the net foreign assets and the trade balance is given by*

$$NFA_H(t) = -\frac{1 - e^{-\rho(T-t)}}{\rho} TB_H(t). \tag{30}$$

*The net foreign asset position of Home is*

$$NFA_H(t) = \frac{\beta_H(t) - \lambda(t)\,\beta_F}{\alpha_H(t) + \lambda(t)\,\beta_F}\, S(t)\,. \tag{31}$$

We remark that the perfect negative correlation between the current trade balance and the NFA position is due primarily to our assumption that the agents have log-linear preferences that rule out intertemporal hedging motives. It is important to evaluate the robustness of this result under alternative preferences that give rise to intertemporal hedging.

Our congruence relationship may remind the reader of the textbook formula from the intertemporal approach to the current account (Obstfeld and Rogoff (1996), Chapter 2). That formula is derived in the context of an economy with no stocks, under many simplifying assumptions. The economy we are considering is significantly more complicated—featuring endogenous portfolio recompositions and trade in equities under incomplete markets—and yet the relationship between the NFA position and the trade balance comes out similar to that in the classical external adjustment theory.

## 4. Equilibrium Portfolios

As we have been stressing, in our model there is a close relationship between the countries' external accounts and their equilibrium portfolio compositions. These interconnections take place through

[9] It is important to note that the "connection" is between the NFA and the trade balance of goods and services, and not between NFA and the conventional current account.

different mechanisms—related to the terms of trade, asset prices, hedging motives, or portfolio choices. In this section, we present special cases of our model that highlight such mechanisms. To establish a benchmark, we start by replicating some results of the earlier literature—in particular, that under certain conditions financial markets could be irrelevant for the allocation of consumption. In this special case of our model, there is perfect risk sharing and the stock market returns are perfectly correlated around the world. Risk diversification occurs through terms of trade fluctuations. Second, we study the case in which the stock market returns are not perfectly correlated, but the optimal portfolios again fully diversify international risk. In this case, the channel of diversification comes from the countries' holdings, and not (entirely) from the terms of trade. The case we study is one in which shocks to any of the countries have no implications for the conventional current account and the change in net foreign asset positions. In this example, portfolio holdings are constant through time and therefore, capital flows are zero. The third example highlights the valuation effects. We have set up this example in such a way that the value of the hedging portfolio is zero, which in turn implies that the bond holdings and the conventional current account are always equal to zero. However, the capital-gains adjusted current account does vary in response the underlying shocks because asset prices and optimal portfolios are not constant. On a separate note, it is important to mention that even though in this example markets are intrinsically incomplete, the (conventional) current account is always zero. Finally, we study a case in which the value of the hedging portfolio is different from zero. We specialize our setup so that markets are complete, but all external accounts are nonzero. We analyze the relationship between the current account and portfolio compositions in this environment. In summary, the objective of this section is to cumulatively develop the intuitions behind the interconnections that our model exhibits, which we do via examples that capture several different aspects of the workings of the model.

### 4.1. Example 1: The Irrelevance Result

The first example we study is one in which the financial markets' structure is irrelevant, there are no net portfolio flows, and therefore, capital gains on financial assets play no role in the international adjustment process. This is the case considered in Helpman and Razin (1978), Cole and Obstfeld (1991), and Zapatero (1995). The three examples that follow relax some of the assumptions we make here in order to clarify the role that capital gains play.

In our model, we obtain our irrelevance result by specializing the Home consumer's preferences so that $\alpha_H$ and $\beta_H$ are constant ($\sigma_{\alpha_H}(t) = \sigma_{\beta_H}(t) = \mathbf{0}$). As is well-known, under this specification the returns on the two stocks are perfectly correlated in equilibrium. Hence, portfolio allocations into these stocks are indeterminate. Only positions in the composite stock market, $S(t) + S^*(t)$, can be uniquely determined. Using an argument analogous to that we employed in Section 2.2, we can derive the investors' optimal fractions of wealth invested in the composite stock market. One can easily see that the portfolios demanded by the two agents are going to be identical: in the absence of preference shifts, they both demand the mean-variance but no hedging portfolios.

It is also well-known that financial markets are effectively complete in this special case. This is equivalent to saying that $\nu_H(t) = \nu_F(t) = \mathbf{0}$ at all times and hence, from Proposition 2, the weight $\lambda$ is constant. Thus, Pareto optimality obtains despite market incompleteness—or, in other words, markets are effectively complete. Investors are not adversely affected by market incompleteness because they do not make use of financial markets to construct portfolios hedging against fluctuations in any state variables: there is no state variable either agent desires to hold a hedge against. The intuition for why the financial markets are not needed in this case comes from the fact that movements in the terms of trade exactly offset output shocks and hence the values of the dividends on the Home and the Foreign stock markets, $p(t)Y(t)$ and $p^*(t)Y^*(t)$, respectively, are always the same (up to a multiplicative constant). Fluctuations in the terms of trade therefore fully offset the supply shocks; i.e., with no demand uncertainty the capital gains on the two stocks are always perfectly correlated. This feature of our model is due to the way we specified preferences (log-linear) and endowments (shares of trees), and represents a simple benchmark for comparison.

Finally, unlike holdings of individual stocks, the bond holdings of the countries are uniquely determined: there are equal to zero at all times.[10] This is because, the two countries demand the same portfolio and in particular, wish to invest the same fraction of wealth in the bond. For the bond market to clear, this fraction has to be zero. Consequently, $CA(t) = 0$ at all times. Moreover, the capital-gains adjusted current account $CGCA$ is also zero. Each country is holding the same portfolio and stock markets are perfectly correlated. Therefore, the net dividend payments and the net capital gains account have to be zero as well.

[10]This result is not new. See Cass and Pavlova (2004).

### 4.2. Example 2: No Asset Cross-Holdings

The purpose of this example is to illustrate the role of international asset cross-holdings and portfolio rebalancing. We consider a version of our model in which portfolio holdings are unique (as opposed to indeterminate as in the previous case), but portfolio holdings are fixed (hence there is never a rebalancing after any of the shocks). We show that in this case the current account and the capital-gains current account are zero at all times. Hence, movements in asset prices are unrelated to the external adjustment.

To highlight these dynamics, consider the special case of the model in which $\beta_H$ remains constant ($\sigma_{\beta_H}(t) = \mathbf{0}$), and $\alpha_H$ is stochastic. In the presence of preference shifts—even one possible shift, as we specify here—the two stocks are no longer perfectly correlated, the volatility matrix is invertible, and hence the expressions in Propositions 1 and 2 readily apply. It turns out that the equity portfolios of the countries, expressed as numbers of shares, take a particularly simple form

$$s_H(t) = \left(1,\ \frac{\beta_H - \lambda(t)\beta_F}{\lambda(t)\alpha_F + \beta_H}\right) \quad \text{and} \quad s_F(t) = \left(0,\ \frac{\lambda(t)(\alpha_F + \beta_F)}{\lambda(t)\alpha_F + \beta_H}\right),$$

where $s_i \equiv (s_i^S,\ s_i^{S^*})$ are obtained from $x_i$ using Lemma 1.

The hedging portfolio held by Home, in numbers of shares, is

$$h(t) = \left(1,\ -\frac{\alpha_H(t) + \lambda(t)\beta_F}{\lambda(t)\alpha_F + \beta_H}\right).$$

The (instantaneous) gain on this hedging portfolio is given by

$$d\,h(t) = [\ldots]\,dt + \frac{W_H(t)}{\alpha_H(t) + \beta_H}\,\sigma_{\alpha_H}(t)\,d\vec{w}(t),$$

where the drift term need not concern us here. Note that the gain on the hedging portfolio is *perfectly instantaneously correlated* with fluctuations in the preference shifts $\alpha_H$. (Recall that $d\,\alpha_H(t) = \sigma_{\alpha_H}(t)\,d\vec{w}(t)$.) Therefore, despite market incompleteness, the Home investor is able to construct a portfolio perfectly correlated with its preference shock. It is of no surprise then that it turns out that $\nu_H(t) = \mathbf{0}$: the investor is able to achieve the same efficiency of hedging as under complete markets. Consequently (Proposition 2), again, the weight $\lambda$ is constant and markets are effectively complete. In contrast to the no preference shifts case, however, one can see that effective market completeness does not lead to the indeterminacy of equilibrium portfolios.

Having established that $\lambda$ is a constant, we can further simplify the expressions for the countries' portfolios. From Proposition 2, $\lambda = \lambda(0) = \beta_H/\beta_F$, and hence the portfolios of the countries, in numbers of shares, are simply $s_H(t) = (1,\ 0)$ and $s_F(t) = (0,\ 1)$. Note that Home ends up holding the entire supply of the Home stock, and Foreign holds the entire supply of its stock as well.

Note that in this example we have an extreme portfolio home bias (an apparent home bias, because in this case the optimal portfolio is to hold all of the home stock and none of the foreign). It is important to point out that the home bias is coming because the demand shock is affecting home demand for home goods (this is equivalent to explicitly modeling shocks to the non-tradable demand which has been already highlighted in the literature). It is equally important to highlight that the home bias in consumption has *nothing* to do with the home bias in portfolios in this case. The home bias in consumption in country $i$ occurs when $\alpha_i$ is larger than $\beta_i$. However, for this result $\alpha_i$ can be larger or smaller than $\beta_i$ in either country, and the home bias in portfolios will remain as long as the demand shocks affect the preference for the home good (i.e., $\alpha_H$ is stochastic) as opposed to that for the foreign good.

Regarding bond holdings, just like in the previous special case, the countries invest nothing in the bond. To see this, we compute the value of the hedging portfolio $h$ and conclude (from Lemma 1) that it is equal to zero at all times. The result then follows from Lemma 2. Intuitively, the hedging portfolio held by Home is a costless long-short portfolio of the two available stocks. If it were not costless, the investor would need to borrow or lend on his bond account in order to finance it.

Finally, note that

$$CA_i(t) = CGCA_i(t) = 0, \qquad i \in \{H,\ F\}.$$

The conventional current account is zero because none of the countries invests in the bond (Lemma 3). The capital-gains adjusted current account is also zero simply because the countries' end up owning no foreign assets and no bonds. Hence, by definition (equation (27)), both the net foreign asset positions of the countries and their capital-gains adjusted current accounts are zero. In this case, the net capital gains and net expected return accounts are zero because the countries exhibit a 100 percent home bias, and not because the stock markets are perfectly correlated (as they were in the previous case).

### 4.3. Example 3: Valuation Effects

We now consider a more general case of our model in which the current account is still identically equal to zero but the capital-gains adjusted current account is now different from zero. The difference between the two definitions is, of course, due to the expected and unexpected capital gains on NFA. The external adjustment process in this example is therefore driven entirely by the valuation effects and has nothing to do with traditional channels.

We consider a special case of the model in which the preference shifter $\alpha_H$ is driven only by the Brownian motion $w^\alpha$ and $\beta_H$ by the Brownian motion $w^\beta$—i.e., the demand shocks are completely independent of the supply shocks. Formally, we assume that $\sigma_{\alpha_H}(t) = (0,\, 0,\, \sigma_{\alpha_1}(t),\, 0)$ and $\sigma_{\beta_H}(t) = (0,\, 0,\, 0,\, \sigma_{\beta_2}(t))$, with $\sigma_{\alpha_1} > 0$ and $\sigma_{\beta_2} > 0$. The stockholdings of Home and Foreign, respectively, are as follows:

$$s_H = \frac{1}{G}\left(\frac{\beta_H}{\sigma^2_{\beta_2}}(\lambda\alpha_F + \beta_H) + \frac{\alpha_H}{\sigma^2_{\alpha_1}}(\alpha_H - \lambda\,\alpha_F), \quad \frac{\beta_H}{\sigma^2_{\beta_2}}(\beta_H - \lambda\,\beta_F) + \frac{\alpha_H}{\sigma^2_{\alpha_1}}(\alpha_H + \lambda\beta_F)\right), \tag{32}$$

$$s_F = \frac{\lambda(\alpha_F + \beta_F)}{G}\left(\frac{\alpha_H}{\sigma^2_{\alpha_1}},\, \frac{\beta_H}{\sigma^2_{\beta_2}}\right), \tag{33}$$

where $G \equiv \frac{\beta_H}{\sigma^2_{\beta_2}}(\lambda\alpha_F + \beta_H) + \frac{\alpha_H}{\sigma^2_{\alpha_1}}(\alpha_H + \lambda\,\beta_F)$. In the expressions above and for the remainder of this section, we suppress the argument $t$. It already becomes clear at this point that depending on parameter values, our model can produce large gross portfolios.[11]

The hedging portfolio, $h$, held by Home, in numbers of shares, is

$$h = \frac{\frac{\beta_H}{\sigma^2_{\beta_2}} - \frac{\alpha_H}{\sigma^2_{\alpha_1}}}{G}\,(\lambda\alpha_F + \beta_H, \quad -(\alpha_H + \lambda\beta_F))\,. \tag{34}$$

Consider again the gain on the hedging portfolio:

$$dh = [\ldots]\,dt + \frac{W_H\beta_H\left(\frac{\beta_H}{\sigma^2_{\beta_2}} - \frac{\alpha_H}{\sigma^2_{\alpha_1}}\right)}{(\alpha_H + \beta_H)\left(\frac{\beta_H^2}{\sigma^2_{\beta_2}} + \frac{\alpha_H^2}{\sigma^2_{\alpha_1}}\right)}\,(0,\, 0,\, \sigma_{\alpha_1},\, \sigma_{\beta_2})\,d\vec{w},$$

and compare it to the fluctuations in the state variable, $\alpha_H + \beta_H$, that Home desires to hedge against:

$$d(\alpha_H + \beta_H) = (0,\; 0,\; \sigma_{\alpha_1},\; \sigma_{\beta_2})\,d\vec{w}.$$

(Recall that $\alpha_H + \beta_H$ is the state variable entering Home's objective function (9).) Unlike in the previous special case, the hedge is no longer perfect. If $\beta_H/\sigma^2_{\beta_2} > \alpha_H/\sigma^2_{\alpha_1}$, the hedging portfolio

[11] The shareholdings simplify considerably in the case of full symmetry ($\alpha_H = \alpha_F = \alpha$, $\beta_H = \beta_F = \beta$, $\lambda = 1$, and $\sigma_{\alpha_1} = \sigma_{\beta_2} = \sigma$).

$$s_H = \frac{1}{\alpha + \beta}\,(\beta,\; \alpha)\,, \quad s_F = \frac{1}{\alpha + \beta}\,(\alpha,\; \beta)\,,$$

In this case, the portfolio holdings of the two countries are mirror images of each other and the extent of the portfolio home bias is directly related to the degree of consumption home bias.

gains in response to a positive shock in $\alpha_H$ (an innovation to $w^\alpha$). But it loses value if the economy is hit by a positive $\beta_H$ shock. The opposite is true for $\beta_H/\sigma^2_{\beta_2} < \alpha_H/\sigma^2_{\alpha_1}$. In any event, Home is able to perfectly hedge against an $\alpha_H$ or a $\beta_H$ shock but not both. The condition determining which shock to focus on reflects the relative importance of a shock. Ceteris paribus, if the volatility of, say, the $\alpha_H$ shock, $\sigma_{\alpha_1}$, is high, Home holds a hedging portfolio that is positively correlated with $\alpha_H$; otherwise, it prefers instead a portfolio positively correlated with the $\beta_H$ shock. Note that the holdings of the two stocks in the hedging portfolio have the opposite sign, and this sign depends on the sign of $\beta_H/\sigma^2_{\beta_2} - \alpha_H/\sigma^2_{\alpha_1}$. This implies that, depending on the relative importance of the two demand shocks, our model can produce a home bias or a reverse home bias in portfolios. Note that the condition determining the direction of the bias is not same one as that for the home bias in consumption ($\alpha_H > \beta_H$), as is often thought.

The inability to hedge perfectly is indicative of the fact that market incompleteness matters. Indeed, in equilibrium,

$$\nu_H = \left(0, \quad 0, \quad -\frac{\alpha_H\,\sigma_{\alpha_1}\,\sigma^2_{\beta_2}}{\beta^2_H\,\sigma^2_{\alpha_1} + \alpha^2_H\,\sigma^2_{\beta_2}}, \quad -\frac{\beta_H\,\sigma^2_{\alpha_1}\,\sigma_{\beta_2}}{\beta^2_H\,\sigma^2_{\alpha_1} + \alpha^2_H\,\sigma^2_{\beta_2}}\right), \tag{35}$$

and hence the weight $\lambda$ follows a stochastic process. The zeros in the first two positions of $\nu_H$ are not accidental. Since the preference shifts that the Home country faces are uncorrelated with the output shocks, it demands a hedge correlated with the Brownian motions $w^\alpha$ and $w^\beta$ but not $w$ and $w^*$. Constructing such a hedging portfolio is possible: one can easily show that any zero-cost portfolio of the two stocks is going to be uncorrelated with the output shocks. The hedging portfolio $h$ must then have a value of zero, and one can easily verify from (34) and Lemma 1 that this is indeed the case. As a corollary, none of the countries holds any bond (Lemma 2) and hence their current accounts are zero (Lemma 3).

Now let us examine the capital-gains adjusted current account of, say, Home. Recall from Lemma 4 that $NFA_H = \frac{\beta_H - \lambda\,\beta_F}{\alpha_H + \lambda\,\beta_F}\,S$. Unlike in the two previous special cases, where it turned out that $\beta_H - \lambda\,\beta_F = 0$, Home's net foreign asset position is no longer zero at all times. We find that in our model, the sign of the responses of the capital-gains adjusted current account to the underlying innovations depends on whether the country is a net debtor or a net creditor. We thus need the following condition:

**Condition A1: Home is Net Creditor.** $\beta_H - \lambda\beta_F > 0$.

We can then sign (or characterize) the direction the valuation effects. Table 1 presents the unexpected gains/losses on the stocks and the ensuing gains/losses on the NFA position of the Home country. An *unexpected* change in a variable $Z$ at time $t$ is defined as the diffusion component, $\sigma_Z(t)$, in the dynamics $dZ(t) = \mu_Z(t)dt + \sigma_Z(t)d\vec{w}(t)$. (The growth in $Z$ due to the trend term, $\mu_Z(t)$, is the *expected* change.)

| | $dw$ | $dw^*$ | $dw^\alpha$ | $dw^\beta$ |
|---|---|---|---|---|
| Unexpected change in $S$ | $+$ | $+$ | $+$ | $-$ |
| Unexpected change in $S^*$ | $+$ | $+$ | $-$ | $+$ |
| $CGCA_H$ | $+^{A1}$ | $+^{A1}$ | $-^{A1}$ | $+^{A1}$ |

Table 1: The valuation effects: Unexpected gains on stocks and Home's the net foreign assets ($CGCA_H$) in response to the underlying shocks. The superscript $^{A1}$ indicates that a sign is valid under Condition A1.

For the signs of the unexpected gain/loss on the Home's net foreign asset position in response to the output shocks, Condition A1 is necessary and sufficient. This observation is similar to the one made in Kraay and Ventura (2000). However, the condition is only a sufficient one for the signs of the responses to the demand shocks, which give rise to more complex dynamics.

Table 1 reveals that on impact, both stocks yield unexpected capital gains in response to a positive output shock in either country ($dw$ or $dw^*$).[12] This is because a positive output shock in say, Home, raises the dividend on the Home tree. At the same time, it causes a deterioration of Home's terms of trade because the Home good becomes less scarce. This in turn improves Foreign's terms of trade and hence raises the value of the output of the Foreign tree. Hence, both stock markets go up. The remaining signs of the unexpected gains on the stocks are summarized in the last two columns. The reaction of stock prices to preference shifts has a distinctly different pattern: the preference shifts make the stock prices always move in opposite directions. As Home shifts its preference towards the Home good (in response to a positive realization of $dw^\alpha$), there is an excess demand for the Home good in the world. This pushes the price of the Home good up, or equivalently, causes an appreciation of the terms of trade, $q$. This raises the value of the Home output relative to Foreign. Consequently, the price of the Home stock increases, while that of the Foreign stock falls.

To understand the intuition behind the impact of the valuation effects on the capital-gains adjusted current account, note first that the budget constraint of Home (6) can be equivalently

[12]To establish this result, we have explicitly computed the diffusion terms $\sigma_S S$ and $\sigma_{S^*} S^*$ in the equilibrium processes for the stock prices (4)–(5) and signed them. See the proof of Proposition 2 in the Appendix.

represented as

$$
\begin{aligned}
dW_H(t) &= \Big[s_H^B(t)\, B(t)\, r(t) + s_H^S(t) S(t)\mu_S(t) + s_H^{S^*}(t) S^*(t)\mu_{S^*}(t)\Big] dt \\
&\quad + \Big[s_H^S(t) S(t)\sigma_S(t) + s_H^{S^*}(t) S^*(t)\sigma_{S^*}(t)\Big] d\vec{w}(t) + \Big[TB_H(t) - p(t)Y(t)\Big] dt\,.
\end{aligned}
$$

We can then substitute this expression into (28) and use (4) and the stock market clearing

$$
\begin{aligned}
CGCA_H(t) &= \Big[TB_H(t) + s_H^{S^*}(t) S^*(t)\mu_{S^*}(t) - s_F^S(t) S(t)\mu_S(t) + s_H^B(t) B(t) r(t)\Big] dt \\
&\quad + \Big[s_H^{S^*}(t) S^*(t)\sigma_{S^*}(t) - s_F^S(t) S(t)\sigma_S(t)\Big] d\vec{w}(t), \qquad (36)
\end{aligned}
$$

Now note that Foreign always holds a positive position in Home stock and, under Condition A1, Home has a positive position in the Foreign stock (see equations (32)–(33)). It is then immediate from (36) and the first two rows of Table 1 that the unexpected component term has to be negative for the $dw^\alpha$ and positive for the $dw^\beta$ shock. In the case of a preference shift towards the Home good due to a positive $dw^\alpha$, Home loses on its investment in the Foreign stock while Foreign gains on its investment in the Home stock—hence the fall in the current account surplus of Home. The opposite is true for a preference shift towards the Foreign good due to a positive $dw^\beta$.

Finally, it turns out that in this example the directions of the agents' portfolio reallocations in response to the underlying shocks are unambiguous. In particular, portfolios respond to demand shocks, but not to the supply shocks, as reported in Table 2.

| Variable/ Effects of | $dw(t)$ | $dw^*(t)$ | $dw^\alpha(t)$ | $dw^\beta(t)$ |
|---|---|---|---|---|
| $ds_H^S$ | 0 | 0 | $-$ | $+$ |
| $ds_H^{S^*}$ | 0 | 0 | $+$ | $-$ |

Table 2: Impact responses of Home's portfolio holdings to the underlying shocks.

### 4.4. Example 4: Bondholdings and the Current Account

This final example illustrates what is required for the current account to have a nontrivial dynamics. As we can anticipate from the previous examples, the key implication that produces such dynamics is that the countries have nontrivial bondholdings.

To induce the countries to trade in bonds, we now allow for the correlation between the preference shifts and the output shocks. To keep the model tractable, however, we reduce the number of Brownian motions driving the economy from four to two. In particular, we shut down Brownian motions $w^{\alpha}$ and $w^{\beta}$ and require that all processes are adapted to the filtration generated by the output shocks $w$ and $w^{*}$. Under this modification, all four-dimensional vectors in our analysis in Section 2 and the Appendix become two-dimensional. This implies further that the volatility matrix of stock returns $\sigma$ is a $2 \times 2$ square matrix. If this matrix is nondegenerate—which is always the case in the presence of stochastic preference shifts—financial markets are complete. Equilibrium allocation is then Pareto optimal and the weight $\lambda$ is constant.

In the interest of space, we do not report the countries' portfolios in this case. It suffices to say that now portfolios depend on all of the parameters of the model except for the drifts of outputs. As to be expected, the gain on the hedging portfolio in this case is perfectly correlated with the fluctuations in $\alpha_H + \beta_H$:

$$d\,h = [\ldots]\,dt + \frac{W_H}{\alpha_H + \beta_H}\left(\sigma_{\alpha_H} + \sigma_{\beta_H}\right) d\vec{w}.$$

In contrast to all the special cases we have considered so far however, the value of the hedging portfolio is not equal to zero. Lemma 2 then implies that now the countries engage in borrowing and lending. Furthermore, for some special cases of this economy, the bondholdings always have a unique sign. For example, if we set $\sigma_{\alpha_H} = (\sigma_{\alpha_1},\, 0)$ and $\sigma_{\beta_H} = (0,\, \sigma_{\beta_2})$, with $\sigma_{\alpha_1} > 0$ and $\sigma_{\beta_2} > 0$, the value of the bondholdings of the Home country becomes

$$-\frac{\left(1 - e^{-\rho(T-t)}\right)}{\rho}\, \frac{Y\,Y^{*}\lambda\left(\alpha_F + \beta_F\right)\sigma_{\alpha_1}\sigma_{\beta_2}}{(1-a)Y\left(\lambda\alpha_F + \beta_H\right)^2 \sigma_{Y^{*}}\sigma_{\alpha_1} + aY^{*}\left(\alpha_H + \lambda\beta_F\right)^2 \sigma_Y\ \sigma_{\beta_2}}.$$

Home borrows from Foreign to finance its hedging portfolio, whose value is always greater than zero in this case. This example demonstrates that in our model it is possible to have a negative bond position forever. This does not in any way contradict sustainability of a country's external position: if an equilibrium exists, the budget constraints of both countries are always satisfied, and so a negative position in the bond account is offset by positive positions in the stocks.

It follows from Lemma 3 that the countries current accounts are nozero. This is the first time we encounter a nonzero current account in this section. As the case we are considering here demonstrates, enough heterogeneity in hedging demands that is sufficient to give rise to trade in bonds for risk sharing purposes guarantees that the current account deviates from zero.

The analysis of the capital-gains adjusted current account is less transparent when the preference shifts depend on the output shocks. This is because the capital gains on the stocks in response to all shocks no longer have unique signs. Recall from our earlier discussions (Section 4.3) that the

Home stock responds positively to an output shock in either country, positively to the preference shift towards the Home good ($d\alpha_H > 0$), but negatively to the preference shift towards the Foreign good. The demand and supply effect reinforce each other for the case of the Home output shock (because $\alpha_H$ loads positively on $dw$), but they go in the opposite direction for the case of the Foreign output shock (because $\beta_H$ loads positively on the Foreign output shock). The analogous argument holds for the Foreign stock.

## 5. Discussion

Some of the recent literature has drawn attention to the relevance of the quality of international assets for the discussion of global imbalances (see Caballero, Farhi, and Gourinchas (2006), Dooley, Folkerts-Landau, and Garber (2004), and Blanchard, Giavazzi, and Sa (2005) for the link to global imbalances, and Kouri (1982) for an earlier discussion). Because in our model the entire output of each country is capitalizable and there are no restrictions on capital flows, financial assets do not vary in their quality. But we believe that differences in asset quality is an important feature of international capital markets, and therefore it would be interesting to extend our framework to include this element into the analysis.

Extending the framework beyond log-linear preferences may also prove fruitful. This would introduce some of the intertemporal hedging motives that have been shut down in our model. Moving away from the log-linear specification, however, has the drawback that the model loses its tractability. For instance, for the case of CES preferences, it is not possible to obtain closed-form characterizations for portfolios and asset prices.[13] There are three ways in which one can tackle such a model. First, one can attempt to solve the model numerically. To our knowledge, this has been done only for the complete-markets case (Gourinchas and Rey (2006))—an extension to the incomplete-markets case is a daunting task. Second, one can follow, for example, Devereux and Sutherland (2006) and Tille and van Wincoop (2007) and approximate around a deterministic steady state. Finally, one can recognize that log-linear preferences are a special case of CES preferences and build on our model to find an approximate solution for the CES case. To do so, one can perturb the equilibrium in our economy by expanding around the unitary elasticity of substitution, for which the solutions are analytical.[14] The advantage of this approach is that the

[13]The analysis of the NFA position is analytically tractable only for the case of complete markets (see ).

[14]This idea is closely related to the works of Judd (1998) and Kogan and Uppal (2003) who develop applications of perturbation methods to solving problems in economics and finance.

approximation is done around a stochastic equilibrium as opposed to a deterministic steady state.

## 6. Conclusion

In his Harms Lecture at the Kiel Institute, Obstfeld (2004) stresses that "recent changes in the functioning of international capital markets require a new view of external adjustment" and moreover, that any notion of "external balance adjustment cannot be defined without reference to the structure of national portfolios." In this paper, we take a step in that direction. We develop an open economy model with endogenous portfolio decisions, in which we investigate the interaction between capital markets and the external adjustment process.

From the methodological point of view, our contribution is to construct a framework that is rich enough to include multiple risky assets, incomplete markets, and supply- and demand-side uncertainty, while at the same time simple enough to allow for closed-form characterizations of asset prices, net foreign asset positions, and equity portfolios. It is within this framework that we are able to establish the interconnections between the real side of the economy represented by the trade balance, current account, and consumption allocations and the financial side such as portfolio holdings, stock prices, and valuation changes.

From the policy point of view, one surprising result in our paper is that even though valuation effects play an important role in the adjustment process, there is a tight link between the trade balance of good and services—the traditional and the preferred policy target—and the new measure of external sustainability based on the market value of net foreign assets—that is extremely difficult to measure and target. On the other hand, this relationship does not exist between the current account and the valuation-effects adjusted measures. Hence, the discussion regarding the disconnect between the new measures of sustainability and the classical ones is far from over.

Of course, the implications that we highlight have been derived in the context of our model, and as any model, it is a highly simplified depiction of reality. Future research must go beyond our stylized framework and establish tighter links with the data.

# Appendix

**Proof of Proposition 1.** In this proof, we closely follow He and Pearson (1991). Their analysis is presented in the context of a single-good economy, but this does not present a difficulty for us because (in the main text) we have reduced our problem to a representation that is equivalent to a familiar single-good one. In particular, the first-order conditions for the consumer problem (11)–(12) have the familiar form

$$e^{-\rho t}\frac{\alpha_i(t)+\beta_i(t)}{\overline{\mathbb{C}}_i(t)} = y_i \xi_{\nu_i}(t)\,, \qquad i \in \{H,\, F\}, \tag{A.1}$$

where the Lagrange multiplier $y_i$ is such that the budget constraint evaluated at the optimal consumption expenditure, $\overline{\mathbb{C}}$, is satisfied with equality:

$$E\left[\int_0^T \xi_\nu(t)\overline{\mathbb{C}}_H(t)dt\right] = W_H(0)\,, \qquad i \in \{H,\, F\}.$$

It follows that, by no-arbitrage, the time-$t$ wealth of a consumer is given by

$$W_i(t) \;=\; E_t\left[\int_t^T \frac{1}{\xi_{\nu_i}(t)}\xi_{\nu_i}(s)\overline{\mathbb{C}}_i(s)ds\right]\,, \qquad i \in \{H,\, F\},$$

and hence, making use of (A.1) and the assumption that $\alpha_i$ and $\beta_i$ are martingales, we have

$$W_i(t) = \frac{\alpha_i(t)+\beta_i(t)}{y_i\xi_{\nu_i}(t)}\,\frac{e^{-\rho t}-e^{-\rho T}}{\rho}, \qquad i \in \{H,\, F\}. \tag{A.2}$$

Of course, for the case of the Foreign country, the arguments $\alpha_F$ and $\beta_F$ are constant over time.

To find optimal portfolios, we apply Itô's lemma to (A.2) and match the corresponding diffusion term to that in the dynamic budget constraint (10). This operation yields

$$x_i^\top(t)\sigma(t) = \frac{\sigma_{\alpha_i}(t)+\sigma_{\beta_i}(t)}{\alpha_i(t)+\beta_i(t)} + (m(t)+\nu_i(t))^\top, \tag{A.3}$$

where we have used equation (14). Recall that in incomplete markets the matrix $\sigma$ is not a square matrix, and hence the above system of equations contains 4 equations (dimensionality of the vector of Brownian motions) in 2 unknowns (the number of stocks). It has a solution if and only if its right-hand side lies in Span($\sigma$). This entails a restriction

$$(I_{4\times 4} - \sigma(t)^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t))\frac{\sigma_{\alpha_i}(t)^\top+\sigma_{\beta_i}(t)^\top}{\alpha_i(t)+\beta_i(t)} + \nu_i(t) = 0, \tag{A.4}$$

where we have applied the projection operator $I_{4\times 4} - \sigma(t)^\top(\sigma(t)\sigma(t)^\top)^{-1}\sigma(t)$. Equation (17) then follows immediately. Note that, for the case of Foreign, equation (A.4) simplifies to yield $\nu_F(t) = \mathbf{0}$ because $\sigma_{\alpha_F}(t)$ and $\sigma_{\beta_F}(t)$ are both equal to zero.

The optimal portfolios are obtained from (A.3) via simple algebraic manipulations that, in particular, make use of the property that $\sigma(t)\nu_i(t) = 0$. ■

**Proof of Lemma 1.** We use the construct of the representative agent to value stocks in the economy. The representative agent's utility evaluated at the aggregate output is given by

$$u(Y(t),\, Y^*(t); \lambda(t)) = \max_{\substack{C_H(t) + C_F(t) = Y(t), \\ C^*_H(t) + C^*_F(t) = Y^*(t)}} u_H(C_H(t),\, C^*_H(t)) + \lambda(t) u_F(C_F(t),\, C^*_F(t)).$$

It follows from this definition that the marginal utilities of the representative agent and the individual agents, evaluated at the optimum, are related as

$$\nabla u(Y(t),\, Y^*(t); \lambda(t)) = \nabla u_H(\overline{C}_H(t),\, \overline{C}^*_H(t)) = \lambda(t) \nabla u_F(\overline{C}_F(t),\, \overline{C}^*_F(t)),$$

where the symbol $\nabla$ is used to denote the gradient. From the first-order conditions of the Home consumer,

$$\nabla u_H(\overline{C}_H(t),\, \overline{C}^*_H(t)) = \Big( y_H p(t) \xi_{\nu_H}(t),\; y_H p^*(t) \xi_{\nu_H}(t) \Big).$$

To derive this we used the fact that $\nabla u_H(\overline{C}_H(t),\, \overline{C}^*_H(t)) = (\alpha_H(t)/\overline{C}_H(t),\, \beta_H(t)/\overline{C}^*_H(t))$ combined with (8) and (A.1). Substituting the sharing rules of the representative agent (18), we can then derive the personalized state price density of the Home consumer and hence that of the representative agent:

$$\xi_{\nu_H}(t) = e^{-\rho t} \frac{p(0)}{p(t)} \frac{\overline{C}_H(0)}{\overline{C}_H(t)} = e^{-\rho t} \frac{p(0) Y(0)}{p(t) Y(t)} \frac{\alpha_H(t) + \lambda(t) \beta_F}{\alpha_H(0) + \lambda(0)\, \beta_F}. \tag{A.5}$$

This state price density can be used to price assets by no-arbitrage:

$$S(t) = \frac{1}{\xi_{\nu_H}(t)} E_t \left[ \int_t^T \xi_{\nu_H}(s)\, p(s)\, Y(s)\, ds \right], \quad S^*(t) = \frac{1}{\xi_{\nu_H}(t)} E_t \left[ \int_t^T \xi_{\nu_H}(s)\, p^*(s)\, Y^*(s)\, ds \right].$$

Hence, the price of the Home stock is

$$\begin{aligned} S(t) &= \frac{e^{\rho t} p(t)\, Y(t)}{\alpha_H(t) + \lambda(t) \beta_F} E_t \left[ \int_t^T e^{-\rho s} (\alpha_H(s) + \lambda(s) \beta_F) ds \right] \\ &= \frac{1 - e^{-\rho(T-t)}}{\rho} p(t) Y(t) + \frac{e^{\rho t} \beta_F\, p(t)\, Y(t)}{\alpha_H(t) + \lambda(t) \beta_F} E_t \left[ \int_t^T e^{-\rho s} (\lambda(s) - \lambda(t)) ds \right], \end{aligned} \tag{A.6}$$

where we used the fact that $\alpha_H$ is a martingale (i.e., $E_t[\alpha_H(s)] = \alpha_H(t)$). Analogously, using the fact that $\beta_H$ is a martingale, we find the price of the Foreign stock to be

$$\begin{aligned} S^*(t) &= \frac{e^{\rho t} p^*(t)\, Y^*(t)}{\beta_H(t) + \lambda(t) \alpha_F} E_t \left[ \int_t^T e^{-\rho s} (\beta_H(s) + \lambda(s) \alpha_F) ds \right] \\ &= \frac{1 - e^{-\rho(T-t)}}{\rho} p^*(t)\, Y^*(t) + \frac{e^{\rho t} \alpha_F\, p^*(t)\, Y^*(t)}{\beta_H(t) + \lambda(t) \alpha_F} E_t \left[ \int_t^T e^{-\rho s} (\lambda(s) - \lambda(t)) ds \right]. \end{aligned} \tag{A.7}$$

There are two ways to proceed in evaluating the above conditional expectations. The first is to assume that $\lambda$ is a martingale (and hence $E_t[\lambda(s) - \lambda(t)] = 0$) and then verify that it is indeed the case in equilibrium. From Proposition 2, however, we can only conclude that $\lambda$ is a local martingale. In all special cases that we consider in Section 4, it is easy to verify that $\lambda$ is also a true martingale under some additional mild regularity conditions imposed on the preference shifts.[15] However, for the general case it is not immediate how to show it.

An alternative approach is to use the following, less direct, argument based on market clearing. In particular, from (A.1)–(A.2), we have

$$W_H(t) + W_F(t) = (\overline{\mathbb{C}}_H(t) + \overline{\mathbb{C}}_F(t))\frac{1 - e^{-\rho(T-t)}}{\rho} = p(t)\,Y(t)\,\frac{1 - e^{-\rho(T-t)}}{\rho} + p^*(t)\,Y^*(t)\,\frac{1 - e^{-\rho(T-t)}}{\rho},$$

where in the last equality we used the fact that the total consumption expenditure at time $t$ equals $p(t)Y(t) + p^*(t)Y^*(t)$. On the other hand, from stock market clearing, we have

$$W_H(t) + W_F(t) = S(t) + S^*(t).$$

Combining the resulting restriction that

$$S(t) + S^*(t) = p(t)\,Y(t)\,\frac{1 - e^{-\rho(T-t)}}{\rho} + p^*(t)\,Y^*(t)\,\frac{1 - e^{-\rho(T-t)}}{\rho}$$

with (A.6)–(A.7), we conclude that

$$S(t) = p(t)\,Y(t)\,\frac{1 - e^{-\rho(T-t)}}{\rho} \quad \text{and} \quad S^*(t) = p^*(t)\,Y^*(t)\,\frac{1 - e^{-\rho(T-t)}}{\rho}.$$

This establishes (21)–(22).

To derive (23), we combine (A.1)–(A.2) with (18)–(19) and use the representation of the stock prices $S$ and $S^*$ derived in this lemma. ■

**Proof of Lemma 4.** Equation (31) follows from $NFA_H(t) = W_H(t) - S(t)$ and Lemma 1. To derive (30), we use the definition of the trade balance, $TB_H(t) = p(t)(Y(t) - C_H(t)) - p^*(t)Y^*(t)$, substitute the equilibrium expressions for consumption and the terms of trade, (18) and (20), and simplify. ■

Before we proceed to the rest of the proofs, we need to define several auxiliary vectors to be used throughout the remainder of this appendix. Let

$$i_1 \equiv (1, 0, 0, 0), \quad i_2 \equiv (0, 1, 0, 0), \quad \text{and} \tag{A.8}$$

$$A(t) \equiv \frac{\sigma_{\alpha_H}(t) - \beta_F\,\lambda(t)\,\nu(t)^\top}{\alpha_H(t) + \lambda(t)\,\beta_F} - \frac{\sigma_{\beta_H}(t) - \alpha_F\,\lambda(t)\nu(t)^\top}{\beta_H(t) + \lambda(t)\,\alpha_F} - \sigma_Y(t)\,i_1 + \sigma_{Y^*}(t)\,i_2. \tag{A.9}$$

[15] The only special case that requires these additional assumptions is that presented in Section 3.3. In particular, one needs to bound the preference shifts in such a way that the expression in (35) satisfies $E\left[e^{\frac{1}{2}\int_0^T \nu(t)^\top \nu(t)dt}\right] < \infty$. This condition is known as the Novikov condition.

**Proof of Proposition 2.** By substituting (A.2) into (24) we derive

$$\lambda(t) = \frac{y_H\,\xi_{\nu_H}(t)}{y_F\,\xi_{\nu_F}(t)}.$$

Applying Itô's lemma and using the representation of the countries' state price densities from (14), we have

$$d\lambda(t) = -\lambda(t)m(t)^\top \nu_H(t)dt - \lambda(t)\nu_H(t)d\vec{w}(t), \tag{A.10}$$

where we have substituted the finding that $\nu_F(t) = \mathbf{0}$ established in Proposition 1. To show that the drift term in (A.10) is equal to zero, we use the definition of $m$ from (13) and the restriction that $\sigma(t)\nu_H(t) = 0$.

To determine $\lambda(0)$, note from Lemma 1 that the initial financial wealth of, say, the Home country is given by (23) evaluated at $t = 0$. On the other hand, $W_H(0) = S(0)$ because the initial portfolio of Home consists of one share of the Home stock. This allows us to pin down $\lambda(0)$. It is easy to show that $\lambda(0) = \beta_H(0)/\beta_F$.

We now report the volatility matrix of stock returns.

$$\sigma(t) = \begin{bmatrix} \frac{1-a}{a\,q(t)+1-a}\,A(t) + \sigma_Y(t)\,i_1 \\ -\frac{a\,q(t)}{a\,q(t)+1-a}\,A(t) + \sigma_{Y^*}(t)\,i_2 \end{bmatrix},$$

where $A(t)$, $i_1$, and $i_2$ are defined in (A.8)–(A.9). This volatility matrix is obtained by applying Itô's lemma to the closed-form expressions for the stock prices (21)–(22).

The market price of risk process $m$ can be derived from the dynamics of $\xi_H$ in (14). Using the identity $\xi_H(t) = ap(t)\xi_H(t) + (1-a)p^*(t)\xi_H(t)$ and equations (20) and (A.5), we derive

$$\xi_H(t) = ae^{-\rho t}p(0)\frac{\alpha_H(t) + \lambda(t)\,\beta_F}{\alpha_H(0) + \lambda(0)\,\beta_F}\left(\frac{Y(0)}{Y(t)}\right) + (1-a)e^{-\rho t}p(0)\,\frac{\beta_H(t) + \lambda(t)\,\alpha_F}{\alpha_H(0) + \lambda(0)\,\beta_F}\,\left(\frac{Y(0)}{Y^*(t)}\right). \tag{A.11}$$

Applying Itô's lemma and identifying the diffusion term with that in the representation of $\xi_H$ in (14), we obtain

$$\begin{aligned} m(t) \quad = \quad & -\frac{p(0)e^{-\rho t}}{\xi_H(t)}\,\frac{Y(0)}{\alpha_H(0) + \lambda(0)\,\beta_F}\Big[a\,\frac{\sigma_{\alpha_H}(t) - \beta_F\lambda(t)\,\nu_H(t)}{Y(t)} - a\,\frac{\alpha_H(t) + \beta_F\lambda(t)}{Y(t)}\sigma_Y(t)i_1 \\ & +(1-a)\,\frac{\sigma_{\beta_H}(t) - \alpha_F\lambda(t)\,\nu_H(t)}{Y^*(t)} - (1-a)\,\frac{\beta_H(t) + \alpha_F\lambda(t)}{Y^*(t)}\sigma_{Y^*}(t)i_2\Big] - \nu_H(t). \end{aligned}$$

This completes the proof of the proposition. ■

**Derivation of the remaining equilibrium quantities.** We first report the interest rate $r$ and the stocks' expected returns $\mu_S$ and $\mu_{S^*}$ and then explain how we derived these expressions.

$$r(t) = \rho + \frac{aq(t)}{aq(t)+1-a}\left(\mu_Y(t) - \sigma_Y(t)^2 + \frac{\sigma_Y(t) i_1 (\sigma_{\alpha_H}(t)^\top - \lambda(t)\,\nu(t)\,\beta_F)}{\alpha_H(t)+\lambda\beta_F(t)}\right)$$
$$+ \frac{1-a}{aq(t)+1-a}\left(\mu_{Y^*}(t) - \sigma_{Y^*}(t)^2 + \frac{\sigma_{Y^*}(t) i_2 (\sigma_{\beta_H}(t)^\top - \lambda(t)\,\nu(t)\,\alpha_F)}{\beta_H(t)+\lambda\alpha_F}\right) \tag{A.12}$$

$$\mu_S(t) = \rho + \mu_Y(t) + \frac{1-a}{aq(t)+1-a}\left(\mu_q(t) - \frac{aq(t)}{aq(t)+1-a}||A(t)||^2 + \sigma_Y(t)A(t)i_1^\top\right), \tag{A.13}$$

$$\mu_{S^*}(t) = \rho + \mu_{Y^*}(t) + \frac{aq(t)}{aq(t)+1-a}\left(-\mu_q(t) + \frac{aq(t)}{aq(t)+1-a}||A(t)||^2 - \sigma_{Y^*}(t)A(t)i_2^\top\right), \tag{A.14}$$

where $\mu_q$ is the expected improvement in the terms of trade, given by

$$\begin{aligned}\mu_q(t) \;=\; & \mu_{Y^*}(t) - \mu_Y(t) + \frac{1}{2}||A(t)||^2 - \frac{1}{2}\frac{||\sigma_{\alpha_H}(t)^\top - \lambda(t)\,\nu(t)\,\beta_F||^2}{(\alpha_H(t)+\lambda\beta_F(t))^2} \\ & + \frac{1}{2}\frac{||\sigma_{\beta_H}(t)^\top - \lambda(t)\,\nu(t)\,\alpha_F||^2}{(\beta_H(t)+\lambda\,\alpha_F(t))^2} + \frac{1}{2}\sigma_Y(t)^2 - \frac{1}{2}\sigma_{Y^*}(t)^2,\end{aligned}$$

and where $A(t)$, $i_1$, and $i_2$ are defined in (A.8)–(A.9).

The interest rate $r$ in (A.12) is equal to the drift term from the Itô expansion of the equilibrium state price density reported in (A.11). The formulas in (A.13)–(A.14) are obtained by applying Itô's lemma to the closed-form expressions for the stock prices (21)–(22) and the terms of trade (20), and then using the definitions of $\mu_S$ and $\mu_{S^*}$ from (4)–(5). ■

**Derivations for Section 4.** All derivations for the special cases examined in Section 4 are tedious but straightforward. Perhaps the easiest way to obtain the formulas and signs reported in that section is to use *Mathematica* to simplify the expressions derived above and manipulate them in *Mathematica* to verify the desired properties. Our programs are available upon request.

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