Economics 131
Section Notes
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Preference Relations, Social Decision Rules, Single-Peakedness, and Social Welfare Functions

1 Preference Relations

1.1 Binary Relations

A preference relation is a special type of *binary relation*. A binary relation is essentially just any set of ordered pairs. Let A and B be sets and define their **Cartesian product** to be the set of all pairwise combinations of members of A and B

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

where $\in$ means "is an element of." A **binary relation** R over A and B is then any subset of $A \times B$, i.e.

$$R \text{ is a binary relation } \Leftrightarrow R \subseteq A \times B \Leftrightarrow ((a, b) \in R \Rightarrow (a, b) \in A \times B)$$

If a pair (a, b) belongs to R we can either write $(a, b) \in R$ or more commonly aRb. The number of potential relations is huge: if A has m elements and B has n elements then $A \times B$ has mn elements and the number of potential relations is then 2^{mn}, which can be very large. Remember that order matters very much in a relation.

Example 1 *Let L be the relation "loves" over the sets $A = B = P$ where P is a set of people. We can write "Anne loves Bill" as $(a, b) \in L$ or just aLb where $a =$ Anne, and $b =$ Bill. Of course Bill might love Anne back in which case $(b, a) \in L$, i.e., bLa, but if Bill does not love Anne then $(b, a) \notin L$. If everyone loves everyone then $L = P \times P$, and if no one loves anyone, then $L = \varnothing$, the empty set.*

Example 2 *Every single-valued function is a binary relation. Let D be the domain and R the range, and f the function. Then we can write $f \subseteq D \times R$, where $(a, b) \in f$ means that $f(a) = b$. All functions are relations, but not all relations are functions, as functions must be single-valued, i.e., each member of D can only be paired with one member of R. $(a, b) \in f$ and $(a, c) \in f \Rightarrow b = c$. This is actually how the concept of a function is defined.*

1.2 Preference Relations

The first primitive of rational choice theory is that there is a **choice set** X of options to choose form. The choice set X need not include just a person's own consumption, but preferences over any state of the world. For example,

$$x = (\textit{Anne} \text{ gets 1 banana, } \textit{Bill} \text{ gets 3 kiwis , the President of France lives in a blue house})$$
$$y = (\textit{Anne} \text{ gets 1 banana, } \textit{Bill} \text{ gets 4 kiwis , the President of France lives in a blue house})$$
$$z = (\textit{Anne} \text{ gets 1 banana, } \textit{Bill} \text{ gets 4 kiwis , the President of France lives in a red house})$$

As you might imagine the size of a choice set, say m, can be extremely large.

A **preference relation** $\succcurlyeq$ is a subset of $X \times X$, that satisfies the following two *rationality* properties

Completeness $\forall x \in X$ (where $\forall$ means "for all") and $\forall y \in X$ either $x \succcurlyeq y$ or $y \succcurlyeq x$, i.e. $(x, y) \in \succcurlyeq$ or $(y, x) \in \succcurlyeq$. This means that it is always possible to say whether or not you would prefer one choice to another.

Transitivity $\forall x \in X, \forall y \in X$, and $\forall z \in X$ if $x \succcurlyeq y$ and $y \succcurlyeq z$ then $x \succcurlyeq z$, i.e. $(x, y) \in \succcurlyeq$ and $(y, z) \in \succcurlyeq$ $\Rightarrow (x, z) \in \succcurlyeq$. The implication of transitivity is that you can order your choices from best to worst, allowing for ties.

In English $x \succcurlyeq y$ means "x is (weakly) preferred to y." Of course there are some binary relations that are not preference relations, although there are a large number that are.

Two other important relations can be derived from $\succcurlyeq$. The first is the **indifference relation** $\sim$, where $x \tilde{} y$ iff both $x \succcurlyeq y$ and $y \succcurlyeq x$. The second is the **strict preference relation** $\succ$ where $x \succ y$ iff both $x \succcurlyeq y$ and not $y \succcurlyeq x$.[1]

Example 3 *Clearly love is not a preference relation. Completeness is not satisfied, as it is not necessary for either Anne to love Bill or Bill to love Anne. Transitivity is also violated: if Anne loves Bill and Bill loves Clea, that does not necessarily imply that Anne loves Clea.* [2]

1.3 Utility Functions

A **utility function** is a function which assigns a number to any given choice depending on how it falls, or formally $U : X \to R$, or if you like $U \subseteq X \times R$, where R is the set of real numbers. A utility function U is said to **represent** a preference relation $\succcurlyeq$ iff

$$\forall x \in X,\ \forall y \in X,\ U(x) \geq U(y) \Leftrightarrow x \succcurlyeq y \qquad \text{(Utility Rep)}$$

Assuming $\succcurlyeq$ is a rational preference relation that satisfies a technical condition known as "continuity"[3] then a utility function that represents $\succcurlyeq$ will always exist. Just as a preference relation can give birth to a utility function, a utility function can give birth to preference relation using the same condition. Despite this apparent sameness (or "isomorphism") between utility functions and preference relations, a utility function can actually contain more information than a preference relation.

Two types of utility are generally considered

Ordinal Utility An *ordinal utility function*, is a utility function where differences between $U(x)$ and $U(y)$ are meaningless. Only the fact that $U(x) \gtreqless U(y)$ has any meaning. This is reflected in the fact that you can subject an ordinal utility function to an increasing transformation $f(u)$ to get a new utility function $\tilde{U}(x) = f(U(x))$ which will represent the same preference relation. Since f is increasing $f(U(x)) \geq f(U(y)) \Leftrightarrow U(x) \geq U(y) \Leftrightarrow x \succcurlyeq y$.

Cardinal Utility A *cardinal utility function* is one in which differences between $U(x)$ and $U(y)$ are meaningful as they reflect the *intensity* of preferences. For example, say $U(x) - U(z) = 2[U(y) - U(z)]$, in which case we could say that an individual prefers x over z *twice as much* as she prefers y to z : .this type of cardinal utility is invariant only to increasing "affine" transformations $f(u) = a + bu$ An even stronger statement would be something like $U(x) = 3utils$, where "$utils$" is some intrinsic measure of happiness. This type of cardinal utility is not invariant to any transformation.

Another consideration is whether or not utility is **interpersonally comparable**. Loosely speaking, utility is interpersonally comparable if differences in utility between individual makes sense. Note that cardinal utility does not necessarily imply utility is interpersonally comparable. Imagine we start with a cardinal utility functions (U_1, U_2) for two different individuals that are interpersonally comparable, but that these are transformed by the same non-affine transformation $f(u)$. The resulting utility functions $\tilde{U}_1(x) = f(U_1(x))$ and $\tilde{U}_2(x) = f(U_2(x))$ will still be interpersonally comparable, but will no longer be cardinal. Utility functions which are ordinal and do not allow for interpersonal comparisons do not contain more information than preference relations. For more on all of this see Roemer (1998).

[1] The indifference relationship is not complete but it is transitive. It also has the two properties of **reflexivity** ($\forall x$, $x \tilde{} x$) and **symmetry** ($\forall x \forall y$, $x \tilde{} y \Rightarrow y \tilde{} x$) Strict preference is also transitive and not complete. It has the property of **irreflexivity** ($\forall x$, never $x \succ x$).

[2] It would be nice if love were *symmetric* i.e. $\forall a \forall b$, $aLb \Rightarrow bLa$ or at least *transitive* $\forall a$,aLa, but, alas, 'tis not the case.

[3] **Continuity** implies that the preference relation is *closed:* if a sequence of ordered pairs belongs in the relation then so does its limit if it has one. i.e. $(x_n, y_n) \in \succcurlyeq \forall n = 1, 2, 3, ..$and $(x_n, y_n) \to (x, y)$ then $(x_n, y_n) \in \succcurlyeq$.

2 Social Decision Rules

2.1 Definition

The central aim of social choice theory is to aggregate preferences. Assume there are N individuals indexed by $i = 1, ..., N$. Each has their own rational preference $\succcurlyeq_i$over some choice set X. The goal is to put all of these preferences $(\succcurlyeq_1, ..., \succcurlyeq_N)$ together to come up with some single **social preference relation** $\succcurlyeq_S$to decide matters of policy and evaluate welfare. We can then say "x is socially preferred to y" as $x \succcurlyeq_S y$. The trick is that different sets of individual preferences should generally produce a different $\succcurlyeq_S$, i.e. it should be *responsive* to what people want, just as an election will determine a voter depending on the preferences of the electorate. Formally, we want a **social decision rule** F whereby we can write

$$\succcurlyeq_S = F(\succcurlyeq_1, ..., \succcurlyeq_N)$$

F can be thought of as a voting rule, where voting could be potentially be much more sophisticated then what we are used to as voting. We write $xF(\succcurlyeq_1, ..., \succcurlyeq_N)\, y$ to say that this rule tells us that x is socially preferred to y. Considering the number of people and all of the possible outcomes X finding a good F can be a formidable task.

Example 4 ***Pairwise majority voting*** *is an example of F. Say there is a single vote between x and y. Then $xF(\succcurlyeq_1, ..., \succcurlyeq_N)\, y$ if and only if the number of people who have $x \succcurlyeq_i y$ is greater than or equal to $N/2$, with strict preference if it is strictly greater.*

2.2 Properties of a Social Decision Rule

What kinds of rules do we think would make sense to aggregate preferences by? What kind of F would make a good F? Here are a few important properties

Rationality $\succcurlyeq_S = F(\succcurlyeq_1, ..., \succcurlyeq_N)$ should always be *complete* and *transitive* for any profile $(\succcurlyeq_1, ..., \succcurlyeq_N)$. This way society can make a decision about any choice and can rank all choices.

Paretian If everyone unanimously prefers x to y then so should society. In other words if $\forall i$, $x \succcurlyeq_i y$ then $xF(\succcurlyeq_1, ..., \succcurlyeq_N)\, y$. This is the only condition we consider here that says F should pick out choices people "like the most."

Universal Domain (UD) No matter what kind of wacky preferences people may have, so long as they are rational, F has to be able to deal with them. In other words we put no restrictions on the profile of preferences $(\succcurlyeq_1, ..., \succcurlyeq_N)$ that goes into F.

Independence of Irrelevant Alternatives (IIA) This says that whether or not society prefers x to y should depend only on how individuals prefer to x to y , i.e. it should not depend on what people think of some other "irrelevant " alternative z. This stated formally by saying that we have two profiles of individual preferences $(\succcurlyeq_1, ..., \succcurlyeq_N)$ and $(\succcurlyeq'_1, ..., \succcurlyeq'_N)$ such that in each profile we have exactly the same people who prefer x to y, i.e. $x \succcurlyeq_i y$ iff $x \succcurlyeq'_i y$, or alternatively $\{i : x \succcurlyeq_i y\} = \{i : x \succcurlyeq'_i y\}$. Then if for the first profile society prefers x to y , $xF(\succcurlyeq_1, ..., \succcurlyeq_N)\, y$, then for the second profile society should also prefer x to y , $xF(\succcurlyeq'_1, ..., \succcurlyeq'_N)\, y$.

Nondictatorial Because of its politically charged wording this is perhaps the least understood property. To understand this correctly we first need to consider the definition of *decisive.* An individual i *is said to be* **decisive** *over a particular pair x and y* if whenever i prefers x to y society prefers x to y, i.e. $xF(\succcurlyeq_1, ..., \succcurlyeq_N)\, y \Leftrightarrow x \succcurlyeq_i y$. An individual i *is said to be* **dictatorial** if she is decisive over *all* pairs x and y. While it might be okay for an individual to be decisive over some choices (e.g. where x and y differ only in that an individual i lives in a red house or a blue house) it is does not seem okay for one individual to be decisive over *everything.* F is said to be **nondictatorial** if it does not produce a dictator, i.e. no individual i is a dictator.

Example 5 *Majority voting will satisfy all 5 of these above properties when X contains only 2 elements, say x and y. Majority voting also satisfies an additional property known as* ***symmetry*** *as the actual identities of the voters are unimportant - we can switch around the profiles without changing the results. For example $F(\succcurlyeq_1, \succcurlyeq_2, \succcurlyeq_3) = F(\succcurlyeq_3, \succcurlyeq_1, \succcurlyeq_2)$. In fact a result known as "May's Theorem" (1952) says that majority voting is the only social decision rule F that satisfies all of these properties when X contains 2 elements.*

2.3 Arrow's Impossibility Theorem

Arrow (1951) discovered a very negative result when he was trying to find a sensible F to aggregate preferences. He wanted to find an F that would satisfy the five properties highlighted above for when X contains 3 or more elements and found it to be logically impossible. Therefore any method of aggregating preferences using a a social decision rule F must violate one of the above properties.

There are generally two ways out of this multi-horned dilemma. The first option is to live with the fact that some of the properties have to be violated and to cut ones losses by abandoning one or more of the above properties in favor of weaker substitutes. Those properties most commonly dropped are weakened are that of Universal Domain (UD) and the Independence of Irrelevant Alternatives (IIA). The second option is to demand more information than just the profile of preferences $(\succcurlyeq_1, ..., \succcurlyeq_N)$ to aggregate preferences with. For example we could ask for the profile of utility functions $(U_1, ..., U_N)$ which will tell us more about peoples preferences or how to aggregate them than $(\succcurlyeq_1, ..., \succcurlyeq_N)$ if utility is cardinal or is interpersonally comparable (or both).

Example 6 *Condorcet (1785) first hinted at the problem of aggregating preferences with his famous Condorcet "paradox" in which majority voting can lead to an intransitive preference. Let $N = 3$, let there be 3 choices $X = \{x, y, z\}$, and let F be the rule assigned by pairwise majority voting. This is within the context of strict preference $\succ$, with the idea that if F is given strict preferences it should produce a strict preference (which is fine) Say individuals have the following preferences*

$$x \succ_1 y \succ_1 z$$
$$y \succ_2 z \succ_2 x$$
$$z \succ_3 x \succ_3 y$$

Pairwise majority voting entails that $xF(\succ_1, \succ_2, \succ_3)\, y$, $yF(\succ_1, \succ_2, \succ_3)\, z$, and, $zF(\succ_1, \succ_2, \succ_3)\, x$. This violates the principle of transitivity, which implies $xF(\succ_1, \succ_2, \succ_3)\, z$ and therefore NOT $zF(\succ_1, \succ_2, \succ_3)\, x$ (remember we rule out indifference) making $F(\succ_1, \succ_2, \succ_3)$ irrational.

3 Single-Peaked Preferences

3.1 Definition

If we abandon the property of universal domain then we can impose the restriction of single-peaked preferences. This requires that we be able to order the choice set X according to some *linear order*, $\geq$, which could rank things from highest to lowest.[4] Preferences are said to be **single-peaked** (or unimodal) if for each individual i there exists a **"bliss point"** x_i^* such that choices further away from x_i^* are disliked more and more:

$$\forall y \in X, \forall y' \in X, \quad y' \geq y \geq x_i^* \text{ or } x_i^* \geq y \geq y' \;\Rightarrow x_i^* \succcurlyeq_i y \succcurlyeq_i y' \qquad \text{((SP))}$$

Single peaked preferences mean we should be able to make a graph with the choices in X along the x-axis, with a utility representation of $\succcurlyeq_i$having a nice smooth, single-peaked shape. Notice that when X has only 2 elements then preferences are always single-peaked.

Example 7 *The preferences in the Condorcet example are not single-peaked no matter how we order X. For example if we order $x > y > z$, then individual 3 will not have single peaked preferences as her bliss point is z but $z \succ_3 x \succ_3 y$, violating the above condition. Similar difficulties arise with any other ordering of X.*

[4] A typical order like $\geq$ that you've known since kindergarten has to satisfy the property of completeness, transitivity and also **weak anti-symmetry** namely if $x \geq y$ and $y \geq x$ then $x = y$.

3.2 Median Voter Theorem

Imagine lining up every individual's bliss point x_i^* in a line and suppose that N is odd. The median voter is defined as the voter whose bliss point x_m^* is located in the $(N+1)/2$ place. Black (1948) showed that when preferences are single-peaked, x_m^* will always win in any pairwise majority vote with any other alternative (i.e., it is the "**Condorcet winner**"), as there will always be $(N+1)/2 > N/2$ voters that will prefer it to any other alternative. Thus, x_m^* is socially preferred to any other alternative. It is also possible to show that majority voting is complete (who ever wins a pairwise vote), transitive (harder), Paretian (unanimity always wins), satisfies IIA (straight from the definition of pairwise voting), and non-dictatorial (the median voter changes with different preference profiles).

4 Social Welfare Functions

If instead of getting the profile of preferences, say we get a profile of utility functions $(U_1, ..., U_N)$ then we can construct a **social welfare function** (SWF) $W(u_1, ..., u_N)$ which takes individual utilities $u_i = U_i(x)$. A social optimum will solve the following problem

$$\max_{x \in X} W\left[U_1(x), ..., U_N(x)\right]$$

The choice of an appropriate W is very similar to the problem of finding the right F considered above. It is possible to come up with analogues for the properties above imposed on social decision rules. If all of these analogue properties hold *and* the profiles $(U_1, ..., U_N)$ do not allow for interpersonal comparisons, then a result very similar to Arrow's Impossibility Theorem will arise: W will be just a transformation of some U_i's utility, where i is essentially a dictator.

If the utility functions are cardinal and allow for interpersonal comparisons then there is a social welfare function that satisfies Arrow's conditions, namely a *weighted utilitarian* SWF

$$W(u_1, ..., u_N) = \sum_{i=1}^{N} \lambda_i u_i$$

where λ_i are weights on different individuals. This is shown by Harsanyi (1955). If we impose the condition of symmetry, then $\lambda_i = 1$ for all i, and we have the old-fashioned "Benthamite" SWF [5]

$$W(u_1, ..., u_N) = \sum_{i=1}^{N} u_i$$

which means that society should rank choices x according to how they maximize the sum of individual utilities. There are many more interesting issues related to SWF's not discussed here.

[5] Jeremey Bentham is well known for his book *Utilitarianism* which prescibes that every social action should seek to find "the greatest happiness for the greatest number." Upon his request, after his death Mr. Bentham's waxed remains were put on display in a glass case in University College, London where he remains unto this day.