Economics 131
Section Notes
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Nash Equilibrium, Pareto Optimality and Public Goods with Two Agents

1 Nash Equilibrium

Consider the case where the case with $N = 2$ agents, indexed by $i = 1, 2$. Most of what we consider here is generalizable for larger N but working with 2 agents makes things much easier. Let agent 1's utility depends on his own action a_1 ("action" is defined very broadly here) as well as agent 2's action, so we can write $U_1(a_1, a_2)$, and similarly for agent 2 $U_2(a_1, a_2)$.

1.1 Definition

A set of actions $\left(a_1^N, a_2^N\right)$ constitutes a *Nash equilibrium* iff

$$U_1\left(a_1^N, a_2^N\right) \geq U_1\left(a_1, a_2^N\right) \quad \text{for all } a_1 \text{, and}$$
$$U_2\left(a_1^N, a_2^N\right) \geq U_1\left(a_1^N, a_2\right) \quad \text{for all } a_2$$

In other words a set of actions is a Nash equilibrium if each agent cannot do better for herself playing her Nash equilibrium action given other people play their Nash equilibrium action.

1.2 Solving for Nash Equilibria

Solving the Nash equilibrium requires solving two maximization problems, namely

$$\max_{a_1} U_1(a_1, a_2) \quad \text{and} \quad \max_{a_2} U_2(a_1, a_2)$$

where each person takes each other action as given. Oftentimes finding a Nash involves checking all the possible combinations (a_1, a_2) and asking yourself "is this a Nash equilibrium?" Sometimes it is possible to eliminate dominated actions iteratively (see a book on game theory) to narrow the cases that need to be checked. However, assuming everything is nicely differentiable and a_1^N and a_2^N are both positive, we can take first order conditions. The first order condition for each first agent is just

$$\frac{\partial U_1\left(a_1^N, a_2^N\right)}{\partial a_1} = 0 \quad \text{and} \quad \frac{\partial U_2\left(a_1^N, a_2^N\right)}{\partial a_2} = 0 \qquad \text{(Nash FOC)}$$

which is a system of 2 equations in 2 unknowns a_1^N, a_2^N, and so usually a little algebra will yield the solution.

1.3 Reaction Curves

By the implicit function theorem the FOC for agent 1 defines what she will play given a_2 (not just at the Nash), i.e. agent 1's *reaction curve* $a_1 = r_1(a_2)$ so that $\frac{\partial U_1(r_1(a_2), a_2)}{\partial a_1} = 0$. A similar reaction curve $r_2(a_1)$ can be defined for agent 2. A Nash equilibrium can be seen as where

$$a_1^N = r_1\left(a_2^N\right) \quad \text{and} \quad a_2^N = r_2\left(a_1^N\right)$$

This is where the reaction curves cross in a graph with a_1 on one axis and a_2 on the other.

1.4 Strategic Complements and Substitutes

It is useful to know how one agent will react if the other agent changes her action. Differentiating totally the expression $\frac{\partial U_1(r_1(a_2),a_2)}{\partial a_1} = 0$ with respect to a_2 we get

$$\frac{d}{da_2}\left[\frac{\partial U_1\left(r_1\left(a_2\right),a_2\right)}{\partial a_1}\right] = \frac{\partial^2 U_1}{\partial a_1^2}\frac{dr_1}{da_2} + \frac{\partial^2 U_1}{\partial a_2 \partial a_1} = 0$$

and so solving for the slope of the reaction curve

$$\frac{dr_1}{da_2} = -\left(\frac{\partial^2 U_1}{\partial a_1^2}\right)^{-1}\frac{\partial^2 U_1}{\partial a_2 \partial a_1}$$

The sign of this expression depends on the sign of the second derivatives of the utility function. Cases where $\frac{dr_1}{da_2} > 0$, where a greater action by 2 elicits more of a response by 1, identifies a situation where a_1 and a_2 are called *strategic complements*. The alternate case where $\frac{dr_1}{da_2} < 0$, is where a_1 and a_2 are called *strategic substitutes*.

2 Pareto Optimality

2.1 Definition

The set of feasible actions $\left(a_1^P, a_2^P\right)$ is *Pareto optimal* if there does not exist another of feasible actions $(\tilde{a}_1, \tilde{a}_2)$ such that

$$U_1\left(\tilde{a}_1, \tilde{a}_2\right) \geq U_1\left(a_1^P, a_2^P\right) \text{ and}$$
$$U_2\left(\tilde{a}_1, \tilde{a}_2\right) \geq U_2\left(a_1^P, a_2^P\right)$$

with at least one above inequality strict. In other words there does not exist an allocation that makes both as well off and making one strictly better off. A logically equivalent condition is that for any feasible set of actions $(\tilde{a}_1, \tilde{a}_2)$

$$U_1\left(\tilde{a}_1, \tilde{a}_2\right) > U_1\left(a_1^P, a_2^P\right) \Rightarrow U_2\left(\tilde{a}_1, \tilde{a}_2\right) < U_2\left(a_1^P, a_2^P\right)$$

A set of actions that makes agent 1 strictly better off must make agent 2 strictly worse off. *Important Note: Except for the trivial case of one person, Pareto optima and Nash equilibria do not necessarily coincide: plenty of Nash equilibria that are not Pareto optima and vice-versa (remember the Prisoner's Dilemma!)*

2.2 Solving for Pareto Optima

Consider a social planner who attaches a relative weight λ to agent 1 relative to agent 2 where $\lambda \gtreqless 1$ depending whether the planner values agent 1 more or less than agent 2. A theorem from mathematics says that "pretty much" any Pareto optimal allocation can be found by maximizing the weighted utilities

$$\max_{a_1,a_2} \lambda U_1\left(a_1,a_2\right) + U_2\left(a_1,a_2\right)$$

for some λ. Different λ will give different Pareto optimal allocations. A popular favorite is to choose $\lambda = 1$, which corresponds to the Utilitarian scoial welfare function. Assuming everything is smooth and the Pareto optimal actions are positive the following FOC must hold at $\left(a_1^P, a_2^P\right)$

$$\lambda\frac{\partial U_1}{\partial a_1} + \frac{\partial U_2}{\partial a_1} = 0 \qquad \text{and} \qquad \lambda\frac{\partial U_1}{\partial a_2} + \frac{\partial U_2}{\partial a_2} = 0 \qquad \text{(Pareto FOC)}$$

Compare this condition to the Nash FOC and you can see that the Pareto optimal actions take into account $\partial U_2/\partial a_1$ and $\partial U_1/\partial a_2$, i.e., that actions of agent 1 have an effect on agent 2 and vice-versa. These externalities are ignored in the Nash equilibrium and so the Nash equilibrium is only optimal if $\partial U_2/\partial a_1 = \partial U_1/\partial a_2 = 0$. Solving each FOC equation for $-\lambda$ and rearranging we see

$$-\lambda = \frac{\frac{\partial U_2}{\partial a_1}}{\frac{\partial U_1}{\partial a_1}} = \frac{\frac{\partial U_2}{\partial a_2}}{\frac{\partial U_1}{\partial a_2}} \Rightarrow \frac{\frac{\partial U_1}{\partial a_2}}{\frac{\partial U_1}{\partial a_1}} = \frac{\frac{\partial U_2}{\partial a_2}}{\frac{\partial U_2}{\partial a_1}}$$

so the marginal rates of substitution between each action for each agent are equal, i.e. $MRS^1_{a_1a_2} = MRS^2_{a_1a_2}$. At the Nash equilibrium the marginal rates of substitution are typically perpendicular as $MRS^1_{a_1a_2} = \infty$ and $MRS^1_{a_1a_2} = 0$.

2.3 Utility Possibility Set

One can imagine the set of all pairs of utility (U_1, U_2) given by all of the different actions a_1 and a_2. The *utility possibility set* is that collection

$$\mathfrak{U} = \{(U_1, U_2) : U_1 = U_1(a_1, a_2), U_2 = U_2(a_1, a_2) \text{ for any feasible } a_1, a_2\}$$

which can usually be represented by a graph with U_1 on the x-axis and U_2 on the y-axis.

By its very nature a Pareto optimum should be on the very edge of that set - that is its "frontier ". More formally the *utility possibility frontier* is the set

$$\mathfrak{U}_F = \{(U_1, U_2) \in \mathfrak{U} : \text{there is no } \left(\tilde{U}_1, \tilde{U}_2\right) \in \mathfrak{U} \text{ such that } \tilde{U}_1 \geq U_1 \text{ and } \tilde{U}_2 \geq U_2 \}$$

The difference between the utility possibility frontier and the set of Pareto optima, is that the set of Pareto optima refers to an outcome or allocation while the frontier refers only to utilities. Also, Pareto optima require that at least one inequality is strict. All Pareto optima will yield utilities on the frontier, however not quite all points on the frontier will relate to a Pareto optimum since it may contain points where one agent (not both) may do better without it costing the other agent.

Say we are at a Pareto optimum. This means that the objective function is given by $\lambda U_1^P + U_2^P$ where $U_i^P = U_i\left(a_1^P, a_2^P\right)$. Just around the optimum $\left(U_1^P, U_2^P\right)$ we can assume that the sum $\lambda U_1^P + U_2^P = \bar{U}$ is constant. Using the implicit function theorem again we can treat U_2^P as a function of U_1^P and differentiate $\lambda + \frac{dU_2^P}{dU_1^P} = 0$ which gives us the slope of the utility possibility set $\frac{dU_2^P}{dU_1^P} = -\lambda$. Thus we can imagine a social planner with straight, parallel indifference curves, each with slope $-\lambda$, in a graph. A Pareto optimum will be found where an indifference curve is tangent to the utility possibility frontier, with slope $\frac{dU_2^P}{dU_1^P}$, outlining $\mathfrak{U}$.

2.4 Minimum Utility Formulation

If you don't like the idea of pulling λ out of a hat, consider an alternate formulation where agent 1 is guaranteed a minimum amount of utility $\bar{u}_1$, and agent 2 has her utility maximized. In other words

$$\max_{a_1,a_2} U_2(a_1, a_2) \quad \text{s.t. } U_1(a_1, a_2) \geq \bar{u}_1$$

If we let λ be the Lagrange multiplier on the constraint to get

$$U_2(a_1, a_2) + \lambda [U_1(a_1, a_2) - \bar{u}_1]$$

then we get the same FOC as the Pareto FOC (it's the same problem!) except that now λ has to be solved for, rather than imposed. The constraint $U_1(a_1, a_2) = \bar{u}_1$ adds a third equation so that we can solve for all three $\left(a_1^P, a_2^P, \lambda\right)$.[1]

[1] If you don't like Lagrange multipliers consider the case where a_2 is irrelevant and so the constraint is $U_2(a_1) = \bar{u}$, which inverted is $a_1 = U_2^{-1}(\bar{u})$. Utility for agent 1 is then $U_1\left[U_2^{-1}(\bar{u})\right]$ and so differentiating implies $\frac{dU_1}{d\bar{u}} = \frac{dU(a_1)}{da_1} / \frac{dU_2(a_1)}{da_2}$

3 Public Goods

Each agent has utility $U_i(G, x_i)$ where x_i is private consumption and public good $G = \sum_{i=1}^{N} g_i$ where g_i is agent i's provision of the public good. The public good, by definition is *nonrival*, consumption by one agent does not reduce it's benefit to another agent, and *nonexcludable*, i.e., it is prohibitively expensive to keep agents from consuming it. Assume that total consumption $X = \sum_{i=1}^{N} x_i$ is produced via a production function F from the public good, where the total amount of public good available is $\bar{G}$, so $X = F\left(\bar{G} - G\right)$ with $F(0) = 0$, $F'(\cdot) > 0$ $F''(\cdot) < 0$, and so the marginal rate of transformation of public good into private good $MRS_{GX} = -\frac{dX}{DG} = F'\left(\bar{G} - G\right)$.

3.1 Pareto Optimal Provision

Back to the case where $N = 2$, then we have $x_1 + x_2 = F\left(\bar{G} - G\right)$ or $x_2 = F\left(\bar{G} - G\right) - x_1$. Then we can write for utility for the individuals as $U_1(x_1, G)$ and $U_2\left(F\left(\bar{G} - G\right) - x_1, G\right)$. As we saw above we solve for the Pareto optimum by solving

$$\max_{x_1, G} \lambda U_1(G, x_1) + U_2\left(G, F\left(\bar{G} - G\right) - x_1\right)$$

Assuming $x_1^P, G^P > 0$ then the following two first order conditions must be satisfied at the optimum $\left(x_1^P, x_2^P, G^P\right)$

$$x_1 : \lambda \frac{\partial U_1}{\partial x} - \frac{\partial U_2}{\partial x} = 0$$
$$G : \lambda \frac{\partial U_1}{\partial G} - \left[\frac{\partial U_2}{\partial G} + \frac{\partial U_2}{\partial x} F'\right] = 0$$

Solving each equation for λ and then solving for F'tells us that

$$\lambda = \frac{\frac{\partial U_2}{\partial x}}{\frac{\partial U_1}{\partial x}} = \frac{\frac{\partial U_2}{\partial G} + \frac{\partial U_2}{\partial x} F'}{\frac{\partial U_1}{\partial G}} \Rightarrow F' = \frac{\frac{\partial U_1}{\partial G}}{\frac{\partial U_1}{\partial x}} + \frac{\frac{\partial U_2}{\partial x}}{\frac{\partial U_2}{\partial x}}$$

Which is the condition that $MRT_{GX} = MRS_{GX}^1 + MRS_{GX}^2$, this is the "Samuelson Rule" that the marginal rate of transformation should equal the sum of the marginal rates of substitution. In the case of constant returns to scale where $F' = p_G$ where p_G can effectively be considered the price of G in terms of x, then $MRS_{GX}^1 + MRS_{GX}^2 = p_G$.

3.2 Reaction Curve and Nash Equilibrium

To ease the notational burden and a few other issues we'll consider the case where F' is constant at $p_G = 1$. Each individual has a budget constraint $x_i + g_i = M_i$,[2] This constraint implies that there is really only one independent solution. Here we let that be g_i and let $x_i = M_i - g_i$. We can even redefine utility to depend on each person's action $\tilde{U}_1(g_1, g_2) = U_1(g_1 + g_2, M_1 - g_1)$ and $\tilde{U}_2(g_1, g_2) = U_2(g_1 + g_2, M_2 - g_2)$ to fit it into the previous framework.

The reaction curve $r_1(g_2, M_1)$ of the first agent, which depnends on g_2 as well as personal income M_1, is determined by the FOC evaluated at $(r_1(g_2, M_1) + g_2, M_1 - r_1(g_2, M_1))$ is

$$\frac{\partial U_1}{\partial G} - \frac{\partial U_1}{\partial x} \leq 0$$

where of course equality holds if $r_1(g_2, M_1) > 0$. Assuming that both x and G are normal goods, then a little effort[3] shows $0 \leq \partial r_1 / \partial M_1 \leq 1$, and $-1 \leq \partial r_1 / \partial g_2 \leq 0$, which means that g_1 and g_2 are strategic subsitutes: i.e. for each unit of G agent 2 gives, agent 1 will reduce her contribution of G, albeit less than

[2]The previous case is just where $\bar{G} = M_1 + M_2$, except now with the Nash equilibrium the initial distribution of resources matters (a general point).

[3]Differentiating the budget constraint $r_1 + x_1 = M_i$ with respect to M_i you get $\partial r_1 / \partial M + \partial x_1 / \partial M = 1$ and so $\partial r_1 / \partial M = 1 - \partial x_1 / \partial M \leq 1$, and by assumption $\partial r_1 / \partial M \geq 0$. Also $\partial r_1 / \partial g_2 = \partial r_1 / \partial M - 1$ and so $-1 \leq \partial r_1 / \partial g_2 \leq 0$.

one-for-one. The possibility $r_1(g_2, M_1) = 0$ is more than a triviality for higher values of g_2 and lower values of M_1.If the solution from the FOC equation is negative, then this means $r_1(g_2, M_1) = 0$.

Assuming the FOC holds with equality this implies $MRS^1_{GX} = \frac{\partial U_1}{\partial G} / \frac{\partial U_1}{\partial x} = 1$. A similar condition holds for agent 2 so that if both contribute $MRS^1_{GX} + MRS^2_{GX} = 2 > 1 = MRT_{GX}$ and hence that the Nash equilibrium is not optimal. The Nash provision is too small. $G^N = g^N_1 + g^N_2 < G^P$.

3.3 Decentralized Solution

Say the government finds provides a subsidy of 1/2 on each unit of G purchased, and levy lump-sum taxes worth a total $G/2$ if it needs to balance its budget. This amounts to taking away money from the individuals and then using that money to provide an incentive to buy a total of G^P on their own. Now $x_i = M_i - g_i/2$ so that the FOC for each agent i, assuming $g_i > 0$ then becomes

$$\frac{\partial U_i}{\partial G} - \frac{1}{2}\frac{\partial U_i}{\partial x} = 0$$

and so $MRS^1_{GX} + MRS^2_{GX} = \frac{1}{2} + \frac{1}{2} = 1 = MRT_{GX}$ and Pareto optimality is restored. All income effects will be eliminated by the lump sum tax placed on each individual of $g^P_i/2$, which is charged independently of what g_i agents actually choose. This presumes the government knows *in advance* what agents will choose.

3.4 Crowding Out

If the government provides the public good directly, and taxes for it (so that there are no income effects) it may "crowd-out" one-for-one the private provision of public goods. For instance say $M_1 = M_2 = M$ and preferences are identical so that intially each agent provided g^N. The government provides $g_0 < G^N = 2g^N$ and levies taxies of $g_0/2$ on each person. Then each agent 1 maximizes

$$U\left(g_1 + (g_0 + g_2), M - g_0/2 - g_1\right)$$

If each agent now provides $g_1 = g_2 = g^N - g_0/2 \geq 0$, then utility will be the same as under the Nash equilibrium.

$$U\left(2g^N, M - g^N\right)$$

As the marginal incentives, i.e. p_G, have not changed the same outcome will occur. If the government provides $g_0 > 2g^N$ then in equilibrium $g_1 = g_2 = 0$ but the just the government supply will be higher than the Nash, and thus socially better so long as g_0 is not inefficiently high.