CHAPTER 15: THE TERM STRUCTURE OF INTEREST RATES

1. **Expectations hypothesis.**
The yields on long-term bonds are geometric averages of present and expected future short rates. An upward sloping curve is explained by expected future short rates being higher than the current short rate. A downward-sloping yield curve implies expected future short rates are lower than the current short rate. Thus bonds of different maturities have different yields if expectations of future short rates are different from the current short rate.

**Liquidity preference hypothesis.**
Yields on long-term bonds are greater than the expected return from rolling-over short-term bonds in order to compensate investors in long-term bonds for bearing interest rate risk. Thus bonds of different maturities can have different yields even if expected future short rates are all equal to the current short rate. An upward sloping yield curve can be consistent even with expectations of falling short rates if liquidity premiums are high enough. If, however, the yield curve is downward sloping and liquidity premiums are assumed to be positive, then we can conclude that future short rates are expected to be lower than the current short rate.

**Segmentation hypothesis.**
This hypothesis would explain a sloping yield curve as an imbalance between supply and demand for bonds of different maturities. An upward sloping yield curve is evidence of supply pressure in the long-term market and demand pressure in the short-term market. According to the segmentation hypothesis, expectations of future rates have little to do with the shape of the yield curve.

2. d.

3. b.

4. True. Under the expectations hypothesis, there are no risk premia built into bond prices. The only reason for long-term yields to exceed short-term yields is an expectation of higher short-term rates in the future.

5. Uncertain. Lower inflation will usually lead to lower nominal interest rates. Nevertheless, if the liquidity premium is sufficiently great, long-term yields may exceed short-term yields *despite* expectations of falling short rates.
6. | Maturity | Price  | YTM | Forward Rates |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$943.40</td>
<td>6.00%</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>$898.47</td>
<td>5.50%</td>
<td>5.00% (1.055^2/1.06 – 1)</td>
</tr>
<tr>
<td>3</td>
<td>$847.62</td>
<td>5.67%</td>
<td>6.00% (1.0567^3/1.055^2 – 1)</td>
</tr>
<tr>
<td>4</td>
<td>$792.16</td>
<td>6.00%</td>
<td>7.00% (1.06^4/1.0567^3 – 1)</td>
</tr>
</tbody>
</table>

7. The expected price path of the 4-year zero coupon bond is as follows. (We discount the face value by the appropriate sequence of forward rates implied by this year’s yield curve.)

<table>
<thead>
<tr>
<th>Beginning of Year</th>
<th>Expected Price</th>
<th>Expected Rate of Return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$792.16</td>
<td>6.00% (839.69/792.16 – 1)</td>
</tr>
<tr>
<td>2</td>
<td>1000/1.05 × 1.06 × 1.07 = 839.69</td>
<td>5.00% (881.68/839.69 – 1)</td>
</tr>
<tr>
<td>3</td>
<td>1000/1.06 × 1.07 = 881.68</td>
<td>6.00% (934.58/881.68 – 1)</td>
</tr>
<tr>
<td>4</td>
<td>1000/1.07 = 934.58</td>
<td>7.00% (1000/934.58 – 1)</td>
</tr>
</tbody>
</table>

8. a. \((1+y_4)^4 = (1+y_3)^3 (1+f_4)\)

\((1.055)^4 = (1.05)^3 (1+f_4)\)

1.2388 = 1.1576 (1 + f_4)

\(f_4 = .0701, \text{ or } 7.01\%\)

b. The conditions would be those that underlie the pure expectations theory of the term structure: risk neutral market participants who are willing to substitute among maturities solely on the basis of yield differentials. This behavior would rule out liquidity or term premia relating to risk as well as market segmentation based on maturity preferences.

c. Under the expectations hypothesis, lower implied forward rates would indicate lower expected future spot rates for the corresponding period. Since the lower expected future rates embodied in the term structure are nominal rates, either lower expected future real rates or lower expected future inflation rates would be consistent with the specified change in the observed (implied) forward rate.
9. You should expect it to lie above the curve since the bond must offer a premium to investors to compensate them for the option granted to the issuer.

10. The interest rates are annual, but each period is a half-year. Therefore, the per period spot rates are 2.5% on one-year bonds and 2% on six-month bonds. The semiannual forward rate can be obtained by solving:

\[ 1 + f = \frac{1.025^2}{1.02} = 1.03 \]

which means that the forward rate is \( .03 = 3\% \) semiannually, or 6% annually. Therefore, choice d is correct.

11. The present value of each bond's payments can be derived by discounting each cash flow by rates from the spot interest rate (i.e., the pure yield) curve.

Bond A: \( PV = \frac{10}{1.05} + \frac{10}{1.08^2} + \frac{110}{1.11^3} = \$98.53 \)

Bond A: \( PV = \frac{6}{1.05} + \frac{6}{1.08^2} + \frac{106}{1.11^3} = \$88.36 \)

Bond A sells for $.13 (i.e., .13% of par value) less than the present value of its stripped payments. Bond B sells for $.02 less than the present value of its stripped payments. Bond A seems to be more attractively priced.

12. a. We obtain forward rates from the following table:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>YTM</th>
<th>Forward rate</th>
<th>Price (for parts c, d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>10%</td>
<td></td>
<td>909.09 (1000/1.10)</td>
</tr>
<tr>
<td>2</td>
<td>11%</td>
<td>12.01% (1.11^2/1.10 – 1)</td>
<td>811.62 (1000/1.11^2)</td>
</tr>
<tr>
<td>3</td>
<td>12%</td>
<td>14.03% (1.12^3/1.11^2 – 1)</td>
<td>711.78 (1000/1.12^3)</td>
</tr>
</tbody>
</table>

b. We obtain next year’s prices and yields by discounting each zero’s face value at the forward rates for next year that we derived in part (a):

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Price</th>
<th>YTM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>892.78 [= 1000/1.1201]</td>
<td>12.01%</td>
</tr>
<tr>
<td>2 years</td>
<td>782.93 [= 1000/(1.1201 \times 1.1403)]</td>
<td>13.02%</td>
</tr>
</tbody>
</table>

Note that this year’s upward sloping yield curve implies, according to the expectations hypothesis, a shift upward in next year’s curve.
c. Next year, the 2-year zero will be a 1-year zero, and will therefore sell at $1000/1.1201 = $892.78. Similarly, the current 3-year zero will be a 2-year zero and will sell for $782.93.

Expected total rate of return:

2-year bond: \[\frac{892.78}{811.62} - 1 = 1.1000 \text{ or } 10\%\]

3-year bond: \[\frac{782.93}{711.78} - 1 = 1.1000 \text{ or } 10\%\]

d. The current price of the bond should equal the value of each payment times the present value of $1 to be received at the “maturity” of that payment. The present value schedule can be taken directly from the prices of zero-coupon bonds calculated above.

Current price = 120 \times (.90909) + 120 \times (.81162) + 1,120 \times (.71178)
= 109.0908 + 97.3944 + 797.1936
= $1,003.68

Similarly, the expected prices of zeros in 1 year can be used to calculate the expected bond value at that time:

Expected price 1 year from now = 120 \times .89278 + 1120 \times .78293
= 107.1336 + 876.8816
= $984.02

Total expected rate of return = \[\frac{120 + (984.02 - 1003.68)}{1003.68}\]
= \[\frac{120 - 19.66}{1003.68}\] = .1000 or 10\%

13. a. A 3-year zero with face value $100 will sell today at a yield of 6% and a price of $100/1.06^3 = $83.96. Next year, the bond will have a two-year maturity, and therefore a yield of 6% (reading from next year’s forecasted yield curve). The price will be $89.00, resulting in a holding period return of 6%.

b. The forward rates based on today’s yield curve are as follows:
Year | Forward Rate  \\
---|-------------------
2  | 6.01% $(1.05^2/1.04 - 1)$  \\
3  | 8.03% $(1.06^3/1.05^2 - 1)$

Using the forward rates, the yield curve next year is forecast as:

<table>
<thead>
<tr>
<th>Maturity</th>
<th>YTM</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6.01%</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>7.02%</td>
<td>$(1.0601 \times 1.0803)^{1/2} - 1$</td>
</tr>
</tbody>
</table>

The market forecast is for a higher YTM on 2–year bonds than your forecast. Thus, the market predicts a lower price and higher rate of return.

14. a. $P = \frac{9}{1.07} + \frac{109}{(1.08)^2} = 101.86$

b. YTM = 7.958%, which is the solution to:

$$\frac{9}{1+y} + \frac{109}{(1+y)^2} = 101.86$$

[On your calculator, input n = 2; FV = 100; PMT = 9; PV = (-)101.86; compute i]

c. The forward rate for next year derived from the zero-coupon yield curve is:

$$1 + f_2 = \frac{(1.08)^2}{1.07} = 1.0901$$ which implies $f_2 = 9.01%$.

Therefore, using an expected rate for next year of $r_2 = 9.01%$, we find that the forecast bond price is

$$P = \frac{109}{1.0901} = 99.99$$

d. If the liquidity premium is 1% then the forecast interest rate is :

$$E(r_2) = f_2 - \text{liquidity premium} = 9.01% - 1% = 8.01%$$

and you forecast the bond to sell at $\frac{109}{1.0801} = 100.92$. 

15-5
15. The coupon bonds may be viewed as portfolios of stripped zeros: each coupon can stand alone as an independent zero-coupon bond. Therefore, yields on coupon bonds will reflect yields on payments with dates corresponding to each coupon. When the yield curve is upward sloping, coupon bonds will have lower yields than zeros with the same maturity, because the yields to maturity on coupon bonds will reflect the yields on the earlier, interim coupon payments.

16. a. The current bond price is $85 \times .9434 + 85 \times .87352 + 1085 \times .81637 = 1040.20$ which implies a yield to maturity of 6.97% [since $85 \times \text{Annuity factor}(6.97\%, 3) + 1000 \times \text{PV factor}(6.97\%, 3) = 1040.20$].

b. If next year, $y = 8\%$, then the bond price will be

$$85 \times \text{Annuity factor}(8\%, 2) + 1000 \times \text{PV factor}(8\%, 2) = 1008.92$$

for a holding period of return equal to $[85 + (1008.92 – 1040.20)]/1040.20 = .0516$ or 5.16%.

17. | Year | Forward rate | PV of $1$ received at period end |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5%</td>
<td>1/1.05 = $.9524</td>
</tr>
<tr>
<td>2</td>
<td>7%</td>
<td>1/(1.05)(1.07) = .8901</td>
</tr>
<tr>
<td>3</td>
<td>8%</td>
<td>1/(1.05)(1.07)(1.08) = .8241</td>
</tr>
</tbody>
</table>

a. Price = $(60 \times .9524) + (60 \times .8901) + (1060 \times .8241) = 984.10$

b. $984.10 = 60 \times \text{Annuity factor}(y, 3) + 1000 \times \text{PV factor}(y, 3)$

which can be solved to show that $y = 6.60\%$

c. | Period | Payment Received at end of period | Will grow by a factor of | To a future value of |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$60</td>
<td>(1.07)(1.08)</td>
<td>69.34</td>
</tr>
<tr>
<td>2</td>
<td>$60</td>
<td>(1.08)</td>
<td>64.80</td>
</tr>
<tr>
<td>3</td>
<td>$1060</td>
<td>1</td>
<td>1060.00</td>
</tr>
</tbody>
</table>

$$984.10 \times (1 + RCY)^3 = 1194.14$$

$$1 + RCY = \left(\frac{1194.14}{984.10}\right)^{1/3} = 1.0666$$

$$RCY = 6.66\%$$
d. Next year, the bond will sell for

\[ 60 \times \text{Annuity factor}(7\%, 2) + 1000 \times \text{PV factor}(7\%, 2) = 981.92 \]

which implies a capital loss of \( 984.10 - 981.92 = 2.18 \).

The holding period return is \( \frac{60 + (-2.18)}{984.10} = 0.0588 \) or 5.88%.

18. The following table shows the expected short-term interest rate based on the projections of Federal Reserve rate cuts, the term premium (which increases at a rate of .10% per 12 months), the forward rate (which is the sum of the expected rate and term premium), and the YTM, which is the geometric average of the forward rates.

<table>
<thead>
<tr>
<th>Time</th>
<th>Expected short rate</th>
<th>Term premium</th>
<th>Forward rate (annual)</th>
<th>Forward rate (semi-annual)</th>
<th>YTM (semi-annual)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5.0%</td>
<td>none</td>
<td>5.00%</td>
<td>2.500%</td>
<td>2.500%</td>
</tr>
<tr>
<td>6 months</td>
<td>4.5</td>
<td>.05%</td>
<td>4.55</td>
<td>2.275</td>
<td>2.387</td>
</tr>
<tr>
<td>12 months</td>
<td>4.0</td>
<td>.10</td>
<td>4.10</td>
<td>2.05</td>
<td>2.275</td>
</tr>
<tr>
<td>18 months</td>
<td>4.0</td>
<td>.15</td>
<td>4.15</td>
<td>2.075</td>
<td>2.225</td>
</tr>
<tr>
<td>24 months</td>
<td>4.0</td>
<td>.20</td>
<td>4.20</td>
<td>2.10</td>
<td>2.200</td>
</tr>
<tr>
<td>30 months</td>
<td>5.0</td>
<td>.25</td>
<td>5.25</td>
<td>2.625</td>
<td>2.271</td>
</tr>
</tbody>
</table>

This analysis is predicated on the liquidity preference theory of the term structure, which asserts that the forward rate in any period is the sum of the expected short rate plus the liquidity premium.

19. a. The return on the one-year bond will be 6.1%. The price of the 4-year zero today is \( \frac{1000}{1.0644} = 940.98 \). Next year, if the yield curve is unchanged, the bond will have a 3-year maturity, a YTM of 6.3%, and therefore sell for \( \frac{1000}{1.0633} = 936.66 \), resulting in a one-year return of 6.7%. The longer-term bond is expected to provide the higher return in this case because its YTM is expected to decline during the holding period.

b. If you believe in the expectations theory, you would not expect that the yield curve next year will be the same as today's curve. The upward slope in today's curve would be evidence that expected short rates are rising and that the yield curve will shift upward, reducing the holding period return on the four-year bond. Under the expectations hypothesis, all bonds have equal expected holding period returns. Therefore, you would predict that the HPR for the 4-year bond would be 6.1%, the same as for the 1-year bond.
20. a. Five-year Spot Rate:

\[
1000 = \frac{70}{(1 + y_1)^1} + \frac{70}{(1 + y_2)^2} + \frac{70}{(1 + y_3)^3} + \frac{70}{(1 + y_4)^4} + \frac{1070}{(1 + y_5)^5}
\]

\[
1000 = \frac{70}{1.05} + \frac{70}{(1.0521)^2} + \frac{70}{(1.0605)^3} + \frac{70}{(1.0716)^4} + \frac{1070}{(1 + y_5)^5}
\]

\[
1000 = 66.67 + 63.24 + 58.69 + 53.08 + \frac{1070}{(1 + y_5)^5}
\]

\[
758.32 = \frac{1070}{(1 + y_5)^5}
\]

\[
(1 + y_5)^5 = \frac{1070}{758.32} \quad \Rightarrow \quad y_5 = \sqrt[5]{1.411} - 1 = 7.13\%
\]

Five-year Forward Rate:

\[
\frac{(1.0713)^5}{(1.0716)^4} - 1 = 1.0701 - 1 = 7.01\%
\]

b. Yield to maturity is the single discount rate that equates the present value of a series of cash flows to a current price. It is the internal rate of return.

The spot rate for a given period is the yield to maturity on a zero-coupon bond which matures at the end of the period. A spot rate is the discount rate for each period. Spot rates are used to discount each cash flow of a coupon bond to calculate a current price. Spot rates are the rates appropriate for discounting future cash flows of different maturities.

A forward rate is the implicit rate that links any two spot rates. Forward rates are directly related to spot rates, and therefore yield to maturity. Some would argue (as in the expectations theory) that forward rates are the market expectations of future interest rates. Regardless, a forward rate represents a break-even rate that links two spot rates. It is important to note that forward rates link spot rates, not yields to maturity.

Yield to maturity is not unique for any particular maturity. In other words, two bonds with the same maturity but different coupon rates may have different yields to maturity. In contrast, spot rates and forward rates for each date are unique.
c. The 4-year spot rate is 7.16%. Therefore, 7.16% is the theoretical yield to maturity for the zero-coupon U.S. Treasury note. The price of the zero-coupon note discounted at 7.16% is the present value of $1000 to be received in 4 years.

Using annual compounding, \[ PV = \frac{1000}{(1.0716)^4} = 758.35 \]

21. The price of the coupon bond, based on its yield to maturity, is

\[ 120 \times \text{Annuity factor}(5.8\%, 2) + 1000 \times \text{PV factor}(5.8\%, 2) = 1113.99. \]

If the coupons were stripped and sold separately as zeros, then based on the yield to maturity of zeros with maturities of one and two years, the coupon payments could be sold separately for

\[ \frac{120}{1.05} + \frac{1120}{1.06^2} = 1111.08. \]

The arbitrage strategy is to buy zeros with face values of $120 and $1120 and respective maturities of one and two years, and simultaneously sell the coupon bond. The profit equals $2.91 on each bond.

22. a. The one-year bond has a yield to maturity of 6%:

\[ 94.34 = \frac{100}{1+y_1} \implies y_1 = .0600 \]

The yield on the two-year zero is 8.472%:

\[ 84.99 = \frac{100}{(1+y_2)^2} \implies y_2 = .08472 \]

The price of the coupon bond is

\[ \frac{12}{1.06} + \frac{112}{(1.08472)^2} = 106.51 \]

Therefore its yield to maturity is 8.333% [on your calculator: n = 2; PV = (-)106.51; FV = 100; PMT = 12]

b. \[ f_2 = \frac{(1 + y_2)^2}{1 + y_1} - 1 = \frac{(1.08472)^2}{1.06} - 1 = .11 = 11\% \]

c. Expected price = \[ \frac{112}{1.11} = 100.90. \] (Note that next year, the coupon bond will have one payment left.)
Expected holding period return = \( \frac{12 + (100.90 - 106.51)}{106.51} = .06 = 6\% \)

which is the same as the return on the one-year zero.

d. If there is a liquidity premium, then

\[ E(r_2) < f_2 \]

\[ E(Price) = \frac{112}{1 + E(r_2)} > 100.90 \]

\[ E(HPR) > 6\% \]

23. a. Maturity (years) | Price  | Forward rate \\
---|---|---
1  | 925.93 | 
2  | 853.39 | .085 
3  | 782.92 | .090 
4  | 715.00 | .095 
5  | 650.00 | .100 

b. For each 3-year zero that you issue today, you can use the proceeds to buy 

\( \frac{782.92}{715} = 1.095 \) four-year zeros. Your cash flows are thus as follows:

<table>
<thead>
<tr>
<th>Time</th>
<th>Cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>-1,000 [The 3-year zero that you issue matures and you pay out face value of $1000.]</td>
</tr>
<tr>
<td>4</td>
<td>+1,095 [The 4-year zeros that you bought mature and you collect face value on each one.]</td>
</tr>
</tbody>
</table>

This is a synthetic one-year loan originating at time 3, with a rate of .095 = 9.5%, precisely the forward rate for year 3.

c. For each 4-year zero that you issue today, you can use the proceeds to buy 

\( \frac{715}{650} = 1.10 \) five-year zeros. Your cash flows are thus as follows:
24. a. For each three-year zero that you buy today, you need to issue $782.92/650 = 1.2045$ five-year zeros to make your time-0 cash flow equal to zero.

b. Your cash flows are thus as follows:

<table>
<thead>
<tr>
<th>Time</th>
<th>Cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>−1,000</td>
</tr>
<tr>
<td>5</td>
<td>+1,204.50</td>
</tr>
</tbody>
</table>

[The 3-year zero that you issue matures and you pay out face value of $1000.]

This is a synthetic two-year loan originating at time 3.

c. The two-year rate on the forward loan is $1,204.50/1,000 − 1 = .2045 = 20.45%$

d. The one-year forward rates for years 4 and 5 are 9.5% and 10%, respectively. Notice that $1.095 \times 1.10 = 1.2045$, which equals $(1 + \text{the two-year forward rate})$ on the 3-year ahead forward loan.

The 5-year YTM is 9.0%. The 3-year YTM is 8.5%. Therefore, another way to derive the 2-year forward rate for a loan starting at time 3 is:

\[1 + f_3(2) = \frac{(1 + y_5)^5}{(1 + y_3)^3} = \frac{(1.09)^5}{(1.085)^3} = 1.2045\]

25. We wish to know what interest rate can be engineered on a loan that initiates in three years with a term of two years. Three-year zero coupon bonds with par value of $1000 sell today for $1000/(1.0619)^3 = $835.12$. Five-year zeros sell for $1000/(1.0651)^5 = $729.54$.

For each 3-year zero that you sell, you can use the proceeds to buy $835.12/729.54 = 1.14472$ 5-year zeros. Your cash flows are thus as follows:
<table>
<thead>
<tr>
<th>Time</th>
<th>Cash flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>–1,000</td>
</tr>
<tr>
<td>5</td>
<td>+1,144.72</td>
</tr>
</tbody>
</table>

This is a synthetic 2-year loan originating at time 3, with a two-year rate of .14472 = 14.472%.

An alternative approach would derive the 2-year forward rate as of year 3 as:

\[ 1 + f_3(2) = \frac{(1 + y_5)^5}{(1 + y_3)^3} = \frac{(1.0651)^5}{(1.0619)^3} = 1.14472 \]